On Topological Pseudo-UP Algebras Based on Pseudo-UP Ideals

Prof. Alias B. Khalaf [a], Mahmoud A.Yousef[b]

[a]Department of Mathematics, College of Science, University of Duhok, Kurdistan Region, Iraq.Email: aliasbkhalaf@uod.ac

[b]Department of Mathematics, College of Basic Education, University of Duhok, Kurdistan Region, Iraq. Email: mahmood.a.y.91@gmail.com

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Abstract: The aim of this paper is to introduce the concept of pseudo-UP ideals defined on a pseudo-UP algebra and by usingthis concept we study a uniform structure on a pseudo-UP algebra. Also, we study some properties of the topology whichis generated by a filter base on pseudo-UP algebra. Moreover, several results are obtained using the concept of pseudo-UPhomomorphisms.

Keywords:UP-algebra; topological UP-algebra; pseudo-UP algebra; topological pseudo-UP algebra; pseudo-UP homomorphism.

I. Introduction

Recently, studying and investigating the topological properties of several types of algebras has becomeof interest to many researchers.The topological concepts of (BCK/BCC/ BE)- algebras are given in [4,1,5]. In 1998, Lee and Ryu investigated topological BCK-algebras and determined several topological featuresof this structure.In 2008, Ahn and Kwon introduced the concept of topological BCC-algebras. In 2017, Mehrshad and Golzarpoorinvestigated some characteristics of the topological BE-algebras and uniformBE-algebras. In this same year, Iampan[2]presented a new class of algebras called UP-algebras whichis an extension of KU-algebras [6] introduced by Prabpayak and Leerawat in 2009. Later in 2019, Satirad and Iampan[10]presented and established further characteristics of the topological UP-algebras.In 2020, Romano [7] introduced another class of algebras called pseudo-UP algebras as an extension of UP- algebras. Also, he studied the concepts of pseudo-UP filers and pseudo-UP ideals of pseudo-UP algebras in [8]. Furthermore, he introduced the concept of homomorphisms between pseudo-UP algebras in [9].

This paper is formatted as follows:In Section 2, we present some definitions and propositions on pseudo-UP algebras and topologies which are needed to develop this paper. In Section 3, we study the congruence relation on pseudo-UP ideals. In Section 4, we study the uniform topology on pseudo-UP algebra.We employ the congruence relationship for the uniform topology to create uniform structures based onpseudo-UP ideals of pseudo-UP algebras. Also, we show that topological pseudo-UP algebra is pseudo-UP algebra with uniform topology. Additionally, several characteristics are acquired. In Section 5, weintroduce the filter base on pseudo-UP algebra to generate a topology on pseudo-UP algebra.

2. Preliminaries

In this section, we provide some basic information and observations on pseudo-UP algebras andtopological concepts which are essential for this paper.

Definition 2.1.[7]A pseudo-UP algebra is a structure $((X,\leq),\ \cdot,\ *,0)$ where $\leq$ is a binary operation on a set X, $\cdot$ and $*$ are two binary operations on X if X satisfies the following axioms: for all $x,y,z\in X$,

1. $(y\cdot z)\leq(x\cdot y)*(x\cdot z)$ and $(y*z)\leq(x*y)\cdot(x*z)$.
2. $x\leq y$ and $y\leq x$ then $x=y$.
3. $(y\cdot 0)*x=x$ and $(y*0)\cdot x=x$.
4. $x\leq y\Leftrightarrow x\cdot y=0$ and $x\leq y\Leftrightarrow x*y=0$.

Proposition 2.2. [7] In a pseudo-UP algebra $((X,\leq),\cdot,*,0)$ the following statements hold: for all $x\in X$,

1. $x\cdot 0=0$ and $x*0=0$,
2. $0\cdot x=x$ and $0*x=x$, and

3. $x \cdot x = 0$ and $x * x = 0$.

Proposition 2.3. [7] In a pseudo-UP algebra $((X, \leq), \cdot, *, 0)$ the following statements hold: for all $x, y \in X$,

1. $x \leq y \cdot x$, and
2. $x \leq y * x$.

Definition 2.4. [8] A non-empty subset $\mathcal{I}$ of a pseudo-UP algebra X is said to be a pseudo-UP ideal of X if it satisfies: for all $x, y \in X$,

1. $0 \in \mathcal{I}$,
2. $x \cdot (x * z) \in \mathcal{I}$ and $y \in \mathcal{I}$ then $x \cdot z \in \mathcal{I}$, and
3. $x * (x \cdot z) \in \mathcal{I}$ and $y \in \mathcal{I}$ then $x * z \in \mathcal{I}$.

Proposition 2.5. [8] Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X, then the following statements hold: for all $x, y \in X$,

1. if $x \in \mathcal{I}$ and $x \cdot y \in \mathcal{I}$ then $y \in \mathcal{I}$, and
2. if $x \in \mathcal{I}$ and $x * y \in \mathcal{I}$ then $y \in \mathcal{I}$.

Definition 2.6. [8] A non-empty subset $\mathcal{F}$ of a pseudo-UP algebra X is said to be a pseudo-UP filter of X if it satisfies: for all $x, y \in X$,

1. $0 \in \mathcal{F}$,
2. $x \cdot y \in \mathcal{F}$ and $x \in \mathcal{F}$ then $y \in \mathcal{F}$, and
3. $x * y \in \mathcal{F}$ and $x \in \mathcal{F}$ then $y \in \mathcal{F}$.

Definition 2.7. [9] Let $((X, \leq), \cdot, *, 0)$ and $((Y, \leq_Y), \cdot_Y, *_Y, 0_Y)$be two pseudo-UP algebras. A map $f: X \rightarrow Y$ is a pseudo-UP homomorphism if

$$f(x \cdot y) = f(x) \cdot_Y f(y) \text{ and} f(x \cdot y) = f(x) *_Y f(y),$$

for all $x, y \in X$. Moreover, f is a pseudo-UP isomorphism if it is bijective.

Definition 2.8. A pseudo-UP algebra X is said to be negative implicative if it satisfies the condition: for all $x, y, z \in X$,

$$(x \cdot y) \cdot (x \cdot z) = x \cdot (y \cdot z) \text{ } and \text{ } (x * y) * (x * y) = x * (y * z).$$

Example 2.9. Let $X = \{0, a, b, c\}$ and the two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

$*$	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	a	b	0

Table1. A negative implicative pseudo-UP algebra

Then it isclear that $((X, \leq), \cdot, *, 0)$ is a pesudo-UP algebra and satisfies the negative implicative condition.

In the remainder of this section, we introduce some topological concepts, by (X, τ) or X we mean a topological space. Let A be a subset X, the closure of A is defined by$cl(A) = \{x \in X: \forall\ O \in \tau$ such that $x \in O, O \cap A \neq \phi\}$. The set of all interior points of A denoted by $int(A)$ and defined as $int(A) = \bigcup\{U: U \in \tau$ and $U \subseteq A\}$. Let $f: (X, \tau) \rightarrow (Y, \tau_Y)$ be a function, then f is continuous if the opposite image of each open set in Y is open X. Also, f is called closed (resp., open) map if the image of each closed (resp., open) set in X is closed (resp., open) set in Y. Furthermore, f is homeomorphism if f is isomorphism, continuous, and open. All topological concepts above can be found in all texts of general topology.

Definition 2.10. [11] A pseudo-UP algebra $((X, \leq), \cdot, *, 0)$ with a topology τ is called a topological pesudo-UP algebra (for short TPUP-algebra) if for each open set O containing $x \cdot y$ and for each open set W containing $x * y$, there exist open sets U_1 and V_1 (U_2 and V_2) containing x and y respectively such that $U_1 \cdot V_1 \subseteq O$ and $U_2 * V_2 \subseteq W$ for all $x, y \in X$.

Proposition 2.11. [11] Let $((X, \leq), \cdot, *, 0, \tau)$ be a TPUP-algebra and M_0 be the minimal open set containing 0. If $x \in M_0$ then M_0 is the minimal open set containing x.

Proposition 2.11. [11] Let $((X, \leq), \cdot, *, 0, \tau)$ be a TPUP-algebra and M_x, $M_y$be two minimal open sets containing x, y respectively. If $x \cdot y, x * y \notin M_0$, then $y \notin M_x$ and $x \notin M_y$ where $x \neq 0$ and $y \neq 0$.

3. On pseudo-UP ideals

In this section, we give some properties of pseudo-UP ideals of pseudo-UP algebras.

Proposition 3.1. Let X be a pseudo-UP algebra and $\{\mathcal{I}_i\}_{i\in A}$ be a family of pseudo-UP ideals of X. Then $\cap_{i\in A} \mathcal{I}_i$ is a UP-ideal of X.

Proof. Clearly, if $0 \in \mathcal{I}_i$ for all $i \in A$, then $0 \in \cap_{i\in A} \mathcal{I}_i$. Let $x, y, z \in X$ such that $x \cdot (y * z) \in \cap_{i\in A} \mathcal{I}_i, x * (y \cdot z) \in \cap_{i\in A} \mathcal{I}_i$ and $y \in \cap_{i\in A} \mathcal{I}_i$.Hence, $x \cdot (y * z) \in \mathcal{I}_i$, $x * (y \cdot z) \in \mathcal{I}_i$ and $y \in \mathcal{I}_i$ for all $i \in A$. Since $\mathcal{I}_i$is a pseudo-UP ideal of X, then $x \cdot z \in \mathcal{I}_i$and $x * z \in \mathcal{I}_i$for all $i \in A$. Therefore, $x \cdot z \in \cap_{i\in A} \mathcal{I}_i$and $x * z \in \cap_{i\in A} \mathcal{I}_i$Hence,$\cap_{i\in A} \mathcal{I}_i$is a pseudo-UP ideal of X.

Definition 3.2. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X. Define the relation $\sim_{\mathcal{I}}$on X as follows: for all $x, y \in X$,

$$x \sim_{\mathcal{I}} y \text{ if and only if } x \cdot y, x * y \in \mathcal{I} \text{ and } y \cdot x, y * x \in \mathcal{I}.$$

Example 3.3. Let $X = \{0, a, b, c\}$ with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

·	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	b	0

*	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

Table2. A psudo-UP algebra

By easy calculation, we can check that$((X, \leq), \cdot, *, 0)$ is a pseudo-UP algebra and $\mathcal{I} = \{0, a, c\}$ is a pseudo-UP ideal of a pseudo-UP algebra X. Then

$$\sim_{\mathcal{I}}= \{(0,0), (0,\text{a}), (\text{a},0), (0,\text{c}), (\text{c},0), (\text{a},\text{c}), (\text{c},\text{a}), (\text{a},\text{a}),\ \text{b},\text{b}), (\text{c},\text{c})\},$$

so, we can see that $\sim_{\mathcal{I}}$ is an equivalence relation on X.

Definition 3.4. An equivalence relation R on a pseudo-UP algebra X is called a congruence relation if for all $x, y, u, v \in X$,

$$x\, R\, u \text{ and } y\, R\, v \text{ implies } x \cdot y\, R\, u \cdot v \text{ and } x * y\, R\, u * v.$$

Proposition 3.5. If $\mathcal{I}$ is a pseudo-UP ideal of a pseudo-UP algebra X, then the binary relation $\sim_{\mathcal{I}}$ defined as in Definition 3.4, is a congruence relation on X.

Proof.*Reflexive:* For all $x \in X, x \cdot x = 0$ and $x * x = 0$. Since $\mathcal{I}$ is a pseudo-UP ideal of X, then $x \cdot x = 0 \in \mathcal{I}$ and $x * x = 0 \in \mathcal{I}$. Therefore, $x \sim_{\mathcal{I}} x$.

Symmetric: Let $x, y \in X$ such that $x \sim_{\mathcal{I}} y$. Then we have $x \cdot y,\ x * y \in \mathcal{I}$, and $y \cdot x,\ y * x \in \mathcal{I}$, so $y \cdot x,\ y * x \in \mathcal{I}$ and $x \cdot y,\ x * y \in \mathcal{I}$. Therefore, $y \sim_{\mathcal{I}} x$.

Transitive: Let $x, y, z \in X$ such that $x \sim_{\mathcal{I}} y$ and $y \sim_{\mathcal{I}} z$. Then we have

$$x \cdot y, x * y, y \cdot z, y * z \in \mathcal{I} \text{ and } y \cdot x, y * x, z \cdot y, z * y \in \mathcal{I}.$$

Since $\mathcal{I}$ is a pseudo-UP ideal of X, we have

$$(y \cdot z) * [(x \cdot y) * (x \cdot z)] = 0 \in \mathcal{I} \text{ and } (y * z) \cdot [(x * y) \cdot (x * z)] = 0 \in \mathcal{I}.$$

Since, $(y \cdot z), (y * z) \in \mathcal{I}$ then by Proposition 2.5, $(x \cdot y) * (x \cdot z), (x * y) \cdot (x * z) \in \mathcal{I}$. Again, since $(x \cdot y), (x * y) \in \mathcal{I}$ then by Proposition 2.5, $(x \cdot z), (x * z) \in \mathcal{I}$. Similarly, $\mathcal{I}$ is a pseudo-UP ideal of X,

$$(y \cdot x) * [(z \cdot y) * (z \cdot x)] = 0 \in \mathcal{I} \text{ and } (y * x) \cdot [(z * y) \cdot (z * x)] = 0 \in \mathcal{I}.$$

Since $(y \cdot x), (y * x) \in \mathcal{I}$ then by Proposition 2.5, $(z \cdot y) * (z \cdot x), (z * y) \cdot (z * x) \in \mathcal{I}$. Again, since $(z \cdot y), (z * y) \in \mathcal{I}$ then by Proposition 2.5, $(z \cdot x), (z * x) \in \mathcal{I}$. Therefore, $x \sim_{\mathcal{I}} y$.

Thus, $\sim_{\mathcal{I}}$ is an equivalent relation on X.

Now, let $x, y, u, v \in X$ such that $x \sim_{\mathcal{I}} u$ and $y \sim_{\mathcal{I}} v$. Then we have

$$x \cdot u, x * u, y \cdot v, y * v \in \mathcal{I} \text{ and } u \cdot x, u * x, y \cdot v, y * v \in \mathcal{I}.$$

Since $\mathcal{I}$ is a pseudo-UP ideal of X, we have

$$(v \cdot y) * [(x \cdot v) * (x \cdot y)] = 0 \in \mathcal{I} \text{ and } (v * y) \cdot [(x * v) \cdot (x * y)] = 0 \in \mathcal{I}.$$

Since, $(v \cdot y), (v * y) \in \mathcal{I}$ then by Proposition 2.5, $(x \cdot v) * (x \cdot y), (x * v) \cdot (x * y) \in \mathcal{I}$. Similarly, since $\mathcal{I}$ is a pseudo-UP ideal of X, we have

$$(y \cdot v) * [(x \cdot y) * (x \cdot v)] = 0 \ \in \mathcal{I} \text{ and } (y * v) \cdot [(x * y) \cdot (x * v)] = 0 \in \mathcal{I}.$$

Since $(y \cdot v), (y * v) \in \mathcal{I}$ then by Proposition 2.5, $(x \cdot y) * (x \cdot v), (x * y) \cdot (x * v) \in \mathcal{I}$. Therefore, $x \cdot y \ \sim_{\mathcal{I}} \ u \cdot v$ and $x * y \ \sim_{\mathcal{I}} \ u * v$. On the other hand, since $\mathcal{I}$ is a pseudo-UP ideal of X, we have

$(u \cdot v) * [(x \cdot u) * (x \cdot v)] = 0 \ \in \mathcal{I}$ and $(u * v) \cdot [(x * u) \cdot (x * v)] = 0 \in \mathcal{I}$.

Since $(x \cdot u), (x * u) \ \in \mathcal{I}$, then $(u \cdot v) * (x \cdot v), (u * v) \cdot (x * v) \in \mathcal{I}$. Similarly, since $\mathcal{I}$ is a pseudo-UP ideal of X, we have

$$(x \cdot v) * [(u \cdot x) * (u \cdot v)] = 0 \ \in \mathcal{I} \text{ and } (x * v) \cdot [(u * x) \cdot (u * v)] = 0 \in \mathcal{I}.$$

Since, $(u \cdot x), (u * x) \in \mathcal{I}$ then $(x \cdot v) * (u \cdot v)$, $(x * v) \cdot (u * v) \ \in \mathcal{I}$. Therefore, $x \cdot v \ \sim_{\mathcal{I}} \ u \cdot v$ and $x * v \ \sim_{\mathcal{I}} u * v$. By transitive of $\sim_{\mathcal{I}}$ we have $x \cdot y \sim_{\mathcal{I}} u \cdot v$ and $x * y \sim_{\mathcal{I}} u * v$. Hence, $\sim_{\mathcal{I}}$ is a congruence relation on X.

4. Uniform topology on pseudo-UP algebras

Suppose that X is a pseudo-UP algebra and $U, V \subseteq X \times X$, consider the following notations:

$U[x] = \{y \in X : (x, y) \in U\}$,

$U \circ V = \{(x, y) \in X \times X \mid \mathit{for\ some}\ z \in X, (x, z) \in U\ \mathit{and}\ (z, y) \in V\}$,

$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$,

$\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 4.1. [3] A uniformity on X is defined as a collection $\mathcal{K}$ of subsets of $X \times X$ that satisfy the following conditions: for all $U, V \in \mathcal{K}$,

$(U_1) \Delta \subseteq U$,

$(U_2) U^{-1} \subseteq \mathcal{K}$,

$(U_3) W \circ W \subseteq U$ for some $W \in \mathcal{K}$,

(U_4) $U \cap V \in \mathcal{K}$, and

(U_5) if $U \subseteq W \subseteq X \times X$, then $W \in \mathcal{K}$.

Then $(X, \mathcal{K})$ is said to be a uniform space (or uniform stracture).

Definition 4.2. Suppose that Λ is an arbitrary family of pseudo-UP ideals of a pseudo-UP algebra X, $U \subseteq X \times X$ and $A \subseteq X$, then we define the following sets:

1. $U_{\mathcal{I}} = \{(x, y) \in X \times X : x \cdot y, x * y \in \mathcal{I}\ \mathit{and}\ y \cdot x, y * x \in \mathcal{I}\}$,
2. $U_{\mathcal{I}}[x] = \{y \in X : (x, y) \in U_{\mathcal{I}}\}$, and $U_{\mathcal{I}}[A] = \bigcup_{a \in A} U_{\mathcal{I}}[a]$,
3. $\mathcal{K}^* = \{U_{\mathcal{I}} : \mathcal{I} \in \Lambda\}$,
4. $\mathcal{K} = \{U \subseteq X \times X : U_{\mathcal{I}} \subseteq U\ \mathit{for\ some}\ U_{\mathcal{I}} \in \mathcal{K}^*\}$.

Proposition4.3. Let Λ be a family of pseudo-UP ideals of a pseudo-UP-algebra X, then $\mathcal{K}^*$ satisfies the conditions $(U_1) - (U_4)$.

Proof. (U_1): Since $\mathcal{J}$ is a pseudo-UP ideal of X, then for all $x \in X$, we have $x \sim_{\mathcal{J}} x$. Hence, $\Delta \subseteq U_{\mathcal{J}}$, for any $U_{\mathcal{J}} \in \mathcal{K}^*$.

(U_2): Let $U_{\mathcal{J}} \in \mathcal{K}^*$, we have

$$(x,y) \in (U_{\mathcal{J}})^{-1} \Leftrightarrow (y,x) \in U_{\mathcal{J}} \Leftrightarrow y \sim_{\mathcal{J}} x \Leftrightarrow x \sim_{\mathcal{J}} y \Leftrightarrow (x,y) \in U_{\mathcal{J}}.$$

Hence, $U^{-1} \subseteq \mathcal{K}^*$.

(U_3):Let $U_{\mathcal{J}} \in \mathcal{K}^*$ and $(x,z) \in U_{\mathcal{J}} \circ U_{\mathcal{J}}$, then there exists $y \in X$ such that $(x,y),(y,z) \in U_{\mathcal{J}}$ implies that $x \sim_{\mathcal{J}} y$ and $y \sim_{\mathcal{J}} z$. By transitive of $\sim_{\mathcal{J}}$ we have $x \sim_{\mathcal{J}} z$. Therefore, $(x,z) \in U_{\mathcal{J}}$ and hence $U_{\mathcal{J}} \circ U_{\mathcal{J}} \subseteq U_{\mathcal{J}}$.

(U_4): Let $U_{\mathcal{J}}, U_{\mathcal{N}} \in \mathcal{K}^*$. We claim that $U_{\mathcal{J}} \cap U_{\mathcal{N}} \in \mathcal{K}^*$. Let

$$(x,y) \in U_{\mathcal{J}} \cap U_{\mathcal{N}} \Leftrightarrow (x,y) \in U_{\mathcal{J}} \; and \; (x,y) \in U_{\mathcal{N}} \Leftrightarrow x \cdot y, x * y \in \mathcal{J} \cap \mathcal{N} \; and \; y \cdot x, \\ y * x \in \mathcal{J} \cap \mathcal{N} \Leftrightarrow x \sim_{\mathcal{J} \cap \mathcal{N}} y \Leftrightarrow (x,y) \in U_{\mathcal{J} \cap \mathcal{N}}.$$

Therefore, $U_{\mathcal{J}} \cap U_{\mathcal{N}} = U_{\mathcal{J} \cap \mathcal{N}}$. Since, $U_{\mathcal{J}}, U_{\mathcal{N}} \in \Lambda$, then we have $U_{\mathcal{J}} \cap U_{\mathcal{N}} \in \mathcal{K}^*$ and so $U_{\mathcal{J} \cap \mathcal{N}} \in \mathcal{K}^*$.

The following example explain that$\mathcal{K}^*$ is not uniform structure.

Example 4.4. Let $X = \{0, a, b, c, d\}$ and the two binary operations $\cdot$ and $*$defined by the following Cayley tables:

·	**0**	**a**	**b**	**c**	**d**
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

*	**0**	**a**	**b**	**c**	**d**
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Table 3. A pseudo-UP ideal of a pseudo-UPalgebra

Then it is clear that $((X,\leq), \cdot, *, 0)$ is a pesudo-UP algebra, $\{0\}, X, \mathcal{J}_1 = \{0, a, b\}$and $\mathcal{J}_2 = \{0, a, c\}$ are pseudo-UP ideals of X. Hence, $\mathcal{K}^* = \{U_{\{0\}}, U_X, U_{\mathcal{J}_1}, U_{\mathcal{J}_2}\}$ where $U_{\{0\}} = \Delta$, $U_X = X \times X$,

$$U_{\mathcal{J}_1} = \{(0,0),(a,a),(b,b),(c,c),(d,d),(0,a),(a,0),(0,b),(b,0),(a,b),(b,a)\},$$

and

$$U_{\mathcal{J}_2} = \{(0,0),(a,a),(b,b),(c,c),(d,d),(0,a),(a,0),(0,c),(a,c),(c,a)\}.$$

Let $M = \mathcal{J}_1 \cap \{d\} = \{0, a, b, d\}$, then $U_M = U_{\mathcal{J}_1} \cup \{(d,d),(a,d),(d,a), (b,d),(d,b),(c,d),(d,c)\}$. We have $U_{\mathcal{J}_1} \subseteq U_M \subseteq X \times X$. Moreover, $M \notin \Lambda$ since $d \cdot c = 0 \in M, d * c = 0 \in M$ and $d \in M$ but $c \notin M$. Therefore, $U_M \notin \mathcal{K}^*$.This means that$\mathcal{K}^*$is not satisfying the condition(U_5) from Definition 4.1.

Proposition 4.5. Let Λ be a family of pseudo-UP ideals of a pseudo-UP-algebra X, then$(X,\mathcal{K})$is a uniform structure.

Proof. From Proposition 4.3, we obtain that $\mathcal{K}$satisfying the conditions$(U_1) - (U_4)$. It is enough to show that $\mathcal{K}$ satisfying(U_5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then there exists a $U_{\mathcal{J}} \subseteq U \subseteq V$, which means $V \in \mathcal{K}$. Hence the proof.

Lemma 4.6. Let X be a pseudo-UP algebra and let$U, V \in \mathcal{K}$ where $U \subseteq V$, then $U[x] \subseteq V[x]$ for every $x \in X$.

Proof. Suppose that $U, V \in \mathcal{K}$ where $U \subseteq V$ and let$y \in X$. Let $b \in U[y]$, then $(y,b) \in U \subseteq V$and so$(y,b) \in V$. Thus, $b \in V[y]$ and hence $U[y] \subseteq V[y]$.

Proposition 4.7. Let $(X,\mathcal{K})$ be a uniform structure, then

$$\mathcal{T} := \{G \subseteq X : \forall\, x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\},$$

is a topology on X.

Proof. Let $(X,\mathcal{K})$be a uniform structure, for all $x \in X$ and $U \in \mathcal{K}$, $U[x] \subseteq X$. Hence, $X \in \mathcal{T}$ and also $\phi \in \mathcal{T}$ by definition. Let $x \in \bigcup_{G_i \in \mathcal{T}, i \in M} G_i$, then there exists $j \in M$ such that $x \in G_j$. Since $G_j \in \mathcal{T}$, there exist $U_j \in \mathcal{K}$ such that $U_j[x] \subseteq G_j[x]$. This implies that $U_j[x] \subseteq \bigcup_{G_i \in \mathcal{T}, i \in M} G_i$. Hence, $\bigcup_{G_i \in \mathcal{T}, i \in M} G_i \in \mathcal{T}$.

Suppose that $G, H \in \mathcal{T}$ such that $x \in G \cap H$, then there exist $U, V \in \mathcal{K}$ such that $U[x] \in G$ and $V[x] \in H$. Let $W := U \cap V$ so by Definition 4.1, $W \in \mathcal{K}$. Let $y \in W[x]$, then $(x,y) \in U$ and $(x,y) \in V$. Therefore, $y \in U[x]$ and $y \in V[x]$. Hence, $W[x] \subseteq U[x] \cap V[x]$. Therefore, we have $W[x] \subseteq U[x] \subseteq G$ and $W[x] \subseteq V[x] \subseteq H$. Hence, $W[x] \subseteq G \cap H$ which implies that $G \cap H \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on X.

Note that $U[x]$ is an open set containing x for all $x \in X$. Moreover, we refer the uniform topology obtained by an arbitrary family Λ, by $\mathcal{T}_\Lambda$ and if $\Lambda = \mathcal{J}$, we refer to it by $\mathcal{T}_\mathcal{J}$.

Definition 4.8. If $(X,\mathcal{K})$ is a uniform structure, the topology $\mathcal{T}$ is called uniform topology on X induced by $\mathcal{K}$.

Example 4.9. From Example 3.3, consider $\mathcal{J} = \{0, a, c\}$ and $\Lambda = \{\mathcal{J}\}$then we have

$$\mathcal{K}^* = \{\, U_\mathcal{J}\} = \{(x,y) \mid x \sim_\mathcal{J} y\}$$
$$= \{(0,0),(0,a),(a,0),(0,c),(c,0),(a,c),(c,a),(a,a),(b,b),(c,c)\}.$$

Then it is easy to check that $(X,\mathcal{K})$ is a uniform structure, where $\mathcal{K} = \{U \mid U_\mathcal{J} \subseteq U\}$. Therefore, the open sets are

$$U_\mathcal{J}[a] = \{0,a,c\}$$
$$U_\mathcal{J}[b] = \{b\}$$
$$U_\mathcal{J}[c] = \{0,a,c\}$$
$$U_\mathcal{J}[0] = \{0,a,c\}.$$

From above we obtain that$\mathcal{T}_\mathcal{J} = \{\phi, \{b\}, \{0,a,c\}, X\}$. Hence, $(X,\mathcal{T}_\mathcal{J})$ is a uniform topological space.

Proposition 4.10. In a pseudo-UP algebra X, $(X,\mathcal{T}_\Lambda)$is a TPUP-algebra.

Proof.Suppose that G, H are open sets containing $x \cdot y$ and $x * y$ for all $x, y \in X$. Then there is $U \in \mathcal{K}$, such that $U[x \cdot y] \subseteq G$, $U[x * y] \subseteq H$ and a pseudo-UP ideal $\mathcal{J}$ of X such that $U_\mathcal{J} \subseteq U$. We claim that the following relation holds:

$$U_\mathcal{J}[x] \cdot U_\mathcal{J}[y] \subseteq U[x \cdot y] \; and \; U_\mathcal{J}[x] * U_\mathcal{J}[y] \subseteq U[x * y].$$

Let $a \in U_\mathcal{J}[x]$ and $b \in U_\mathcal{J}[y]$, then we have $x \sim_\mathcal{J} a$ and $y \sim_\mathcal{J} b$. Since $\sim_\mathcal{J}$ is a congruence relation, it follows that $x \cdot y \sim_\mathcal{J} a \cdot b$ and $x * y \sim_\mathcal{J} a * b$. Thus, $(x \cdot y, a \cdot b) \in U_\mathcal{J} \subseteq U$ and $(x * y, a * b) \in U_\mathcal{J} \subseteq U$. Hence, $a \cdot b \in U_\mathcal{J}[x \cdot y] \subseteq U[x \cdot y]$ and $a * b \in U_\mathcal{J}[x * y] \subseteq U[x * y]$. Therefore, $a \cdot b \in G$ and $a * b \in H$. Clearly, $U_\mathcal{J}[x]$ and $U_\mathcal{J}[y]$ are open sets containing x and y respectively. Hence, $(X,\mathcal{T}_\Lambda)$ is a TPUP-algebra.

Proposition 4.11. [3] Let X be any set and $\mathfrak{S} \subseteq \mathcal{P}(X \times X)$ be a family where the following conditions hold: for all $U \in \mathfrak{S}$,

1. $\Delta \subseteq U$,
2. U^{-1}includes an element of $\mathfrak{S}$, and
3. there is a$V \in \mathcal{S}$ such that $V \circ V \subseteq U$.

So, there is a unique uniformity $\mathcal{U}$, of which $\mathcal{S}$ is a subbase.

Proposition 4.12. Let $\mathfrak{D} := \{U_\mathcal{J} : \mathcal{J}$ be a pseudo-UP ideal of a pseudo-UP algebra $X\}$, then $\mathfrak{D}$ is the subbase for the uniformity of X. We refer to its correlating topology by$\mathfrak{S}$.

Proof.Clearly $\mathfrak{D}$ satisfies all conditions of Proposition 4.11 because $\sim_\mathcal{J}$ is an equivalence relation.

Example 4.13. In Example 4.9, it is clear that $(X,\mathcal{T}_\mathcal{J})$ is a TPUP-algebra.

Proposition 4.14. Let $\mathcal{J}$ be a pseudo-UP ideal of a pseudo-UP algebra X. $\mathcal{J} = \{0\}$ if and only if $U_\mathcal{J} = U_{\{0\}}$.

Proof.Suppose that $\mathcal{J} \neq \{0\}$, then there exist $z \in \mathcal{J}$ such that $z \neq 0$. Since $z \cdot 0 = 0 \in \mathcal{J}, 0 \cdot z = z \in \mathcal{J}$ and $z * 0 = 0 \in \mathcal{J}, 0 * z = z \in \mathcal{J}$. Hence,$0 \in U_\mathcal{J}[z]$ and so $(z,0) \in U_\mathcal{J}$. On the other hand since $z \neq 0$, $(z,0) \notin U_{\{0\}}$. Therefore, if $U_\mathcal{J} = U_{\{0\}}$, then $\mathcal{J} = \{0\}$.

Proposition 4.15. Let Λ be a family of pseudo-UP ideals of a pseudo-UP algebra X. Then any pseudo-UP ideal in the collection Λ is a clopen subset of X.

Proof. Let $\mathcal{I}$ be a pseudo-UP ideal of X in Λ and $y \in \mathcal{I}^c$. Then $y \in U_{\mathcal{I}}[y]$ and we have $\mathcal{I}^c \subseteq \bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}^c\}$. We claim that $U_{\mathcal{I}}[y] \subseteq \mathcal{I}^c$for all $y \in \mathcal{I}^c$. Let $z \in U_{\mathcal{I}}[y]$, then $z \sim_{\mathcal{I}} y$ and so $z \cdot y, z * y \in \mathcal{I}$. If $z \in \mathcal{I}$ then $y \in \mathcal{I}$, which is a contradiction. Thus, $z \in \mathcal{I}^c$ and we have $\bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}^c\} \subseteq \mathcal{I}^c$.Hence, $\mathcal{I}^c = \bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}^c\}$.Since,$U_{\mathcal{I}}[y]$ is an open for any $y \in X$, then $\mathcal{I}$ is a closed subset of X. Next, we have to prove that $\mathcal{I} = \bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}\}$. If $y \in \mathcal{I}$, then $y \in U_{\mathcal{I}}[y]$ and hence $\mathcal{I} \subseteq \bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}\}$.Let $y \in \mathcal{I}$, if $z \in U_{\mathcal{I}}[y]$ then $y \sim_{\mathcal{I}} z$ and so $y \cdot z, y * z \in \mathcal{I}$. Since $y \in \mathcal{I}$, and $\mathcal{I}$ is a pseudo-UP ideal then $z \in \mathcal{I}$. Hence, we have $\bigcup\{U_{\mathcal{I}}[y] \mid y \in \mathcal{I}\} \subseteq \mathcal{I}$. Hence, $\mathcal{I}$ is an open subset of X.

Proposition 4.16. Let Λ be a family of pseudo-UP ideals of a pseudo-UP algebra X, then $U_{\mathcal{I}}[x]$ is clopen subset of X for all $x \in X$ and $\mathcal{I} \in \Lambda$.

Proof. We have to prove $(U_{\mathcal{I}}[x])^c$ is open. If $y \in (U_{\mathcal{I}}[x])^c$, then $x \cdot y, x * y \in \mathcal{I}^c$ or $y \cdot x, y * x \in \mathcal{I}^c$. Let $y \cdot x, y * x \in \mathcal{I}^c$, then by Proposition 4.10 and the proof of Proposition 4.15, we get $(U_{\mathcal{I}}[y] \cdot U_{\mathcal{I}}[x]) \subseteq U_{\mathcal{I}}[y \cdot x] \subseteq \mathcal{I}^c$ and $(U_{\mathcal{I}}[y] * U_{\mathcal{I}}[x]) \subseteq U_{\mathcal{I}}[y * x] \subseteq \mathcal{I}^c$. We claim that $U_{\mathcal{I}}[y] \subseteq (U_{\mathcal{I}}[x])^c$. If $z \in U_{\mathcal{I}}[y]$, then $z \cdot x \in (U_{\mathcal{I}}[z] \cdot U_{\mathcal{I}}[x])$ and $z * x \in (U_{\mathcal{I}}[z] * U_{\mathcal{I}}[x])$. Hence, $z \cdot x, z * x \in \mathcal{I}^c$ then we have $z \in (U_{\mathcal{I}}[x])^c$, hence $(U_{\mathcal{I}}[x])^c$ is open. Thus, $U_{\mathcal{I}}[x]$ is closed. It is clear that $U_{\mathcal{I}}[x]$ is open. Therefore, $U_{\mathcal{I}}[x]$ is clopen subset of X.

A topological space (X, τ) is connected if and only if X and ϕ are only clopen sets in τ. Thus, we get the following result.

Corollary 4.17. $(X, \mathcal{T}_\Lambda)$ is disconnected space.

Proposition 4.18. $\mathcal{T}_\Lambda = \mathcal{T}_{\mathcal{N}}$,where $\mathcal{N} = \bigcap\{\ \mathcal{I}: \mathcal{I} \in \Lambda\}$.

Proof. Let $\mathcal{K}$ and $\mathcal{K}^*$ defined as in Definition 4.1 and 4.2. Now, consider $\Lambda_0 = \{\mathcal{N}\}$ and define

$$\mathcal{K}_0{}^* = \{\mathcal{N}\}\ and\ \mathcal{K}_0 = \{U: U_{\mathcal{N}} \subseteq U\}.$$

Let $G \in \mathcal{T}_\Lambda$, so for every$x \in G$ there is$U \in \mathcal{K}$ such that $U[x] \subseteq G$. Since$\mathcal{N} \subseteq \mathcal{I}$, then we have $U_{\mathcal{N}} \subseteq U_{\mathcal{I}}$, for every pseudo-UP ideal $\mathcal{I}$of Λ. Since, $U \in \mathcal{K}$ there is$\mathcal{I} \in \Lambda$ such that $U_{\mathcal{I}} \subseteq U$. Thus, $U_{\mathcal{N}}[x] \subseteq U_{\mathcal{I}}[x] \subseteq G$. Since $U_{\mathcal{N}} \in \mathcal{K}_0$, $G \in \mathcal{T}_{\mathcal{N}}$. Hence, $\mathcal{T}_\Lambda \subseteq \mathcal{T}_{\mathcal{N}}$.

Conversely, let $H \in \mathcal{T}_{\mathcal{N}}$ then for every$x \in H$, there is $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. Thus, $U_{\mathcal{N}}[x] \subseteq H$ and since Λ is closed under intersection $\mathcal{N} \in \Lambda$. Then we obtain that $U_{\mathcal{N}} \in \mathcal{K}$ and so $H \in \mathcal{T}_\Lambda$. Therefore, $\mathcal{T}_{\mathcal{N}} \subseteq \mathcal{T}_\Lambda$.

Remark 4.19. Let Λ be a family of pseudo-UP ideals of a pseudo-UP algebra X and $\mathcal{N} = \bigcap\{\ \mathcal{I}: \mathcal{I} \in \Lambda\}$.Then the following statements hold:

1. By Proposition 4.18, we have$\mathcal{T}_\Lambda = \mathcal{T}_{\mathcal{N}}$. For all$U \in \mathcal{K}$, and for all$x \in\ X$ we get$U_{\mathcal{N}}[x] \subseteq U[x]$. Hence, $\mathcal{T}_\Lambda$ is equivalent to $\{G \subseteq X: \forall\, x \in G, \exists\ U_{\mathcal{N}}[x] \subseteq G\}$. Therefore, $G \in X$ is an open set if and only if for all $x \in G$, $U_{\mathcal{N}}[x] \subseteq G$ if and only if $G = \bigcup_{x \in G} U_{\mathcal{N}}[x]$.
2. By (1) we get$U_{\mathcal{N}}[x]$is a minimal open set containing x for all $x \in X$.
3. Let $\mathfrak{B}_{\mathcal{N}} = \{U_{\mathcal{N}}[x]: x \in X\}$. By (1), and (2) it is easy to show that $\mathfrak{B}_{\mathcal{N}}$ is a base of $\mathcal{T}_{\mathcal{N}}$.

Proposition 4.20. Let $\mathcal{I}$ and $\mathcal{N}$ be two pseudo-UP ideals of a pseudo-UP algebra X. Then $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_{\mathcal{N}}$ if and only if $\mathcal{N} \subseteq \mathcal{I}$.

Proof. Let $\mathcal{N} \subseteq \mathcal{I}$, and consider:

$$\Lambda_1 = \{\mathcal{I}\}, \mathcal{K}_1{}^* = \{U_{\mathcal{I}}\}, \mathcal{K}_1 = \{U: U_{\mathcal{I}} \subseteq U\}\ \text{and}\ \Lambda_2 = \{\mathcal{N}\}, \mathcal{K}_2{}^* = \{U_{\mathcal{N}}\}, \mathcal{K}_2 = \{U: U_{\mathcal{N}} \subseteq U\}.$$

Let $G \in \mathcal{T}_{\mathcal{I}}$, then for all $x \in G$, there exist $U \in \mathcal{K}_1$ such that $U[x] \in G$. Since $\mathcal{N} \subseteq \mathcal{I}$, then $U_{\mathcal{N}} \subseteq U_{\mathcal{I}}$. Since $U_{\mathcal{I}}[x] \subseteq G$, we have $U_{\mathcal{N}}[x] \subseteq G$. Then, $U_{\mathcal{N}} \in \mathcal{K}_2$ and thus $G \in \mathcal{T}_{\mathcal{N}}$.

Conversely, let $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_{\mathcal{N}}$. Suppose that $a \in \mathcal{N} \setminus \mathcal{I}$, since $\mathcal{I} \in \mathcal{T}_{\mathcal{I}}$ by assumption we have $\mathcal{I} \in \mathcal{T}_{\mathcal{N}}$. Then for all $x \in \mathcal{I}$, there exist $U \in \mathcal{K}_2$ such that $U[x] \subseteq \mathcal{I}$, and so $U_{\mathcal{N}}[x] \subseteq \mathcal{I}$. Then, $U_{\mathcal{N}}[0] \subseteq \mathcal{I}$we have $a \cdot 0 = 0 \in \mathcal{N}$, $0 \cdot a = a \in \mathcal{N}$, $a * 0 = 0 \in \mathcal{N}$ and $0 * a = a \in \mathcal{N}$. Thus, $a \sim_{\mathcal{N}} 0$, and so $a \in U_{\mathcal{N}}[0]$. Therefore $a \in \mathcal{I}$ which is a contradiction.

A uniform structure $(X, \mathcal{K})$is called totally bounded if for every $U \in \mathcal{K}$, there exists $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U[x_i]$. Moreover, $(X, \mathcal{K})$is compact if for every open cover of X has a finite subcover.

Proposition 4.21. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X. Then the following statements are equivalent:

1. $(X,\mathcal{T}_{\mathcal{I}})$ is compact.
2. $(X,\mathcal{T}_{\mathcal{I}})$is totally bounded.
3. There exist $P=\{x_1,x_2,\dots,x_n\}\subseteq X$ such that for every $a\in X$ there exist $x_i\in P$ with $a\cdot x_i, a*x_i\in \mathcal{I}$, and $x_i\cdot a, x_i*a\in\mathcal{I}$.

Proof. $(1\Rightarrow 2)$: Obvious.

$(2\Rightarrow 3)$:Let$U_{\mathcal{I}}\in\mathcal{K}$. Since $(X,\mathcal{T}_{\mathcal{I}})$is totally bounded, so there exists $x_1,x_2,\dots,x_n\in\mathcal{I}$ such that $X=\bigcup_{i=1}^{n}U[x_i]$. Now, let $a\in X$, so there exist x_i such that $a\in\bigcup_{i=1}^{n}U[x_i]$. Therefore,

$$a\cdot x_i, a*x_i\in\mathcal{I}\ and\ x_i\cdot a, x_i*a\in\mathcal{I}.$$

$(3\Rightarrow 1)$: By assumption there exist $x_i\in P$ with $a\cdot x_i, a*x_i\in\mathcal{I}$, and $x_i\cdot a, x_i*a\in\mathcal{I}$ for all $a\in X$.Hence, we get $a\in U_{\mathcal{I}}[x_i]$ and therefore $X=\bigcup_{i=1}^{n}U_{\mathcal{I}}[x_i]$. Now, let $X=\bigcup_{\alpha\in M}O_\alpha$ where $O_\alpha$is an open set in X for each $\alpha\in M$, then for every $x_i\in X$ there exists $x_i\in O_{\alpha_i}$. Since, $O_{\alpha_i}$is open then $U_{\mathcal{I}}[x_i]\subseteq O_{\alpha_i}$ and so $X=\bigcup_{i=1}^{n}U_{\mathcal{I}}[x_i]\subseteq\bigcup_{i=1}^{n}O_{\alpha_i}$. Therefore, $X=\bigcup_{i=1}^{n}O_{\alpha_i}$and hence$(X,\mathcal{T}_{\mathcal{I}})$ is compact.

Proposition 4.22. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X such that $\mathcal{I}^c$ is finite $(X,\mathcal{T}_{\mathcal{I}})$ is compact.

Proof.Suppose that $X=\bigcup_{\alpha\in M}O_\alpha$ where $O_\alpha$is an open set in X for each$\alpha\in M$, and let $\mathcal{I}^c=\{x_1,x_2,\dots,x_n\}$. Then there exists $\alpha,\alpha_1,\alpha_2,\dots,\alpha_n$ such that $0\in O_\alpha, x_1\in O_{\alpha_1}, x_2\in O_{\alpha_2},\dots,x_n\in O_{\alpha_n}$. Then $U_{\mathcal{I}}[0]\subseteq O_\alpha$, but $U_{\mathcal{I}}[0]=\mathcal{I}$. Hence, $X=\bigcup_{i=1}^{n}O_{\alpha_i}\cup O_\alpha$.

Proposition 4.23. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X, then$\mathcal{I}$ is compact in$(X,\mathcal{T}_{\mathcal{I}})$.

Proof.Suppose that $U_{\mathcal{I}}[x]\subseteq\bigcup_{\alpha\in M}O_\alpha$ where O_α is an open set in X for each $\alpha\in M$. Since, $0\in\mathcal{I}$ then there exist $\alpha\in M$ such that $0\in O_\alpha$. Then $\mathcal{I}=U_{\mathcal{I}}[0]\subseteq O_\alpha$ and hence $\mathcal{I}$ is a compact set in $(X,\mathcal{T}_{\mathcal{I}})$.

Proposition 4.24. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X, then for all $x\in X, U_{\mathcal{I}}[x]$is compact in$(X,\mathcal{T}_{\mathcal{I}})$.

Proof.Suppose that for all $x\in X, U_{\mathcal{I}}[x]\subseteq\bigcup_{\alpha\in M}O_\alpha$where O_α is an open set in X for each $\alpha\in M$.Since,$x\in U_{\mathcal{I}}[x]$, then there exist $\alpha\in M$such that $x\in O_\alpha$. Thus, $U_{\mathcal{I}}[x]\subseteq O_\alpha$. Therefore, $U_{\mathcal{I}}[x]$ is compact set in $(X,\mathcal{T}_{\mathcal{I}})$.

Proposition 4.25. Let $\mathcal{I}$ be a pseudo-UP ideal of a pseudo-UP algebra X. Then $(X,\mathcal{T}_{\Lambda})$ is a discrete topology if and only if $U_{\mathcal{I}}[x]=\{x\}$ for all $x\in X$.

Proof.Suppose that$\mathcal{T}_{\Lambda}$ is a discrete topology on X.If for any $\mathcal{I}\in\Lambda$, then there exists $x\in X$ such that $U_{\mathcal{I}}[x]\neq\{x\}$. Let $\mathcal{N}=\cap\{\mathcal{I}:\mathcal{I}\in\Lambda\}$, then $\mathcal{N}\in\Lambda$ so there exist $x_0\in X$ such that $U_{\mathcal{N}}[x_0]\neq\{x_0\}$. It follows there is a $y_0\in U_{\mathcal{I}}[x_0]$ and $x_0\neq y_0$. By Remark 4.19, we have $U_{\mathcal{N}}[x_0]$ is a minimal open set containing x_0. Hence, $\{x_0\}$ is not open subset of X which is a contradiction.

Conversely, for all $x\in X$, there exists $\mathcal{I}\in\Lambda$ such that $U_{\mathcal{I}}[x]=\{x\}$. Hence, $\{x\}$ is open subset of X. Thus, $(X,\mathcal{T}_{\Lambda})$ is a discrete topology.

Proposition 4.26. Let$(X,\mathcal{T}_{\Lambda})$be the topological space where Λ is a family of pseudo-UP ideals of a pseudo-UP algebra X and $\mathcal{I}\in\Lambda$. Then for any $A\subseteq X$, $cl(A)=\cap\{U_{\mathcal{I}}[A]: U_{\mathcal{I}}\in\mathcal{K}^*\}$.

Proof.Let $x\in cl(A)$, we have $U_{\mathcal{I}}[x]$ is an open set containing x and so $U_{\mathcal{I}}[x]\cap A\neq\phi$, for every$\mathcal{I}\in\Lambda$. Thus, there exist $y\in A$ such that $y\in U_{\mathcal{I}}[x]$ and so $(x,y)\in U_{\mathcal{I}}$ for every$\mathcal{I}\in\Lambda$. Therefore, $x\in U_{\mathcal{I}}[y]\subseteq U_{\mathcal{I}}[A]$ for every$\mathcal{I}\in\Lambda$.

Conversely, let $x\in U_{\mathcal{I}}[A]$ for every$\mathcal{I}\in\Lambda$, so there exist $y\in A$ such that $x\in U_{\mathcal{I}}[y]$ and hence $U_{\mathcal{I}}[y]\cap A\neq\phi$. Thus, $x\in cl(A)$.

Proposition 4.27. Let Λ be a family of pseudo-UP ideals of a pseudo-UP algebra X,and W be an open set containing K where $K$is a compact subset of X. Then $K\subseteq U_{\mathcal{I}}[K]\subseteq W$.

Proof.Let W be an open set containing K, then for all $k\in K$ we get$U_{\mathcal{I}_k}[k]\subseteq W$ for some $U_{\mathcal{I}_k}\in\Lambda$. Hence, $K\subseteq\bigcup_{\mathcal{I}_k}U_{\mathcal{I}_k}[k]\subseteq W$. Since K is a compact then there exists $k_1,k_2,\dots,k_n$ such that $K\subseteq U_{\mathcal{I}_{k_1}}[k_1]\cup U_{\mathcal{I}_{k_2}}[k_2]\cup\dots\cup U_{\mathcal{I}_{k_n}}[k_n]$. Take $\mathcal{I}=\bigcap_{i=1}^{n}\mathcal{I}_{k_i}$. We claim that $U_{\mathcal{I}_k}[k]\subseteq W$ for all $k\in K$. Let $k\in K$, so there exists $1\le i\le n$ such that $k\in U_{\mathcal{I}_{k_i}}[k_i]$ and so $k\sim_{\mathcal{I}_{k_i}}k_i$. Let $y\in U_{\mathcal{I}}[k]$ then $y\sim_{\mathcal{I}}k$ and so $y\sim_{\mathcal{I}_{k_i}}k_i$. Hence, $y\in U_{\mathcal{I}_{k_i}}[k_i]\subseteq W$ and so $U_{\mathcal{I}}[K]\subseteq W$ for all $k\in K$. Therefore, $K\subseteq U_{\mathcal{I}}[K]\subseteq W$.

Proposition 4.28. Let Λ be a family of pseudo-UP ideals of a pseudo-UP algebra X, and let $K,P \subseteq X$such that K is a compact and $P$is a closed. If $K \cap P = \phi$, then $U_{\mathcal{I}}[K] \cap U_{\mathcal{I}}[P] = \phi$ for all $\mathcal{I} \in \Lambda$.

Proof.Let$K \cap P = \phi$ and P be a closed, $X\backslash P$ is an open set containing K. By Proposition 4.27, there exists $\mathcal{I} \in \Lambda$ such that $U_{\mathcal{I}}[K] \subseteq X\backslash P$. Suppose that $U_{\mathcal{I}}[K] \cap U_{\mathcal{I}}[P] \neq \phi$, then there exist $y \in X$ such that $y \in U_{\mathcal{I}}[k]$ and $y \in U_{\mathcal{I}}[p]$ for all $k \in K$ and $p \in P$. Hence, $k \sim_{\mathcal{I}} p$ and so $p \in U_{\mathcal{I}}[k] \subseteq U_{\mathcal{I}}[K]$ which is a contradiction with the fact $U_{\mathcal{I}}[K] \subseteq X\backslash P$. Hence, $U_{\mathcal{I}}[K] \cap U_{\mathcal{I}}[P] = \phi$.

From Proposition 2.5, we obtain that every pseudo-UP ideal is a pseudo-UP filter in a pseudo-UP algebra X, then we have the following result:

Corollary 4.29. Let $((X, \leq), \cdot, *, 0, \tau)$be a TPUP-algebra and $\mathcal{I}_0$ is a minimal open set containing 0, then $\mathcal{I}_0$ is a pseudo-UP ideal of X.

Proposition 4.30. Let $((X, \leq), \cdot, *, 0, \tau)$be a TPUP-algebra and$(X, \mathcal{T}_{\mathcal{I}_0})$ be a uniform topology induced by $\mathcal{I}_0$. Then τ is finer than $\mathcal{T}_{\mathcal{I}_0}$.

Proof. Suppose that M_y is the minimal open set in τ containing y. We have to show that $U_{\mathcal{I}_0}[x] = \bigcup_{y \in U_{\mathcal{I}_0}[x]} M_y$ for each $x \in X$. Let $y \in U_{\mathcal{I}_0}[x]$ and $z \in M_y$. If $z \cdot y, z * y \notin \mathcal{I}_0$or $y \cdot z, y * z \notin \mathcal{I}_0$, then by Proposition 2.12, $z \notin M_y$. Therefore, $z \cdot y, z * y \in \mathcal{I}_0$or $y \cdot z, y * z \in \mathcal{I}_0$. Since, $(y \cdot z) * [(x \cdot y) * (x \cdot z)] = 0 \in \mathcal{I}_0$, $(y * z) \cdot [(x * y) \cdot (x * z)] = 0 \in \mathcal{I}_0$ and $(x \cdot y), (x * y) \in \mathcal{I}_0$, then we have $(y \cdot z) * (x \cdot z), (y * z) \cdot (x * z) \in \mathcal{I}_0$. Again, since $y \cdot z, y * z \in \mathcal{I}_0$ then by Proposition 2.5, $x \cdot z, x * z \in \mathcal{I}_0$. By the same way we can get $z \cdot x, z * x \in \mathcal{I}_0$. Hence, $z \in U_{\mathcal{I}_0}[x]$. Thus, $M_y \subseteq U_{\mathcal{I}_0}[x]$ for every $y \in U_{\mathcal{I}_0}[x]$ and hence $\bigcup_{y \in U_{\mathcal{I}_0}[x]} M_y \subseteq U_{\mathcal{I}_0}[x]$. The converse is clear.

Proposition 4.31. Let $((X, \leq), \cdot, *, 0, \tau)$be a TPUP-algebra and$(X, \mathcal{T}_{\mathcal{I}_0})$ be a uniform topology induced by $\mathcal{I}_0$.If there exist $U \in \tau$ such that $U \notin \mathcal{T}_{\mathcal{I}_0}$, then there exist $x \in U$ and $y \in U_{\mathcal{I}_0}[x]$ such that $y \notin U$ and the following statements hold: for all $a \in \mathcal{I}_0$,

1. $x, y \notin \mathcal{I}_0$.
2. $a \cdot y, a * y \notin U_{\mathcal{I}_0}[x] \cap U$.
3. If $d \in U_{\mathcal{I}_0}[x] \cap U$, then $a \cdot d \neq y$ and $a * d \neq y$.

Proof.

1. Let $x \in \mathcal{I}_0$, then by Proposition 2.11, $\mathcal{I}_0 \subseteq U$. Since $x \in \mathcal{I}_0$, $y \in U_{\mathcal{I}_0}[x]$ and $\mathcal{I}_0$is a pseudo-UP ideal, then $y \in \mathcal{I}_0 \subseteq U$ which is a contradiction. Now, let $y \in \mathcal{I}_0$ and since $y \in U_{\mathcal{I}_0}[x]$and $\mathcal{I}_0$ is a pseudo-UP ideal, then $x \in \mathcal{I}_0$ which is a contradiction.
2. Suppose there exist $a \in U_{\mathcal{I}_0}[x]$ such that $a \cdot y, a * y \in U_{\mathcal{I}_0}[x] \cap U$, then there exist two open sets V and W containing a and y respectively such that $V \cdot W \subseteq U_{\mathcal{I}_0}[x] \cap U$ and $V * W \subseteq U_{\mathcal{I}_0}[x] \cap U$. By Proposition 2.11, $\mathcal{I}_0 \subseteq V$ and so $y = 0 \cdot y \in \mathcal{I}_0 \cdot W \subseteq U_{\mathcal{I}_0}[x] \cap U$ and $y = 0 * y \in \mathcal{I}_0 * W \subseteq U_{\mathcal{I}_0}[x] \cap U$. Therefore, $y \in U_{\mathcal{I}_0}[x] \cap U$ which is a contradiction.
3. Suppose that there exist $a \in U_{\mathcal{I}_0}[x]$such that $a \cdot d = y$ and $a * d = y$ for some $d \in U_{\mathcal{I}_0}[x] \cap U$. Since $0 \cdot d = d \in U_{\mathcal{I}_0}[x] \cap U$ and $0 * d = d \in U_{\mathcal{I}_0}[x] \cap U$, then there exist two open sets V and W containing 0 and d such that $V \cdot W \subseteq U_{\mathcal{I}_0}[x] \cap U$ and $V * W \subseteq U_{\mathcal{I}_0}[x] \cap U$. Then we have $y = a \cdot d \in \mathcal{I}_0 \cdot W \subseteq U_{\mathcal{I}_0}[x] \cap U$ and $y = a * d \in \mathcal{I}_0 * W \subseteq U_{\mathcal{I}_0}[x] \cap U$. Therefore, $y \in U_{\mathcal{I}_0}[x] \cap U$which is a contradiction.

Proposition 4.32. Let $((X, \leq), \cdot, *, 0, \tau)$be a TPUP-algebra and$(X, \mathcal{T}_{\mathcal{I}_0})$ be a uniform topology induced by $\mathcal{I}_0$. If $\mathcal{T}_{\mathcal{I}_0} \subsetneq \tau$,then there exist a non-empty $U \in \tau$ such that $U \subsetneq U_{\mathcal{I}_0}[x]$ for some $x \notin X\backslash \mathcal{I}_0$.

Proof. Suppose that $\mathcal{T}_{\mathcal{I}_0} \subsetneq \tau$, then there exist $V \in \tau$such that $V \notin \mathcal{T}_{\mathcal{I}_0}$. Since $(X, \mathcal{T}_{\mathcal{I}_0})$is a uniform topology, then there exist $x \in V$ such that $U_{\mathcal{I}_0}[x] \nsubseteq V$. Therefore, $U_{\mathcal{I}_0}[x] \cap V \nsubseteq U_{\mathcal{I}_0}[x]$. Take $U = U_{\mathcal{I}_0}[x] \cap V$, then $U \in \tau$ and $U \nsubseteq U_{\mathcal{I}_0}[x]$. If $x \in \mathcal{I}_0$, then $U_{\mathcal{I}_0}[x] = \mathcal{I}_0$.Hence, $x \in U$ and by Proposition 2.11, $U_{\mathcal{I}_0}[x] = \mathcal{I}_0 \subseteq U$ which is a contradiction.

Proposition 4.33. [9] Let $f: X \longrightarrow Y$ be a pseudo-UP homomorphism between two pseudo-UP algebras X and Y. Then the following statements hold:

1. If $\mathcal{I}$ is a pseudo-UP ideal in Y, then$f^{-1}(\mathcal{I})$ is a pseudo-UP ideal in X.
2. If f is surjective and $\mathcal{I}$ is a pseudo-UP ideal in X, then $f(\mathcal{I})$ is a pseudo-UP ideal in Y.

Proposition 4.34. Let $f: X \longrightarrow Y$ be a pseudo-UP isomorphism between two pseudo-UP algebras X,Y, and let $\mathcal{J}$ is a pseudo-UP ideal in Y, then for all $x_1,x_2 \in X$,

$$(x_1,x_2) \in U_{f^{-1}(\mathcal{J})} \Leftrightarrow \big(f(x_1),f(x_2)\big) \in U_{\mathcal{J}}.$$

Proof. For all $x_1,x_2 \in X$, we have

$$(x_1,x_2) \in U_{f^{-1}(\mathcal{J})} \Leftrightarrow x_1 \sim_{f^{-1}(\mathcal{J})} x_2 \Leftrightarrow \ f(x_1) \sim_{\mathcal{J}} \ f(x_2) \Leftrightarrow \big(f(x_1),f(x_2)\big) \in U_{\mathcal{J}}.$$

Proposition 4.35. Let $f: X \longrightarrow Y$ be a pseudo-UP isomorphism between two pseudo-UP algebras X,Y, and let $\mathcal{J}$ is a pseudo-UP ideal in Y. Then the following statements hold: for all $x \in X$ and for all $y \in Y$,

1. $f\big(U_{f^{-1}(\mathcal{J})}[x]\big) = U_{\mathcal{J}}[f(x)]$.
2. $f^{-1}(U_{\mathcal{J}}[y]) = U_{f^{-1}(\mathcal{J})}[f^{-1}(y)]$.

Proof.

1. Let $y \in f\big(U_{f^{-1}(\mathcal{J})}[x]\big)$, then there exist$x_1 \in U_{f^{-1}(\mathcal{J})}[x]$ such that $y = f(x_1)$. It follows that
$$x \sim_{f^{-1}(\mathcal{J})} x_1 \Longrightarrow f(x) \sim_{\mathcal{J}} \ f(x_1) \Longrightarrow f(x) \sim_{\mathcal{J}} y \Longrightarrow y \in U_{\mathcal{J}}[f(x)].$$
Conversely, let $y \in U_{\mathcal{J}}[f(x)] \Longrightarrow f(x) \sim_{\mathcal{J}} y \Longrightarrow f^{-1}(f(x) \sim_{\mathcal{J}} y) \Longrightarrow x \sim_{f^{-1}(\mathcal{J})} f^{-1}(y) \Longrightarrow f^{-1}(y) \in U_{f^{-1}(\mathcal{J})}[x] \Longrightarrow y \in f\big(U_{f^{-1}(\mathcal{J})}[x]\big)$.
2. Let$x \in f^{-1}(U_{\mathcal{J}}[y]) \Leftrightarrow f(x) \in U_{\mathcal{J}}[y] \Leftrightarrow f(x) \sim_{\mathcal{J}} y \Leftrightarrow f^{-1}(f(x) \sim_{\mathcal{J}} y) \Leftrightarrow x \sim_{f^{-1}(\mathcal{J})} f^{-1}(y) \Leftrightarrow x \in U_{f^{-1}(\mathcal{J})}[f^{-1}(y)]$.

Proposition 4.36. Let $f: X \longrightarrow Y$ be a pseudo-UP isomorphism between two pseudo-UP algebras X,Y, and let $\mathcal{J}$ is a pseudo-UP ideal in Y. Then f is homeomorphism map from $\big(X, \mathcal{T}_{f^{-1}(\mathcal{J})}\big)$ to $(Y, \mathcal{T}_{\mathcal{J}})$.

Proof.First, we have to show that f is continuous. Let $A \in \mathcal{T}_{\mathcal{J}}$ then by Remark 4.19, we get $A = \bigcup_{a\in A} U_{\mathcal{J}}[a]$. It follows that

$$f^{-1}(A) = f^{-1}(\bigcup_{a\in A} U_{\mathcal{J}}[a]) = \bigcup_{a\in A} f^{-1}(U_{\mathcal{J}}[a]).$$

We claim that if $b \in f^{-1}(U_{\mathcal{J}}[a])$, then we have $U_{f^{-1}(\mathcal{J})}[b] \subseteq f^{-1}(U_{\mathcal{J}}[a])$. Now, let $c \in U_{f^{-1}(\mathcal{J})}[b]$, then $c \sim_{f^{-1}(\mathcal{J})} b$ and so $f(c) \sim_{\mathcal{J}} \ f(b)$. Since $f(b) \in U_{\mathcal{J}}[a]$we have $f(b) \sim_{\mathcal{J}} a$. Therefore, $f(c) \sim_{\mathcal{J}} a$and hence $f(c) \in U_{\mathcal{J}}[a]$. Thus, $c \in f^{-1}(U_{\mathcal{J}}[a])$ and so

$$f^{-1}(U_{\mathcal{J}}[a]) = \bigcup_{b\in f^{-1}(U_{\mathcal{J}}[a])} U_{f^{-1}(\mathcal{J})}[b] \in \mathcal{T}_{f^{-1}(\mathcal{J})}.$$

Therefore, $f^{-1}(A) = f^{-1}(\bigcup_{a\in A} U_{\mathcal{J}}[a]) = \bigcup_{a\in A} f^{-1}(U_{\mathcal{J}}[a]) \in \mathcal{T}_{f^{-1}(\mathcal{J})}$and hence f is continuous.

Finally, we have to show that f is an open map. Let A be an open in $\big(X, \mathcal{T}_{f^{-1}(\mathcal{J})}\big)$. We claim that $f(A)$ is an open set in $(Y, \mathcal{T}_{\mathcal{J}})$. Let $a \in \ f(A)$ we will have to show that $U_{\mathcal{J}}[a] \subseteq \ f(A)$. Now, for all $b \in U_{\mathcal{J}}[a]$ we have $b \sim_{\mathcal{J}} a$. By Proposition 4.34, we have $f^{-1}(b) \sim_{f^{-1}(\mathcal{J})} f^{-1}(a)$. Hence, $f^{-1}(b) \in U_{f^{-1}(\mathcal{J})}[f^{-1}(a)]$. Since f is a one-to-one and $a \in \ f(A)$ then we have $f^{-1}(a) \in A$. By Remark 4.19, we get that $U_{f^{-1}(\mathcal{J})}[f^{-1}(a)] \subseteq A$ and hence $f^{-1}(b) \in A$implies that $b \in \ f(A)$. Therefore, $U_{\mathcal{J}}[a] \subseteq \ f(A)$and thus f is an open map.

5. New topology and related result

In this section we will give a filter base on X to generate a topology on X where X is a pseudo-UP algebra. Let $V \subseteq X$ and $a \in X$ we define $V(a)$ as following:

$$V(a) = \{x \in X: x \cdot a, x * a \in V \ and \ \ a \cdot x, a * x \in V\}.$$

Obviously$V(a) \subseteq U(a)$ when $V \subseteq U \subseteq X$.

Proposition 5.1. Let X be a pseudo-UP algebra satisfying$x \cdot (y \cdot z) = y \cdot (x \cdot z)$, and $x * (y * z) = y * (x * z)$ for all $x,y,z \in X$ and Ωbe a filter base satisfying the following conditions:

1. For every $v \in V \in \Omega$, there exist $U \in \Omega$ such that $U(v) \subseteq V$.
2. If $p,q \in V \in \Omega$ and $p \cdot (q \cdot x) = 0, p * (q * x) = 0$ then $x \in V$ for all $x \in X$.

Then there exist a topology on X for which is a fundamental system of open sets containing 0 and $V(a)$ is an open set for all $V \in \Omega$ and for all $a \in X$. Moreover, (X, τ_{Ω}) is a TPUP-algebra.

Proof. Let $\tau_\Omega = \{O \subseteq X : \forall\, a \in O, \exists\, V \in \Omega \text{ such that } V(a) \subseteq O\}$. First, we have to show that τ_Ω is a topology on X. Clearly, $X, \phi \in \tau_\Omega$. Let $O_\lambda \in \tau_\Omega$ for some $\lambda \in M$ and let $a \in \bigcup_{\lambda \in M} O_\lambda$. Then $a \in O_\lambda$ for some $\lambda \in M$, so there exist V such that $V(a) \subseteq O_\lambda$ and thus $\bigcup_{\lambda \in M} O_\lambda \in \tau_\Omega$. Now, suppose that $O_1, O_2 \in \tau_\Omega$ and let $a \in O_1 \cap O_2$. Thus, there exist V_1 and V_2 such that $V_1(a) \subseteq O_1$ and $V_2(a) \subseteq O_2$. Since Ω is a base filter then there exist V such that $V \subseteq V_1 \cap V_2$. Thus, we have $V(a) \subseteq (V_1 \cap V_2)(a) \subseteq V_1(a) \cap V_2(a) \subseteq O_1 \cap O_2$ and so $O_1 \cap O_2 \in \tau_\Omega$. Then τ_Ω is a topology on X.

Now, we have to show that Ω is a filter base of an open set containing 0 with respect to τ_Ω. Since $p \cdot (q \cdot 0) = 0$ and $p * (q * 0) = 0$, for any $p, q \in V$ then by (2) $0 \in V$ (*i.e.*, each element $V \in \Omega$ contains 0). If $x \in V(p)$, then $x \cdot p, x * p, p \cdot x, p * x \in V$ and so $v = p \cdot x$, and $v = p * x$. Hence, $v \cdot (p \cdot x) = 0$ and $v * (p * x) = 0$ implies that $x \in V$. Thus, $V(p) \subseteq V$ and $V \in \tau_\Omega$. If we suppose that V is an open set containing 0, then there exist a $U \in \Omega$ such that $U(0) \subseteq V$. Then for some $a \in U$ we note that $0 \cdot a, 0 * a \in U$ and $a \cdot 0, a * 0 \in U$. Thus, $a \in U(0)$ and so $0 \in U \subseteq U(0) \subseteq V$. Then Ω is a fundamental system of open sets containing 0 with respect to τ_Ω.

Next, we have to show that $V(a)$ is an open in τ_Ω. Let $x \in V(a)$, then we have $a \cdot x, a * x \in V$ and $x \cdot a, x * a \in V$. Then by (1), there exists $O_1, O_2 \in \Omega$ such that $O_1(a \cdot x) \subseteq V, O_1(a * x) \subseteq V$ and $O_2(x \cdot a) \subseteq V, O_2(x * a) \subseteq V$. Since Ω is a base filter then there exist $W \in \Omega$ such that $W \subseteq O_1 \cap O_2$. Let $y \in W(x)$, then $y \cdot x, y * x \subseteq W$ and $x \cdot y, x * y \in W$. Since, $(x \cdot y) * [(a \cdot x) * (a \cdot y)] = 0$, and $(x * y) \cdot [(a * x) \cdot (a * y)] = 0$. Also, $(x \cdot a) * [(y \cdot x) * (y \cdot a)] = 0$, and $(x * a) \cdot [(y * x) \cdot (y * a)] = 0$. Then by (2), we have $a \cdot y, a * y$ and $y \cdot a, y * a$ are contained in V. Hence, $V(a)$ is an open set.

Finally, to show that (X, τ_Ω) is a TPUP-algebra. Let x and y be two elements in X. Since each open set containing $x \cdot y$ and $x * y$ contains $V(x \cdot y)$ and $V(x * y)$ for $V \in \Omega$. It is enough to show that $V(x) \cdot V(y) \subseteq V(x \cdot y)$ and $V(x) * V(y) \subseteq V(x * y)$. Let $u \cdot v \in V(x) \cdot V(y)$ and $u * v \in V(x) * V(y)$, then $u \in V(x)$ and $v \in V(y)$. Therefore, $u \cdot x, u * x, x \cdot u, x * u, v \cdot y, v * y, y \cdot v, y * v \in V$ and so we have

$$y \cdot v \leq (x \cdot y) * (x \cdot v) \leq (x \cdot y) * [(u \cdot x) * (u \cdot v)] = (u \cdot x) * [(x \cdot y) * (u \cdot v)],$$

and

$$y * v \leq (x * y) \cdot (x * v) \leq (x * y) \cdot [(u * x) \cdot (u * v)] = (u * x) \cdot [(x * y) \cdot (u * v)].$$

Hence, $(y \cdot v) * [(u \cdot x) * ((x \cdot y) * (u \cdot v))] = 0$ and $(y * v) \cdot [(u * x) \cdot ((x * y) \cdot (u * v))] = 0$. Then by (2), we have $(x \cdot y) * (u \cdot v), (x * y) \cdot (u * v) \in V$ and by the same way we can obtain that $(u \cdot v) * (x \cdot y), (u * v) \cdot (x * y) \in V$. Therefore, $u \cdot v \in V(x \cdot v)$ and $u * v \in V(x * y)$ which implies that $V(x) \cdot V(y) \subseteq V(x \cdot y)$ and $V(x) * V(y) \subseteq V(x * y)$. Hence, (X, τ_Ω) is a TPUP-algebra.

Example 5.2. Let $X = \{0, a, b, c\}$ with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

·	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	a	b	0

∗	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

Table 4. A pseudo-UP algebra with: $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ and $x*(y*z) = y*(x*z)$ $\forall$ x,y, z $\in$ X.

Example 5.3. If X is a pseudo-UP algebra satisfying $x \cdot (y \cdot z) = y \cdot (x \cdot z)$, and $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$, then the filter base of pseudo-UP ideal $\mathcal{I}$ of X is a base filter satisfies conditions in Proposition 5.1. Since for every $x \in \mathcal{I}$, $\mathcal{I}(x) \subseteq \mathcal{I}$. Hence, the condition (1) satisfies. Now, if $p, q \in \mathcal{I}$ and $p \cdot (q \cdot x) = 0 \in \mathcal{I}$, $p * (q * x) = 0 \in \mathcal{I}$ then $p \cdot x \in \mathcal{I}$ and $p * x \in \mathcal{I}$. Again, since $p \in \mathcal{I}$, then by Proposition 2.5 $x \in \mathcal{I}$. Hence, the condition (2) satisfies. Therefore, the topology induced by $\mathcal{I}$ is a TPUP-algebra.

Let A be any subset of a TPUP-algebra X whose topology is the topology generated be a filter base satisfies all conditions of Proposition 5.1, we mean that $V(A) = \bigcup_{a \in A} V(a)$ which is clearly an open set containing A. Thus, we have the following result.

Proposition 5.4. Let A be any subset of a TPUP-algebra X, then $cl(A) = \cap \{V(A) : V \in \Omega\}$.

Proof. Let $x \in cl(A)$ and $V \in \Omega$. Since $V(x)$ is an open set containing x, then $V(x) \cap A \neq \phi$. Therefore, there exist $a \in A$ such that $a \cdot x, a * x \in V$ and $x \cdot a, x * a \in V$. Then $x \in V(a)$ and $x \in \cap \{V(A): V \in \Omega\}$.
Conversely, if $x \in \cap \{V(A): V \in \Omega\}$, then for any $U \in \Omega$we have $x \in U(A)$. Therefore, $U(x) \cap A \neq \phi$.
Proposition 5.5. Let A be a compact subset of a TPUP-algebra X. If U is an open set containing A, then there exist $V \in \Omega$ such that $A \subseteq V(A) \subseteq U$.

Proof.Since U is an open set of A, then by Proposition 5.1 for all $a \in A$ there exist $V_a \in \Omega$ such that $V_a(a) \subseteq U$. Since, A is a compact and $A \subseteq \bigcup_{a \in A} V_a(a)$ then there exist $a_1, a_2, \ldots, a_n$such that $A \subseteq V_{a_1}(a_1) \cup V_{a_2}(a_2) \cup \ldots \cup V_{a_n}(a_n)$. Now, let $V = \bigcup_{i=1}^{n} V_{a_i}(a_i)$ so it is enough to show that $V(a) \subseteq U$ for all $a \in A$. Since $a \in V_{a_i}$ for some a_i, then $a \cdot a_i, a * a_i \in V_{a_i}$ and $a_i \cdot a, a_i * a \in V_{a_i}$. If $x \in V(a)$, then we have $a \cdot x, a * x \in V$ and $x \cdot a, x * a \in V$. Since $(a \cdot x) * [(a_i \cdot a) * (a_i \cdot x)] = 0$, and $(a * x) \cdot [(a_i * a) \cdot (a_i * x)] = 0$. Then by Proposition 5.1, we have $a_i \cdot x, a_i * x \in V_{a_i}$. Similarly, $(x \cdot a) * [(a \cdot a_i) * (x \cdot a_i)] = 0$ and $(x * a) \cdot [(a * a_i) \cdot (x * a_i)] = 0$. Hence, $x \cdot a_i, x * a_i \in V_{a_i}$. Therefore, $x \in V_{a_i}(a_i) \subseteq U$ and $V(a) \subseteq U$ and thus $V(A) \subseteq U$.

Proposition 5.6. Let K be a compact subset of a TPUP-algebra X and F be a closed subset of X. If $K \cap V = \phi$, so there exist $V \in \Omega$ such that $V(K) \cap V(F) = \phi$.

Proof. Since $X \backslash F$ is an open set of K, then by Proposition 5.5, there exist $V \in \Omega$ such that $V(K) \subseteq X \backslash F$. Suppose that $V(K) \cap V(F) \neq \phi$for every $V \in \Omega$. Then there exist $x \in V(K) \cap V(F)$. Thus, $x \in V(k)$ and $x \in V(f)$ for some $k \in K$ and for some $f \in F$. Since, $(x \cdot f) * [(k \cdot x) * (k \cdot f)] = 0$, $(x * f) \cdot [(k * x) \cdot (k * f)] = 0$ and $(x \cdot k) * [(f \cdot x) * (f \cdot k)] = 0, (x * k) \cdot [(f * x) \cdot (f * k)] = 0$. Then by Proposition 5.1, we have $k \cdot f, k * f \in V$ and $f \cdot k, f * k \in V$. Therefore, $f \in V(k)$ which is a contradiction. Hence, $V(K) \cap V(F) = \phi$.

6. Conclusion

In this article, some properties of UP-ideals are extended to pseudo UP-ideals. Using pseudo UP-ideals we constructed a uniform structure on pseudo-UP algebras. several topological properties andrelations among pseudo-UP algebras are obtained by using pseudo-UP isomorphisms. In the last sectionwe generated a new topology from a filter base defined a pseudo-UP algebra and several results areobtained.

References

[1]S. S. Ahn, and S. H. Kown, On topological BCC-algebras, Communications of the Korean Mathematical Society, 22 (2) (2008), 169–178.

[2] A. Iampan, A new branch of the logical algebra: UP-algebras, Journal of Algebra and Related Topics, 5 (1) (2017), 35-54.

[3] K. D. Joshi, Introduction to General Topology, New Age International Publisher, India, 1997.

[4] D. S. Lee and D. N. Ryu, Notes on topological BCK-algebras, Scientiae Mathematicae, 1 (2) (1998), 231–235.

[5] S. Mehrshad and J. Golzarpoor, On topological BE-algebras, MathematicaMoravica, 21 (2) (2017), 1–13.

[6] C. Prabpayak and U. Leerawat, On ideals and congruence in KU-algebras, Scientia Magna, 1 (5) (2009), 54-57.

[7] D. Romano, Pseudo-UP algebras, An introduction, Bulletin of International Mathematical Virtual Institute, 10 (2) (2020), 349–355.

[8] D. Romano, Pseudo-UP Ideals and Pseudo-UP Filters in Pseudo-UP Algebras, Mathematical Sciences and Applications E-Notes, 8 (1) (2020), 155-158.

[9] D. Romano, Homomorphism of Pseudo-UP algebras, Bulletin of International Mathematical Virtual Institute, 11 (1) (2021), 47-53.

[10] A. Satirad and A.Iampan, Topological UP-algebras, General Algebra and Applications, 39 (2019), 231-250.

[11] M. A. Yousef and A. B. Khalaf, Topological pseudo-UPalgebras. Submitted.