Analytical solution of nonlinear Van der Pol oscillator using a hybrid intelligent analytical inverse method

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Abstract

Van der Pol (VDP) oscillator is an essential topic in applied dynamics to describe complex dynamical behaviors, chaotic motions, etc. In this study, a novel method is proposed towards the analytical solution of the VDP equation-as a quintessential example of nonlinear ordinary differential equations. The presented technique takes advantage of generic trial functions and their derivatives substituted into the target differential equation. Next, an optimization algorithm is tailored to minimize residual mean. Initial/boundary conditions, on the other hand, are dealt with as the second objective of the optimization problem. The proposed method was compared to two conventional analytical methods. The judgment on their accuracy was made based on the Runge-Kutta numerical method. Accordingly, it was observed that the solutions of the proposed method were in tight agreement with the numerical method. Furthermore, it was observed that the newly developed method circumvented the computational burden encountered when using the rival techniques. The obtained results champion outperformance of this method compared with the rival analytical methods in terms of accuracy, simplicity, and robustness.

Keywords: Analytical solution, Differential equation, Van der Pol oscillator, Inverse method, Optimization

1. Introduction

An ordinary differential equation (ODE) is a relationship between an independent variable and a dependent variable and the derivatives of the dependent variable. On the other hand, the whole real-life phenomena, to be modeled mathematically through differential equations (DEs), comprise indigenous nonlinear behaviors, which greatly obfuscate ascertaining the exact solutions. Meanwhile, not all DEs have accurate analytical solutions, or it may be challenging to find them. Therefore, the best scientists can aim to search for approximate solutions that share almost the same accurate results as do the exact methods. In this regard, a great deal of research has been conducted to analyze approximate analytical solutions to DEs (Vahidi-Moghaddam et al, 2021; Vahidi-Moghaddam et al, 2020)

Van der Pol equation was presented to model complex dynamic systems utilizing a second-order differential equation with nonlinear damping (Kimiaeifar et al, 2010). It was primarily proposed to model an electrical circuit with a triode valve and was later expanded for a broader range of applications (Zhuravlev, 2020). Due to the importance of VDP equations, a large number of works have been proposed attempting to solve these equations using numerical (Rasedee et al, 2021; Yin et al, 2020) or analytical (Ganji et al, 2010; Kimiaeifar et al, 2009; Usta & Sarıkaya, 2019) methods. To assess the lock-in phenomenon of a turbo-machinery blade, (Hoskoti et al, 2020) have used a spring-mounted airfoil along with a VDP oscillator.

(He, 1999) has proposed an analytical method to approximate nonlinear problems, called the variational iteration method (VIM). Unlike perturbation methods (PMs), VIM does not need small perturbations and linearization; that is why VIM has found use in a diverse range of applications (Daeichi & Ahmadian, 2015). Moreover, according to the results obtained, VIM shares a higher convergence rate compared to ADM.

Akbari and Ganji have recently developed an analytical technique, called Akbari-Ganji's method (AGM), to solve nonlinear partial/ordinary DEs (Akbari et al, 2015). Since its debut in 2014, AGM has been used to solve a large variety of nonlinear DEs. Akbari et al. have investigated the application of AGM on three nonlinear ODEs (NODEs) of VDP, Rayleigh, and Duffing (Akbari et al, 2014). (Meresht & Ganji, 2018) have scrutinized analytical solutions to nonlinear hypocycloid motion equation with AGM aid. Akbari et al. investigated the nonlinear vibration of an arched beam used in bridge constructions (Akbari et al, 2017).

Consequently, it can be claimed that the results offered by AGM are reportedly in excellent agreement with those obtained by numerical methods, and, virtually in all the cases, it was even more accurate than the formerly developed analytical methods.

According to the taxonomy presented by (Chaquet & Carmona, 2019), the methods for solving DEs are classified into four main groups: analytical, numerical, heuristic, and hybrid approaches. The previous methods reviewed in the literature may all be categorized among the analytical techniques.

With the advent of artificial intelligence, the computational tasks, which once seemed formidable and time-consuming, could be accomplished more easily, accurately, and often faster than using deterministic paradigms. In this regard, many research endeavors have been geared towards inventing powerful computational tools such as bio-inspired evolutionary optimization algorithms (Simon, 2013). Optimization algorithms are being used in multifaceted areas of engineering applications (Mir Mohammad Sadeghi et al, 2019; MirMohammadSadeghi et al, 2018; Nikzadfar et al, 2019). Recently, scientists have been increasingly paying attention to solving linear/nonlinear, partial/ordinary differential equations through optimization algorithms (Panda & Pani, 2018, Pan 2019; Zhang et al, 2020). Some of the latest works that have analyzed the solution of DEs with intelligent optimization algorithms include (Cao et al, 2000; Karr & Wilson, 2003; Tsoulos & Lagaris, 2006). (Chaquet & Carmona, 2012) have proposed a novel approach for solving different kinds of DEs based on Evolution Strategies (ESs), in which candidate solutions were partial sums of Fourier series. They illustrated that the proposed method performed satisfactorily by reporting experimental results with acceptable values of error. (Nemati et al, 2014) took advantage of the Imperialist Competitive Algorithm (ICA) to cope with initial value (IV) and boundary value (BV) problems. Using numerical proofs, they have

demonstrated that the proposed method was a promising tool for solving a wide range of DEs. (Kamali et al, 2015) have successfully recruited a modified version of Ant Colony Programming (ACP) algorithm to solve various ODEs and PDEs. (Fateh et al, 2017) have developed a differential evolution-based technique to solve BV problems. Implementation of the proposed method on some benchmark problems revealed that this method has a wide range of applicability with a fast convergence rate.
Particle Swarm Optimization (PSO) algorithm is among swarm-based, intelligent, metaheuristic optimization algorithms known well for its simplicity yet efficacious performance (Eberhart & Kennedy, 1995).
(Nemati et al, 2015) have used Learning Automata Particle Swarm Optimization (LA-PSO) algorithm for solving IV and BV differential equations. The given numerical results validated the high accuracy and efficiency of the suggested technique. (Babaei, 2013) has presented a general methodology to achieve approximate analytical solutions of a wide range of DEs by using the PSO algorithm. He took advantage of Fourier series expansion as trail answers for each of the examples. Conducted comparisons between the answers of the proposed method and those of the exact method indicated that the presented method offered solutions with satisfactory accuracy.
(Bangian-Tabrizi & Jaluria, 2018) proposed a method based on optimization techniques to solve a two-dimensional heat source's inverse natural convection problem on a vertical flat plate. PSO algorithm was applied to find the best pair of vertical locations. It was revealed that the hired technique resulted in satisfactory solutions for both source strength and location.
The method presented in this study, namely intelligent Akbari-Ganji's Method (IAGM), takes advantage of the analytical trial functions that are to be tuned by intelligent, meta-heuristic optimization algorithms. In it, a generic function must be selected as a preliminary, trial answer to the DE that is to be solved. Thereafter, the chosen trial function along with its derivatives are embedded into the DE of interest. The initial/boundary conditions, on the other hand, are dealt with as a second objective of the optimization problem - this is contrary to most of the similar previous works in which a penalty function is used to do so. Next, a weighted sum decomposition multi-objective version of the PSO algorithm is recruited to minimize the residual mean of the DE and to search for the best coefficients of the chosen trial function by doing so. That is, the DE of interest is now converted to an optimization problem.
The rest of the article is organized as follows. Section 2 presents brief explanation of the PSO algorithm. In section 3, the presented method is described. Section 4 explains Van der Pol oscillator as a nonlinear ODE to be solved. The obtained results are given in Section 5.

2. Overview of PSO algorithm

PSO algorithm is among the most well-known swarm-based, evolutionary optimization algorithms (Tomita, 2014). The reason behind its popularity lies mainly in its simple execution mechanism along with its efficient performance. The execution mechanism of PSO may be briefly explained as follows: firstly, random positions (candidate solutions) are allocated to every particle. Next, the cost (fitness) values of the allocated positions are evaluated. After that, the algorithm main loop begins in which positions and velocities of the particles are updated, and the new personal and global fittest are stored. The before mentioned loop is reiterated until a termination criterion is met. Finally, the latest global fittest is selected as the final answer to the problem. The position and velocity updating relations are given as below:

$$x_{t+1}{}^{i} = x_{t}{}^{i} + v_{t+1}{}^{i} \quad (1)$$

$$v_{t+1}{}^{i} = w_{i} \times v_{t}{}^{i} + c_{1,t}{}^{i} \times r_{1} \times \left(p_{t}{}^{i} - x_{t}{}^{i}\right) + c_{2,t}{}^{i} \times r_{2} \times \left(p_{t}{}^{g} - x_{t}{}^{i}\right) \quad (2)$$

Where $x_t{}^i$ and $p_t{}^i$ indicate, respectively, the position and the personal best record of the particle i in the iteration t. $p_t{}^g$ denotes the global best record in the t^{th} iteration. r_1 and r_2 are two random numbers uniformly distributed in the range [0, 1]; w is the inertia weight; and c_1 and c_2 are two constants called personal and global learning coefficients, respectively. In this study, c_1 and c_2 account, respectively, for controlling the bias toward exploitation and exploration during the execution of the algorithm.

3. Description of the presented IAGM Method

Consider the general form of the nonhomogeneous differential equations that follow:

$$F\left(t, y, \dot{y}, \ddot{y}, \ldots, y^{(m-1)}, y^{(m)}\right) = G(t) \tag{3}$$

Where $y^{(m)}$ indicates the m-order derivative of the dependent variable y. $G(t)$ is an excitation force and, hence, it just depends on the independent variable t. Subject to the initial/boundary conditions of:

$$y(0) = y_0, y^{(1)}(0) = y_1, \ldots, y^{(m-1)}(0) = y_{(m-1)}, y^{(m)}(0) = y_m \tag{4}$$

Based on the physics of the problem - whether it is excited or non-excited (homogeneous/ nonhomogeneous), vibrational or non-vibrational, and so forth - a trial function must be selected, as a raw answer, with the following general form of:

$$\hat{y}\left(\boldsymbol{t} \mid \boldsymbol{a}\right) \tag{5}$$

The trial function $\hat{y}$ is a function of both time vector ($\boldsymbol{t}$) and the vector of decision variables ($\boldsymbol{a}$). By substituting the selected trial function (Eq. 5) and its derivatives into (Eq. 3), one gets the objective (fitness) function of the minimization problem as follows

$$\min_{\boldsymbol{a}} MSE = [\frac{1}{n}\sum_{i=1}^{n}[F(t, \hat{y}, \dot{\hat{y}}, \ddot{\hat{y}}, \ldots, \hat{y}^{(m-1)}, \hat{y}^{(m)}) - G(t)]^2] + [\frac{1}{n}\sum_{i=1}^{n}(y(0) - y_0)^2 + \ldots + \frac{1}{n}\sum_{i=1}^{n}(\hat{y}^{(m)}(0) - y_m)^2] \tag{6}$$

The fitness value of any individual (candidate solution) implies how close the suggested candidate solution is to the exact solution of the DE. As can be inferred from Eq. 6, in search of the decision variables, i.e. the answer to the DE of interest, the recruited optimization algorithm attempts in minimizing two terms (objectives): 1) the residual between left- and right-hand sides of the DE and 2) the error related to the boundary/initial conditions. For the homogenous DEs, where a right-hand side does not exist, only the left-hand side of the DE should be minimized to zero. According to the performance index used in Eq. 6, i.e., the mean squared error, the formulated objectives are both of the same scales, so there is no need for any scale leveling factor (Hoseini et al, 2021).

4. Experiments on Van der Pol equation

In this section, to verify the effectiveness of IAGM, it will be recruited to solve the VDP oscillator as a quintessential example of nonlinear ODEs. Figure 1 illustrates a schematic of mechanical and electrical VDP oscillator generators. In addition, the offered solutions by IAGM will be.

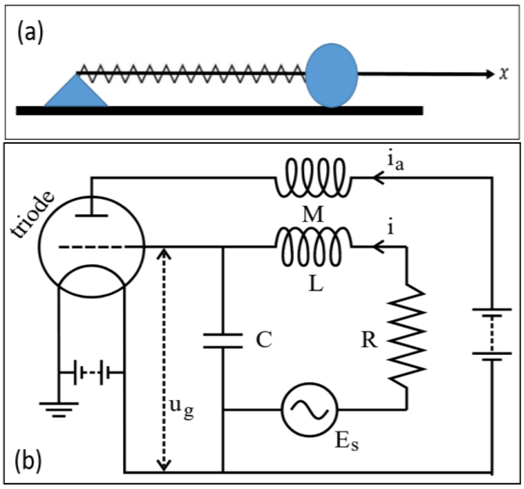

Fig. 1. (a) One-dimensional, mechanical Van der Pol oscillator, (b) electrical circuit generating a forced Van der Pol oscillator (Tomita, 2014).

Compared to those offered by formerly the conventional techniques, including VIM and AGM. Meanwhile, the accuracy of the mentioned methods will be judged, altogether, according to the answer obtained by the numerical Runge-Kutta (4, 5) method.

A 2nd-order, initial value, vibrational, homogenous, nonlinear ODE, Van der pol equation is among the most well-known DEs used to simulate non-conservative oscillators with nonlinear damping. This equation is of the general form that follows

$$\frac{d^2}{dt^2}x(t)+\mu(1-x(t)^2)\frac{d}{dt}x(t)+\beta^2 x(t)=0; t\in[0,50] \tag{7}$$

With the constant coefficients of μ=0.15 and β=1.2, and initial conditions of

$$x(0)=0.2, \dot{x}(0)=0; \tag{8}$$

The trial function used for the Van der pol equation is (Akbari et al, 2014):

$$\hat{x}(t)=e^{-a_1 t}\times\left[a_2\times\cos(a_3 t+a_4)\right] \tag{9}$$

Substituting Eq. 9 into Eq. 7 and applying the boundary conditions given in Eq. 8, we can get the following objective function.

$$\min_{a_1,a_2,a_3,a_4} MSE=\frac{1}{n}\sum_{t=0}^{n}\begin{bmatrix} ((e^{-a_1 t}\times\begin{bmatrix}(a_1^2 a_2-a_2 a_3^2)cos(a_3 t+a_4)+\\ 2a_1 a_2 a_3 sin(a_3 t+a_4)\end{bmatrix}+\\ 0.15\times\begin{bmatrix}(e^{-a_1 t}\begin{bmatrix}(a_1 a_2 cos(a_3 t+a_4)+\\ a_2 a_3 sin(a_3 t+a_4)\end{bmatrix}\\ (e^{-2a_1 t}a_2^2 cos^2(a_3 t+a_4)-1))\end{bmatrix}+\\ 1.44\times(e^{-a_1 t}\times[a_2 cos(a_3 t+a_4)])-0 \\ \end{bmatrix}^2 \tag{10}$$

$$+\frac{1}{n}\sum_{t=0}^{n}(a_2 cos(a_4)-0.2)^2+\frac{1}{n}\sum_{t=0}^{n}(-a_1 a_2 cos(a_4)-a_2 a_3 sin(a_4))^2$$

By minimizing the objective function given in Eq. 10, we will get the analytic solution to this DE as follows

$$x_{IAGM}(t) = 0.20038620e^{-0.074630t} \times \cos(1.200134t - 0.062104) \tag{11}$$

According to (Ledari et al, 2015), VIM, AGM, and Runge-Kutta (4, 5) methods were used for validation of IAGM. These methods were applied to Eq. 7, and the results are shown as follows

$$x_{AGM}(t) = 0.20036097e^{-0.07200t} \times \cos(1.197838t - 0.060036) \tag{12}$$

$$\begin{aligned} x_{VIM}(t) &= 0.01277264\sin(1.197653t)e^{-0.07500t} + 0.2000518\cos(1.197653t)e^{-0.07500t} \\ &-0.0000220\cos(3.592961t)e^{-0.22500t} - 0.0000044\sin(3.592961t)e^{-0.22500t} \\ &-0.0000298\cos(1.197653t)e^{-0.22500t} - 0.0002414\sin(1.197653t)e^{-0.22500t} \end{aligned} \tag{13}$$

It is worth noting that VIM series had 2 terms, and Eq. 9 was used as a trial function for AGM.

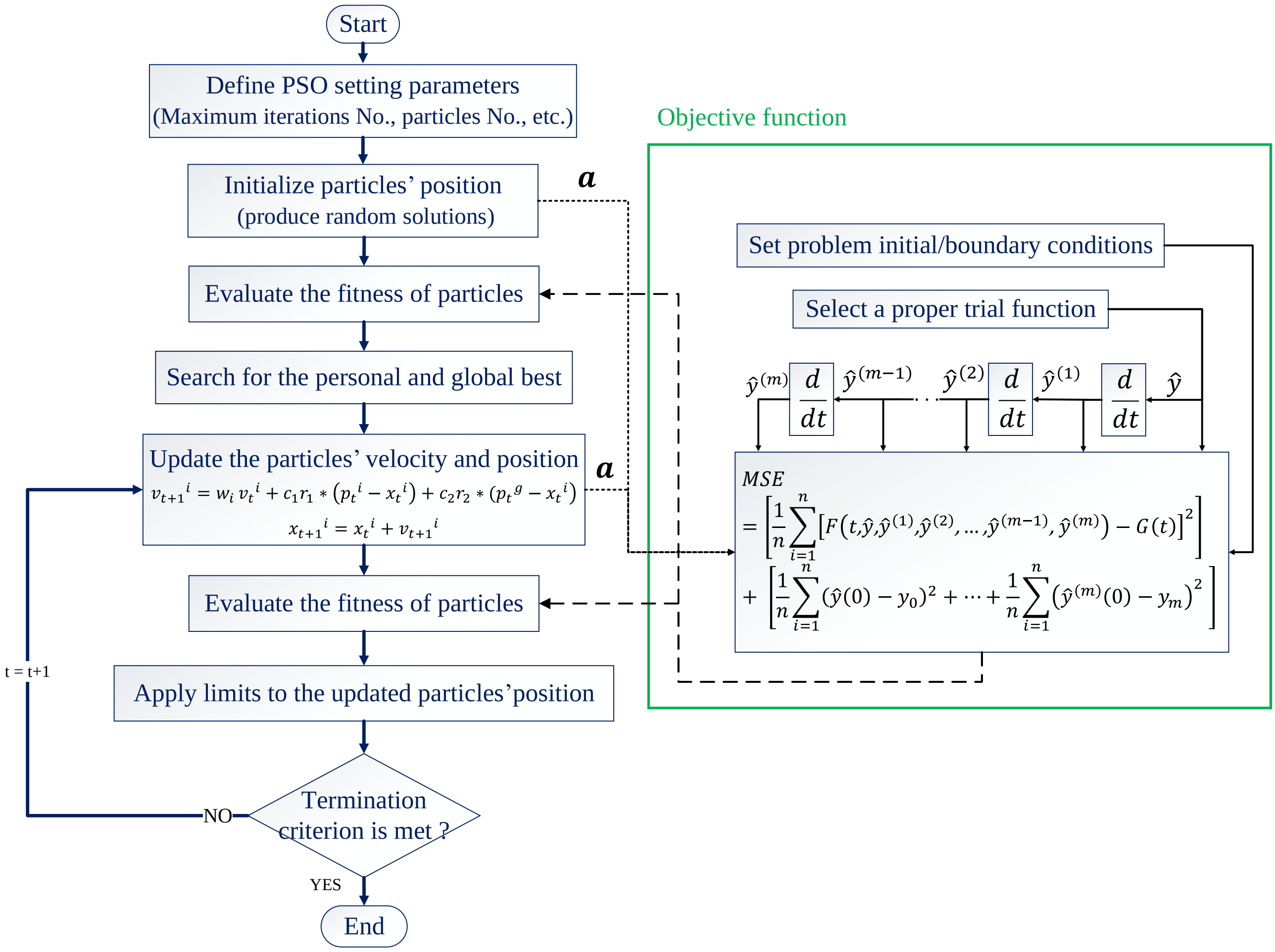

Fig. 1. Flowchart of the IAGM algorithm.

5. Results and Discussion

In this section, the case study described previously will be solved using VIM, AGM, and IAGM; besides, the accuracy of the obtained results will be judged according to the numerical Runge-Kutta (4, 5) method. For the sake of conciseness, however, the authors have sufficed to depict only the plot of the final answers. As explained before, to use IAGM, a PSO algorithm is recruited with a population size of NPop=500 and a maximum number of iterations MaxIt=500 as the termination criterion. In addition,

the personal and global learning coefficients (c1,c2) are equal to 2, and the particles' inertia weight (w) is equal to 0.72. The platform on which the subsequent computations were performed is an Intel^® Core^(TM2)Duo CPU @ 2.50 GHz with 4.00 GB of RAM memory. Like AGM, IAGM needs a trial function as a preliminary, generic answer to the DE that is to be solved. The corresponding trial function is specified according to the physics of the DE. For elaborate discussion on correctly selecting a trial function for DEs of different categories, please refer to (Akbari et al, 2015).
Figure 3 illustrates the analytic solutions offered by each method compared to those of the numerical method. As can be seen, the answer suggested by IAGM are in tight agreement with those of the numerical method. More specifically, IAGM offers better accuracy compared to all methods. Furthermore, in terms of simplicity, IAGM was shown to be among the most straightforward analytical methods presented up to now. Consequently, it can be posited that IAGM is a novel, powerful technique in solving nonlinear ODEs. Table 1 tabulates root-mean-square error (RMSE) of each case study suggested by each of the analytical methods. Accordingly, the bold numbers indicate the least error, and, tabulates the root-mean-square error (RMSE) of each case study suggested by each of the analytical methods hence, the superiority of the corresponding method with respect to the others.

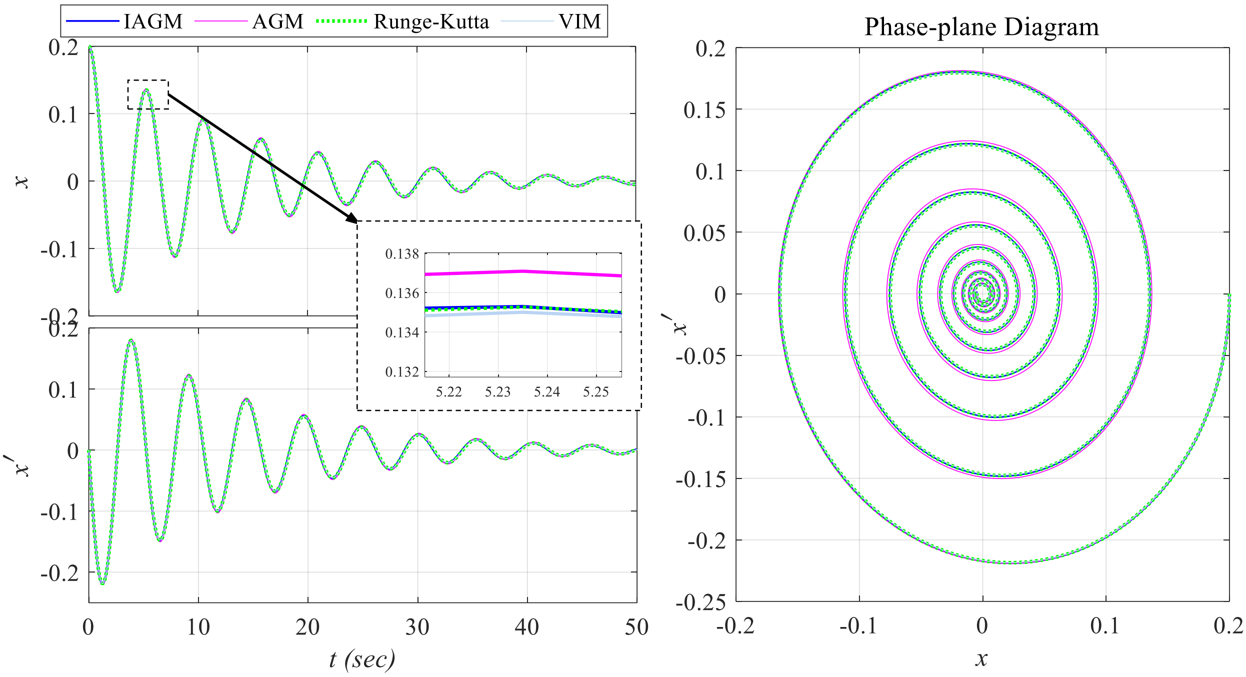

Fig. 2. Results for the case study: comparison of the answers relative to Runge-Kutta.

Table 1. RMSE of case study solved by each analytical method.

	RMSE		
Method	IAGM	AGM	VIM
Accuracy	1.1112e-04	1.3898e-03	2.1144e-04

According to Figure 3, it can be seen that the developed IAGM method offers an acceptable accuracy compared to the numerical, AGM, and VIM methods. Table 1 lists the RMSE accuracies achieved by each of the methods in solving the VDP equation. Accordingly, IAGM has found a more accurate solution than AGM while using the same trial function as AGM. It is important to note that the accuracy of the IAGM can be further improved by fine-tuning the parameters of the optimization algorithm−e.g. increasing the population size, increasing the maximum iteration number, etc. This would be one of the main advantages of this method. Among other advantages of this method is that this method is more straightforward than the other analytical methods.

There are a considerable number of works for further expansion of the current topic in the future. IAGM can solve systems of differential equations and nonlinear partial differential equations as efficaciously as it solved VDP equation in the current study. Furthermore, solving DEs with IAGM can be thought as a simple, straightforward benchmark problem to judge on the efficiency of any newly developed optimization algorithms. In this regard, other meta-heuristic optimization algorithms (Bakhshinezhad et al, 2020; MirMohammad Sadeghi et al, 2019) may be employed to solve various types of DEs, and statistical analysis, e.g. Markov Chain Monte Carlo can be conducted to fairly judge on the performance of each.

6. Conclusion

In this paper, a novel technique was introduced to obtain an approximate, closed-form, analytical solution for the nonlinear Van der Pol equation. For the sake of validation, however, the results suggested suggested by IAGM were compared to those by the conventional analytical methods including VIM and AGM. Accordingly, it was observed that IAGM is more advantageous to rival methods in terms of solution accuracy. In other words, with regard to the numerical method of Runge-Kutta (4, 5), IAGM offered solutions with the least residual compared to the examined analytical methods. The error can be further decreased by increasing the number of terms in trial functions that have been selected. Equally important is that IAGM is the most user-friendly and straightforward method among the other existing analytical methods. Next, searching for the unknown coefficients of the trial function, an optimization algorithm (PSO in this study) minimizes the residual of the DE, and the boundary/initial conditions are considered the second objective to be minimized. Thus, the problem tended to be that of multi-objective type, and it was dealt with using a weighted-sum decomposition method. By doing so, the problem of solving DEs is transformed into a parameter identification one. Therefore, it can be expected that, in terms of accuracy, the answers offered by IAGM would be substantially superior to those by the other existing rival, deterministic methods. Consequently, according to the results obtained, it can be postulated that the developed IAGM technique is more advantageous in comparison with other formerly developed analytical methods.

Table 2. Nomenclature

Abbreviation	Definition
ACP	Ant Colony Programming
ADM	Adomian's decomposition method
AGM	Akbari-Ganji's Method
BVP	Boundary Value Problem
CMAES	Covariance Matrix Adaptation Evolution Strategy
DE	Differential Equation
ES	Evolution Strategy
GPAD	Genetic Programming and Automatic Differentiation
IAGM	Intelligent Akbari-Ganji's Method
HPM	Homotopy Perturbation Method
ICA	Imperialist Competitive Algorithm
LA-PSO	Learning Automata Particle Swarm Optimization
MCMC	Markov Chain Monte Carlo
MSE	Mean Squared Error
NODE	Near Ordinary Differential Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equations

PM	Perturbation Method
PSO	Particle Swarm Optimization
SDE	Systems of Differential Equations
VIM	Variation Iteration Method

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