B-Spline Collocation Solution of One Dimensional Nonlinear Differential Equation Arising in Homogeneous Porous Media

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Abstract: This paper investigates the Numerical solution nonlinear partial differential equation for one dimensional instability phenomenon known as Boussinesq's equation arising in a porous media in oil-water displacement treatment (instability). Its Numerical Solution has been acquired by utilizing B-Spline method with proper boundary and initial conditions. The Numerical solution of Boussinesq's equation using Spline method is very nearer to Exact Solution obtained by analytical method .It is surmise that when distance and time increases, its saturation of injected water is increases . Numerical solution and graphical illustration has been obtained by Matlab.

Keywords: Water-flooding process, Instability, Immiscible displacement, Fluid flow, B-Spline Collocation Method.

1. Introduction

The fingering phenomenon occurs during the secondary recovery process arising in porous media, which is popular in different engineering fields such as soil mechanics, Agriculture, groundwater and hydrology, and petroleum engineering **(Brailovsky, Babchin, Frankel, & Sivashinsky, 2006)**, **(Posadas, Quiroz, Crestana, & Vaz, 2009)**, **(Tullis & Wright, 2007)**. This kind of phenomenon can also be seen in the oil recovery treatment that occurs in oil reservoirs. It is common to practice oil recovery technology to inject water into oil fields at specific locations in an attempt to drive oil into a production well. This stage of oil recovery is referred to as secondary recover.

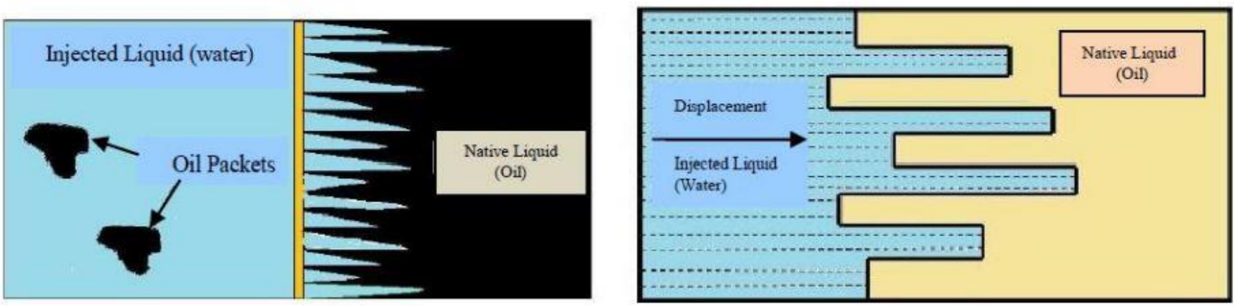


Figure 1:

The fingering process between native oil and injected water flow through a porous medium is visualized in fig (1). Only the average cross-sectional area occupied by the fingers was measured in the statistical treatment of fingering, Neglecting the size and shape of individual fingers **(Scheidegger A. , 1960)**. The statistical behaviour of fingering phenomenon in porous media was studied by **(Scheidegger & Johnson, 1961)**, who used the method of characteristics. With the use of a perturbation solution, **(Verma, 1969)** investigated the stabilization of instabilities in oil-water displacement treatment through heterogeneous porous media with capillary pressure. The confluent hypergeometric function was used by **(Patel D. M., 1998)** to solve the problem. Using the advection-diffusion concept, **(Patel D. M., 1998)** has explained this problem. Many Researchers **(Mehta & Joshi, M. S, 2009)**, **(Pradhan, Mehta, & Patel, 2011)**, and **(Patel & Desai, 2015)** have explained analytical and numerical approaches to the fingering phenomenon arising in homogeneous porous media using various methods such as Rdtm method, Crank-Nicolson method, and Homotopy methods. Using the Spline method, we can obtain a numerical values of a One-Dimensional Boussinesq's equation arising in secondary recovery treatment in homogeneous porous media. Matlab software has been used to get numerical values and graphical demonstrations.

2. Statement of the problem

It is reflected that injected water is uniform into the porous medium, such that the injected water shoots across the native oil and yields are perturbed. Consider permeability and porosity as constant. This occurs in a well-defined finger flow. Due to the water injection, the whole oil at the initial boundary (measured in the direction of displacement) is expatriated over a short distance. Finally, we decide that the initial boundary is saturated with water.

Darcy's law is assumed for mathematical formation of the problem and As a result, only the average behaviour of the two fluids is considered. During recovery process the saturation of water is determine as the cross-sectional area occupied by injected water. The goal of this study is to solve a Boussinesq's equation for one-dimensional instability Phenomenon in a homogeneous porous media at the time of recovery process. Using B spline method we can obtained numerical solution of Boussinesq's equation and the numerical values has been compared with exact values which is obtained by analytic method.

3. Mathematical formulation of the Problem

For two immiscible fluids, we can write down the seepage velocities of injected water (V_{iw}) and native oil (V_{no}) expressed by Darcy's law as (Bear J. , 2013),

$$V_{iw} = -\frac{k_{iw}}{\delta_{iw}} K \left[\frac{\partial P_{iw}}{\partial x} + \rho_{iw} g \sin\alpha \right] \quad (1)$$

$$V_{no} = -\frac{k_{no}}{\delta_{no}} K \left[\frac{\partial P_{no}}{\partial x} + \rho_{no} g \sin\alpha \right] \quad (2)$$

Where,

δ_{iw} =The constant kinematic viscosity of injected water

δ_{no} =The constant kinematic viscosity of native oil

k_{iw} =Relative permeability of injected water

k_{no} = Relative permeability of native oil

K = Permeability of the homogeneous porous medium

P_{iw} =The pressures of water

P_{no} =The pressures of oil

ρ_{iw} =constant density of water

ρ_{no} = constant density of oil

V_{iw} =The seepage velocity of water

V_{no} =The velocity Native oil

α =The inclination of the bed

g =Acceleration due to gravity.

For injected water, Continuity equations can be expressed as:

$$P\frac{\partial S_{iw}}{\partial t} + \frac{\partial V_{iw}}{\partial x} = 0 \quad (3)$$

$$P\frac{\partial S_{no}}{\partial t} + \frac{\partial V_{no}}{\partial x} = 0 \quad (4)$$

Where, P is the porosity. As per phase saturation, Relation between S_{iw} and S_{no} as

$$S_{iw} + S_{no} = 1 \quad (5)$$

The relation between P_c =capillary pressure and S_{iw} , determined by **(Bear J. , 2013)**,

$$P_c = -\beta S_{iw} \tag{6}$$

Where, β is a constant.

$$P_c = P_{no} - P_{iw} \tag{7}$$

For the mathematical formation, Due to **(Scheidegger & Johnson, 1961)** , we use following relations between saturation of injected water and relative permeability of injected water as below:

$$k_{iw} = S_{iw} \tag{8}$$

$$k_{no} = S_{no} = 1 - S_{iw} \tag{9}$$

From equations (1) - (4), the equation Motion for saturation can be expressed as

$$P\frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_{iw}}{\delta_{iw}} K \frac{\partial P_{iw}}{\partial x}\right) \tag{10}$$

$$P\frac{\partial S_{no}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_{no}}{\delta_{no}} K \frac{\partial P_{no}}{\partial x}\right) \tag{11}$$

Using equation (7), equation (10) gives

$$P\frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_{iw}}{\delta_{iw}} K \left(\frac{\partial P_{no}}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right] \tag{12}$$

Using equations (11) and (5), eliminating $\frac{\partial S_{iw}}{\partial t}$ from (12), we have

$$\frac{\partial}{\partial x}\left[K\left(\frac{k_{iw}}{\delta_{iw}} + \frac{k_{no}}{\delta_{no}}\right)\frac{\partial P_{no}}{\partial x} - K\frac{k_{iw}}{\delta_{iw}}\frac{\partial P_c}{\partial x}\right] = 0 \tag{13}$$

Finally after integration both sides, we get,

$$K\left(\frac{k_{iw}}{\delta_{iw}} + \frac{k_{no}}{\delta_{no}}\right)\frac{\partial P_{no}}{\partial x} - K\frac{k_{iw}}{\delta_{iw}}\frac{\partial P_c}{\partial x} = -C \tag{14}$$

Where, C is integrating constant. Solving (14) for $\frac{\partial P_{no}}{\partial x}$

$$\frac{\partial P_{no}}{\partial x} = \frac{-C}{K\left(\frac{k_{iw}}{\delta_{iw}} + \frac{k_{no}}{\delta_{no}}\right)} + \frac{\frac{\partial P_c}{\partial x}}{1 + \frac{k_{no}}{k_{iw}}\frac{\delta_{iw}}{\delta_{no}}} \tag{15}$$

Using (15) in (12) we have

$$P\frac{\partial S_{iw}}{\partial t} + \frac{\partial}{\partial x}\left[\frac{K\frac{k_{no}}{\delta_{no}}\frac{\partial P_c}{\partial x}}{1 + \frac{k_{no}}{k_{iw}}\frac{\delta_{iw}}{\delta_{no}}} + \frac{C}{1 + \frac{k_{no}}{k_{iw}}\frac{\delta_{iw}}{\delta_{no}}}\right] = 0 \tag{16}$$

Replacing the value of the pressure of oil P_{no} , we have

$$P_{no} = \frac{P_{no} + P_{iw}}{2} + \frac{P_{no} - P_{iw}}{2} = \bar{P} + \frac{1}{2}P_c \tag{17}$$

The mean pressure $\bar{P}$ is constant, Hence (14) gives

$$C = \frac{K}{2}\left(\frac{k_{iw}}{\delta_{iw}} - \frac{k_{no}}{\delta_{no}}\right)\frac{\partial P_c}{\partial x} \tag{18}$$

Equation (16) becomes

$$P\frac{\partial S_{iw}}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left[K\frac{k_{iw}}{\delta_{iw}}\frac{\partial P_c}{\partial S_{iw}}\frac{\partial S_{iw}}{\partial x}\right] = 0 \tag{19}$$

Since $k_{iw} = S_{iw}$ and $P_c = -\beta S_{iw}$, we have

$$P\frac{\partial S_{iw}}{\partial t} - \frac{\beta}{2}\frac{K}{\delta_{iw}}\frac{\partial}{\partial x}\left(S_{iw}\frac{\partial S_{iw}}{\partial x}\right) = 0 \tag{20}$$

Using dimensionless variables $X = \frac{x}{L}$, $T = \frac{K\beta t}{2\delta_{iw}L^2 P}$, equation (20) gives Boussinesq's equation as

$$\frac{\partial S_{iw}}{\partial T} = \frac{\partial}{\partial X}\left(S_{iw}\frac{\partial S_{iw}}{\partial X}\right) = S_{iw}\frac{\partial^2 S_{iw}}{\partial X^2} + \left(\frac{\partial S_{iw}}{\partial X}\right)^2 \tag{21}$$

Where $S_{iw}(x,t) = S_{iw}(X,T)$.

For solution of Boussinesq's equation (21) given the set of initial and boundary values as bellows

$$S_{iw}(X,0) = X \quad ;\ \text{initial values of saturation for fixed value } T = 0$$

$$S_{iw}(0,T) = T \quad ;\text{Values of saturation for fixed } T \text{ at } X = 0 \tag{22}$$

$$S_{iw}(1,T) = 1 - T \quad ;\ \text{Values of saturation for fixed } X = 1.$$

4. B-Spline Solution of Boussinesq's equation

In the region [0, 1], we are taking equal partition of the length h such that $0 < X_1 < ... < X_N = 1$. Let $\Phi_m(X)$ be cubic B-splines. Now, basis for functions defined as $\{\Phi_{-1}, \Phi_0, \Phi_1, ..., \Phi_N, \Phi_{N+1}\}$ over [0, 1]. Hence, in the terms of the cubic B-splines as trial functions, the B-Spline solution approximation $S_{iwN}(X,T)$ can be defined as:

$$S_{iwN}(X,T) = \sum_{m=-1}^{N+1} e_m(T)\Phi_m(X), \tag{23}$$

Φ_m : Cubic B-splines for m=-1...N+1, defined as below:

$$\Phi_m = \frac{1}{h^3}\begin{cases} (X - X_{m-2})^3 & [X_{m-2}, X_{m-1}], \\ h^3 + 3h^2(X - X_{m-1}) + 3h(X - X_{m-1})^2 - 3(X - X_{m-1})^3 & [X_{m-1}, X_m], \\ h^3 + 3h^2(X_{m+1} - X) + 3h(X_{m+1} - X)^2 - 3(X_{m+1} - X)^3 & [X_m, X_{m+1}], \\ (X_{m+2} - X)^3 & [X_{m+1}, X_{m+2}] \\ 0 & otherwise \end{cases} \tag{24}$$

Here $h = X_{m+1} - X_m, m = -1, ..., N+1$.

Using equation (23) and cubic splines (24), In the forms of the elements e_m the values of S_{iw}, S'_{iw}, S''_{iw} at the knots are defined by

$$S_{iw_m} = S_{iw}(X_m) = e_{m-1} + 4e_m + e_{m+1}$$

$$S_{iw}^{'} = S_{iw}^{'}(X_m) = \frac{3}{h}\left(e_{m+1} - e_{m-1}\right) \tag{25}$$

$$S_{iw}^{"} = S_{iw}^{"}(X_m) = \frac{6}{h^2}\left(e_{m-1} - 2e_m + e_{m+1}\right)$$

Where,

$S_{iwm}^{'}$ = First time derivative of S_{iwm} w.r.to X .

$S_{iwm}^{"}$ = Two time derivative of S_{iwm} w.r.to X .

The B-Spline solution of S_{iwm} for given Boussinesq's equation

$$S_{iwT} - S_{iw}S_{iwXX} - \left(S_{iwX}\right)^2 = 0 \tag{26}$$

can be obtained by considering the solution of

$$(S_{iwT})_m^n - \Theta\left[\left(S_{iw}S_{iwXX}\right)_m^{n+1} + \left(S_{iw}S_{iwXX}\right)_m^n\right] - (1-\Theta)\left[\left(S_{iwX}^2\right)_m^{n+1} + \left(S_{iwX}^2\right)_m^n\right] = 0 \tag{27}$$

The Spline method to the governing equation (21) with the appropriate conditions of the expression (22) has been employed as under

$$\begin{aligned} & e_{m-1}^{n+1}\left(1 - \Theta\Delta TL_3 - \frac{6\Theta\Delta TL_1}{h^2} + (1-\Theta)\frac{6\Delta TL_2}{h^2}\right) + e_m^{n+1}\left(4 + 4\Theta\Delta TL_3 + \frac{12L_1\Theta\Delta T}{h^2}\right) \\ & + e_{m+1}^{n+1}\left(1 - \Theta\Delta TL_3 - \frac{6\Theta\Delta TL_1}{h^2} - (1-\Theta)\frac{6\Theta\Delta TL_2}{h}\right) \\ & = L_1 \end{aligned} \tag{28}$$

Where

$$L_1 = e_{m-1}^n + 4e_m^n + e_{m+1}^n; \qquad L_2 = \frac{3}{h}\left(e_{m+1}^n - e_{m-1}^n\right)$$

$$L_3 = \frac{6}{h^2}\left(e_{m-1}^n - 2e_m^n + e_{m+1}^n\right)$$

5. Results and Discussion

We used here Θ =0.5 then we have N+1 system of linear equations with the N+3 unknowns $d^n = (e_{-1}^n, e_0^n, e_1^n, e_2^n, ..., e_N^n, e_{N+1}^n)$. For B-Spline Numerical solution to this system we required two values e_{-1}^n and

e_{N+1}^{n}. These values are obtained from the boundary condition. For removal of values e_{-1}^{n}, e_{N+1}^{n} from given system (28) we have to use following equations

$$S_{iw}(X_0) = e_{-1}^{n+1} + 4e_{0}^{n+1} + e_{1}^{n+1} = T$$
$$S_{iw}(X_N) = e_{N-1}^{n+1} + 4e_{N}^{n+1} + e_{N+1}^{n+1} = 1 - T \quad (17)$$

Finally we have $(N+1)\times(N+1)$ matrix system. Now Use of Thomas Algorithm we can solve this system matrix. Table 1 shows the numerical values of injected water saturation for various distances X and times T =0.0011, 0.0022, 0.0033, 0.0044, 0.0055. Figure 2 shows the graphical representation of Table 1 of $S_{iw}(X,T)$ for injected water versus distance X for fixed time T =0.0011, 0.0022, 0.0033, 0.0044, 0.0055.

Table 1: B-Spline Solution of Saturation $S_{iw}(X,T)$ for fixed values of T =0.0011, 0.0022, 0.0033, 0.0044, 0.0055 and $X=0$ to 0.5

X/T	T=0.0011	T=0.0022	T=0.0033	T=0.0044	T=0.0055
0	0.0011	0.0022	0.0033	0.0044	0.0055
0.025	0.026100	0.027200	0.028300	0.029400	0.030500
0.05	0.051100	0.052200	0.053300	0.054400	0.055500
0.075	0.076100	0.077200	0.078300	0.079400	0.080500
0.1	**0.101100**	**0.102200**	**0.103300**	**0.104400**	**0.105500**
0.125	0.126100	0.127200	0.128300	0.129400	0.130500
0.15	0.151100	0.152200	0.153300	0.154400	0.155500
0.175	0.176100	0.177200	0.178300	0.179400	0.180500
0.2	0.201100	0.202200	0.203300	0.204400	0.205500
0.225	0.226100	0.227200	0.228300	0.229400	0.230500
0.25	0.251100	0.252200	0.253300	0.254400	0.255500
0.275	0.276100	0.277200	0.278300	0.279400	0.280500
0.3	0.301100	0.302200	0.303300	0.304400	0.305500
0.325	0.326100	0.327200	0.328300	0.329400	0.330500
0.35	0.351100	0.352200	0.353300	0.354400	0.355500
0.375	0.376100	0.377200	0.378300	0.379400	0.380500
0.4	0.401100	0.402200	0.403300	0.404400	0.405500
0.425	0.426100	0.427200	0.428300	0.429400	0.430500
0.45	0.451100	0.452200	0.453300	0.454400	0.455500
0.475	0.476100	0.477200	0.478300	0.479400	0.480500
0.5	0.501100	0.502200	0.503300	0.504400	0.505500

Figure 2: $S_{iw}(X,T)$ versus distance X at fixed values of T =0.0011, 0.0022, 0.0033, 0.0044, and 0.0055

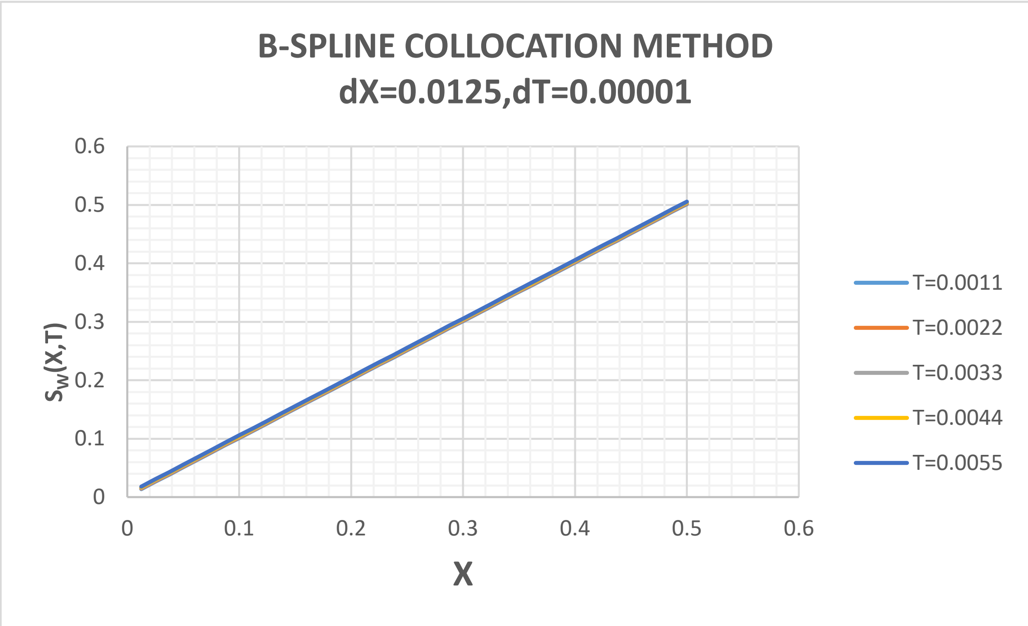


Figure 3: $S_{iw}(X,T)$ of water vs .time T for fixed values X =0.1, 0.2, 0.3, 0.4, and 0.5

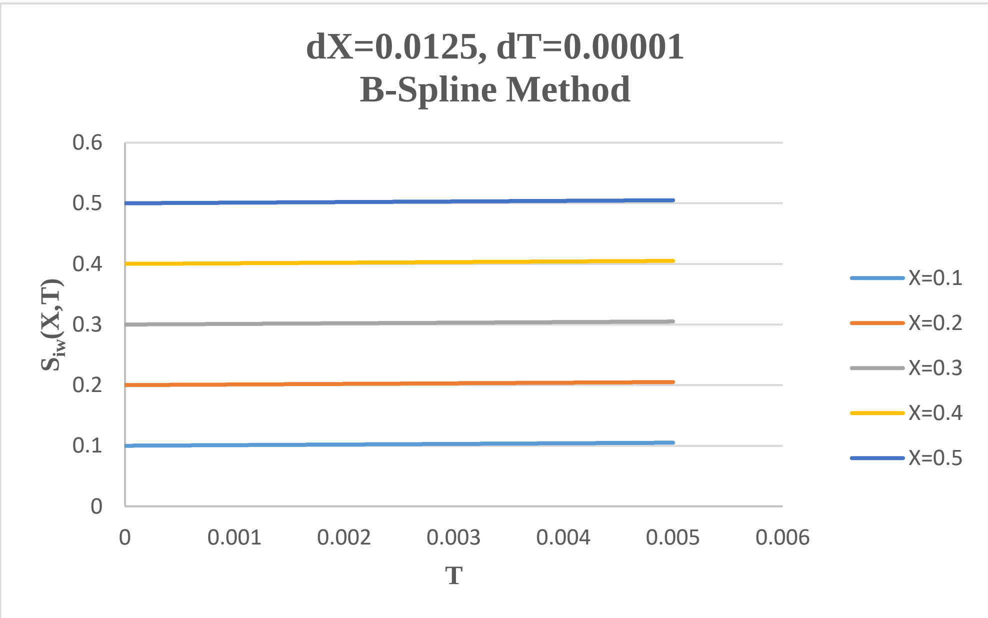
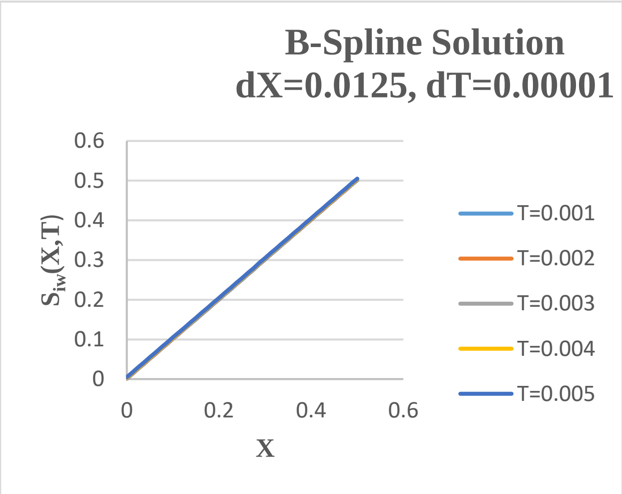
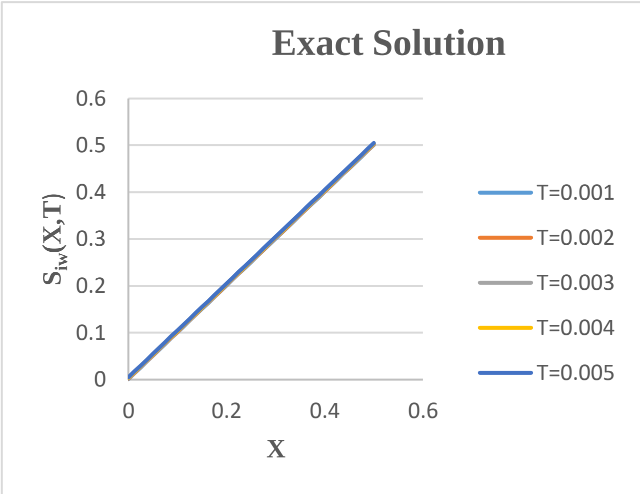


Table 2: Comparative Study of B-Spline and Exact Solution of $S_{iw}(X,T)$

	X=0.1		X=0.2		X=0.3		X=0.4		X=0.5	
T	**B-Spline**	**Exact**	**B-Spline**	**Exact**	**B-Spline**	**Exact**	**B-Spline**	**Exact**	**B-Spline**	**Exact**
0.001	0.101000	0.1008	0.201	0.2006	0.301	0.3004	0.401	0.4002	0.501	0.5
0.002	0.102000	0.1016	0.202	0.2012	0.302	0.3008	0.402	0.4004	0.502	0.5
0.003	0.103000	0.1024	0.203	0.2018	0.303	0.3012	0.403	0.4006	0.503	0.5
0.004	0.104000	0.1032	0.204	0.2024	0.304	0.3016	0.404	0.4008	0.504	0.5
0.005	0.105000	0.104	0.205	0.203	0.305	0.302	0.405	0.401	0.505	0.5

Figure 4: Graph of Exact and B-Spline solution of $S_{iw}(X,T)$ (Saturation of injected water) for fixed values of T =0.001, 0.002, 0.003, 0.004, and 0.005





6. Conclusion

The solutions of Boussinesq's equation by B-Spline Collocation method are presents graphically (figure 2) and in tabular(table 1) using Matlab which observed that the solutions by spline method are convergent to exact solutions for fixed values of T = 0.011,0.022,0.033,0.044,0.055. For accurate B-Spline Solution we have to select proper values of dX = 0.00125 and dT = 0.00001. Figure (4) shows that the comparative study demonstrations that B-Spline Solution of given equation is very close to exact solution. And also shows that saturation of water $S_{iw}(X,T)$ linearly growing when distant X growing for fixed time T = 0.001, 0002, 0.003, 0.004, 0.005.

References

[1]. Bear, J. (1979). *Hydraulics of Groundwater* . McGraw-Hill Intl.
[2]. Bear, J. (2013). *Dynamics of fluids in porous media.* Courier Corporation.
[3]. Brailovsky, I., Babchin, A., Frankel, M., & Sivashinsky, G. (2006). *Fingering instability in water-oil displacement", Transport in Porous Media* (Vol. 63).
[4]. Caldwell, J. (1987). Application of cubic splines to the nonlinear Burgers' equation. *Numerical Methods for Nonlinear Problems, 3*, 253-261.
[5]. Chen, Z., Huan, G., & Ma, Y. (2006). *Computational methods for multiphase flows in porous media.* SIAM.

[6]. Dag, I., Irk, D., & Saka, B. (2005). A numerical solution of the Burgers' equation using cubic B-splines. *Applied Mathematics and Computation, 163*(1), 199-211.
[7]. Desai, N. B. (2002). *The study of problems arises in single phase and multiphase flow through porous media.* Ph. D. Thesis, South Gujarat University, Surat, India.
[8]. Marino, M. A. (1978). Flow against dispersion in nonadsorbing porous media. *Journal of Hydrology, 37*(1-2), 149-158.
[9]. Mehta, M. N., & Joshi, M. S. (2009). Solution by group invariant method of instability phenomenon arising in fluid flow through porous media. *Int. J. of Engg. Research and Industrial App., 2*(1), 35-48.
[10]. Mukherjee, B., & Shome. (2009). An analytic solution of fingering phenomenon arising in fluid flow through porous media by using techniques of. *Journal of Mathematics Research, 1*(2).
[11]. Patel, D. M. (1998). The classical solution approach of problems arising in fluid flow through porous media. *Ph. D. Thesis, South Gujarat University, Surat, India*.
[12]. Patel, M. A., & Desai, N. B. (2015). An approximate analytical solution of nonlinear differential equation arising in fluid flow through homogeneous porous media. *International Journal of Innovative Research in Science, Engineering and Technology, 4*(8), 7655--7662.
[13]. Posadas, A., Quiroz, R. a., Crestana, S., & Vaz, C. (2009). Characterizing water fingering phenomena in soils using magnetic resonance imaging and multifractal theory. *Nonlinear processes in geophysics, 16*(1), 159-168.
[14]. Pradhan, V. H., Mehta, M. N., & Patel, T. (2011). A numerical solution of nonlinear equation representing one dimensional instability phenomena in porous media by finite element technique. *IJAET, 2*, 221-227.
[15]. Prenter, P. M., & others. (2008). *Splines and variational methods.* Courier Corporation.
[16]. Scheidegger, A. (1960). Growth of instabilities on displacement fronts in porous media. *The Physics of Fluids, 3*(1), 94-104.
[17]. Scheidegger, A. E. (1958). The physics of flow through porous media. *Soil Science, 86*(6), 355.
[18]. Scheidegger, A. E. (1961). General theory of dispersion in porous media. *Journal of Geophysical Research, 66*(10), 3273-3278.
[19]. Scheidegger, A. E., & Johnson, E. F. (1961). The statistical behavior of instabilities in displacement processes in porous media. *Canadian Journal of physics, 39*(2), 326-334.
[20]. Smith, G. D., & Smith, D. (1985). *Numerical solution of partial differential equations: finite difference methods.* Oxford university press.
[21]. Tullis, B. P., & Wright, S. J. (2007). Wetting front instabilities: a three-dimensional experimental investigation. *Transport in Porous Media, 70*(3), 335--353.
[22]. Verma, A. P. (1969). Statistical behaviour of fingering in a displacement process in heterogeneous porous medium with capillary pressure. *Canadian J. Physics, 47*(3), 319-324.
[23]. Von Rosenberg, D. U. (1969). *Methods for the numerical solution of partial differential equations.* North-Holland.