Research Article

Symmetric Bi-Derivations on Bitonic Systems

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Abstract: This study was conducted to present the concept of symmetric bi-derivations on bitonic systems as a generalized dual BCC-algebraic system and explain the properties of symmetric bi-derivations on bitonic systems.

Keywords: Bitonic system, symmetric bi-derivation, trace, kernel

1. Introduction

Derivations play a key role in the theory of rings and algebraic systems. Investigating derivations and generalized derivations of bitonic systems, Shule Ayar Ozbal and Yong Ho Yon introduced the concept of bitonic algebraic system in 2018 [5]. BCI algebra, BCC algebra and BCK algebra have been investigated by researchers such as Zhang and Dudek in 1998, Dudek in 1992, Tanaka and Iseki in 1976 and Xin and Meng in 1992. The concept of derivations in the near-ring and ring theory was applied by Xin and Jun to BCI algebra in 2004. In 2006, derivations in BCK algebra were investigated by Alshahri and Hamza. Two years later, equivalent conditions in which derivations were isotone on lattices with the greatest element were presented by Li and Xin. Gy. Maksa introduced the symmetric bi-derivations concept [3], and symmetric bi-derivation on rings was investigated by J.Vukman. H. Bresar introduced generalized derivations of rings [10], which were applied to logical systems, including lattices and BCC algebra, which was in turn introduced as generalized BCK algebra by Komori and Dudek [6-8].

Moreover, the generalized derivations and derivations of BCC algebra were addressed in [9]. Despite addressing bi-derivations in literature, bi- derivations on BCC algebra has not been investigated yet.

The algebraic system of (C,*,1) is dual BCC algebra that satisfies the following axioms for $a, b\ and\ c \epsilon C$:
(i) $1 * a = a$
(ii) $a * a = 1$
(iii) $a * b = 1$, $b * a = 1 \rightarrow a = b$
(iv) $(a * b) * [(b * c) * (a * c)] = 1$

(v) $a * 1 = a$

The concept of dual BCC algebra is the generalized Hilbert algebra, Heyting alge- bra, implication algebra and lattice implication algebra. All of these types of algebra satisfy the following condition:

(S) $a \leq b \rightarrow b * c \leq a * c$, $c * a \leq c * b$

These types of algebra can be therefore considered generalized algebra that satisfies (S).

Bitonic systems play a key role in information technology, networks, coding systems, encoding, security and formal models of secure computer systems. The results of this paper can be applied to bitonic median, bitonic sorters of minimal depth and bitonic networks (Theoretical Computer Science, Journal Elsevier [11]).

Preliminaries

Certain properties and definitions presented were required for obtaining the results. The properties of bitonic systems mentioned in this article were mainly adopted from [5].

Definition 1
The algebraic system of (A,*, 1) is defined as a Bitonic system in which A represents a set, 1 one of its members and * a binary operator acting on the set with axioms as follows being assumed for every a, b and c∈A:
(B1) $a * 1 = 1$
(B2) $1 * a = a$
(B3) $a * b = 1$ and $b * a = 1 \rightarrow a = b$

(B4) $a * b = 1 \rightarrow (b * c) * (a * c) = 1$ and $(c * a) * (c * b) = 1$

Lemma 1

For every a, b and c∈A as a bitonic system:

(i) $b * c = a * b = 1 \to a * c = 1$

(ii) $a * a = 1$

(iii) $a * (b * a) = 1$

Definition 2

For ≤ defined as a binary operator on the bitonic system of A with a,b∈A:

$a * b = 1 \leftrightarrow a \leq b$

According to the axioms, ≤ constitutes a partial order on A and 1 the greatest element in A.

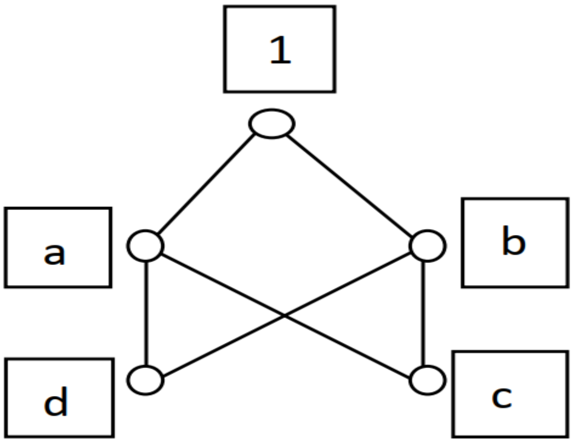

Figure 1: The Hasse diagram of bitonic system G in example 1

Lemma 2

For every a, b and c ∈ A as a bitonic system:

(i) $a \leq b * a$

(ii) $a \leq b \to b * c \leq a * c$ and $c * a \leq c * b$

Example 1

Let * be defined as a binary operator on $G = \{1, a, b, c, d\}$ as per the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	d
c	1	1	1	1	a
d	1	1	1	c	1

(G,*,1) then constitutes a bitonic system whose Hasse diagram is shown in figure 1.

Lemma 3

For every a, b and c ∈ A as a dual BCC algebra:

$a * b = 1 \ \to (b * c) * (a * c) = (c * a) * (c * b) = 1$

Theorem 1

Dual BCC algebra constitutes a bitonic system, although the inversion of this theorem is not true. In fact, bitonic system *G* in example 1 is not dual BCC algebra, since

$$(x * z) * [(z * w) * (x * w)] = z * (x * w) = z * w = 1$$

Theorem 2

The following equivalent conditions can be considered for every a, b and c ∈ A as a bitonic system:

(i) $a * b \leq (b * c) * (a * c)$
(ii) $b * (a * c) = a * (b * c)$

Definition 3
Binary operator ⋁ is defined as follows for all a and b∈A as a bitonic system:

$a \vee b = (a * b) * b$

Proposition 1
Binary operator ⋁ satisfies the following conditions for every a,b,c∈A as a bitonic system:

(i) $a \vee 1 = 1$, $1 \vee a = 1$
(ii) $a \leq b \rightarrow a \vee b = b$
(iii) $b \leq a \vee b$

(iv) $a \leq b$, $b \leq c \rightarrow a \leq c$ (transitivity)

Proof

(iv) $a \leq b$, $b \leq c \rightarrow a * c = 1 * (a * c) = (a * b) * (a * c) = 1 \rightarrow a \leq c$

Definition 4
For every a and $b \in A$ and assuming the mapping of d on A as a bitonic system defined as d: A→A

$d(a * b) = [a * d(b)] \vee [d(a) * b]$ → d=(r,l) derivation of A

$d(a * b) = [d(a) * b] \vee [a * d(b)]$ → d=(1,r) derivation of A

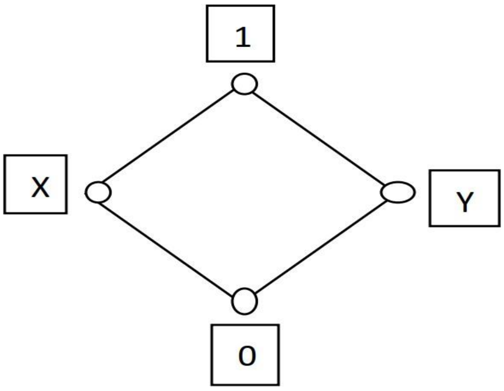

Figure 2: The Hasse diagram of bitonic system B in example 3

Example 2
By taking $d1 = 1, x = a, y = b, z = c,$ and $w = d$ in Example 1, we can define the mapping $d : G \to G$ by $d(1) = 1$, $d(a) = 1$, $d(y) = y$, $d(z) = 1$, $d(w) = w$,

As a result, d constitutes a (r, l) derivation rather than a (l, r)-derivation, as $d(z * w) = 1 = x = (d(z) * w) \vee (z * d(w))$. Moreover, defining the mapping of $\delta : G \to G$ as

$\delta(1) = 1$, $\delta(x) = x$, $\delta(y) = 1$, $\delta(z) = 1$, $\delta(w) = w$.

As a result, δ constitutes a (l, r) derivation rather than a (r, l) derivation, as

$\delta(z * w) = x = 1 = (z * \delta(w)) \vee (\delta(z) * w)$

Example 3

Let * be an operator on B={x,y,0,1} as per the following table:

*	1	x	y	0
1	1	x	y	0
x	1	1	y	y
y	1	x	1	0
0	1	1	1	1

As a result, B constitutes a bitonic system whose Hasse diagram is shown in figure 2. In addition, the mapping of $\boldsymbol{d}: \boldsymbol{B} \to \boldsymbol{B}$ defined using $\boldsymbol{d(0) = 1}$, $\boldsymbol{d(y) = 1}$, $\boldsymbol{d(x) = x}$ and $\boldsymbol{d(1) = 1}$ constitutes both (l,r) and (r,l) derivations of B.

Symmetric bi-derivations of bitonic systems

The basic notions and results are provided as follows:

Definition 5

For every x and $y \in A$, if $D(x,y) = D(y,x)$, the mapping of D(.,.): $A \to A$ with A representing a bitonic system is symmetric.

Definition 6

The mapping of d: $A \to A$ defined using d(x)=dx=D(x,x) for every $x \in A$ as a bitonic system is defined as the trace of D(.,.):$A \to A$ as a symmetric mapping.

Definition 7

If D(x*y,z)=$(x * D(y,z)) \vee (D(x,z) * y)$ for every x, y and z$\in A$, in which A represents a bitonic system and D a symmetric mapping of D(.,.): $A \to A$. As a result, D constitutes a symmetric bi-derivation on A in a way that:

D(x,y*z)=$(D(x,y) * z) \vee (y * D(x,z))$

Example 4

Assume * acts as an operator on A={0,1,a,b} based on the following table:

*	0	b	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Let D(.,.): $A \to A$, in which A clearly represents a bitonic system. In addition,

$$D(x,y) = \begin{cases} a & for\ x = y = 0 \\ b & for\ x = 0\ and\ y = a\ or\ y = 0\ and\ x = a \\ 1 & otherwise \end{cases}$$

As a result, D constitutes a symmetric bi-derivation on A.

Example 5

Assume that A={a,b,0,1} satisfies the following table:

*	a	b	0	1
a	1	1	b	1
b	b	1	a	1
0	1	1	1	1

1	a	b	0	1

D defined as follows therefore constitutes a symmetric bi-derivation on A.

D(x,y)= $\begin{cases} x & for\ y=0 \\ y & for\ x=0 \\ b & for\ y=x=a \\ 1 & otherwise \end{cases}$

Remark

Assume D_1(x,y)=x*y and D_2(x,y)=y*x for B as a bitonic system. D_1 and D_2 cannot be, however, necessarily considered symmetric bi-derivations on B unless B is a lattrice [5].

Proposition 2

For $x \in A$ and d representing the trace of a symmetric bi-derivation D on A as a bitonic system:

(i) D(1,x)=1

(ii) d1=1

Proof

(i) For every x and y ∈ A:

D(1,x)=$D(x*1,x) = \big(x*D(1,x)\big)\vee\big(D(x,x)*1\big) = \big(x*D(1,x)\big)\vee \quad = 1$

(ii) is also proved based on the cited reasoning.

Proposition 3

For every x and y ∈ A as a bitonic system with d representing the trace of a symmetric bi-derivation of D on A:

(i) D(x,y)∨x =D(x,y)

(ii) x,y≤D(x,y)

(iii) dx)∨$x = dx$

(iv) x≤d^2x and x≤dx

(v) d^2x=dx

Proof

i) For every x, y∈A:

D(x,y)=D(1*x,y)=[1*D(x,y)]∨[D(1,y)*x]=D(x,y)∨(1*x)=D(x,y)∨x.

ii) Can be obviously proved based on Lemma 1 and (i).

iii and iv) For every x∈A:

dx=D(x,x)=D(1*x,x)=D(x,x)∨(1*x)=dx∨x.

x≤d(x) for every x∈A. Given that x≤dx≤d^2x, x≤ d^2x.

v) dx≤d^2x→dx*d^2x=1; x≤d^2x→1=x*dx≤d^2x*dx≤1. As a result, d^2x*dx=1. According to definition 1, B(3), d^2x=dx.

Proposition 4

For every x,y∈A as a bitonic system with d representing the trace of a symmetric bi-derivation of D on A:

dx*y≤x*y≤x*dy

Proof

According to Proposition 3, x≤dy and x≤dx. According to Lemma 2, dx*y≤x*y≤x*dy

Lemma 4

For x∈A as a bitonic system with d representing the trace of a symmetric bi-derivation of D on A:
(i) D(dx*x,x)=1
(ii) d(dx*x)=d(x*dx)=1

Proof

i) According to theorem 2, the properties of bitonic systems (B1) and (B2) and the definition of symmetric bi-derivation D of bitonic systems:

D(dx*x,x)=[dx,D(x,x)]∨[D(dx,x)*x)=(dx,dx)∨[D(dx*x)*x]=1

ii) According to the properties of bitonic systems and the definition of the trace of a symmetric bi-derivation D of these systems:

d(dx*x)=D(dx*x,dx*x)=[dx*D(x,dx*x)]∨[D(dx,dx*x)*x]=[dx*(dx*dx)∨(D(x,dx)*x)]∨[D(dx,dx*x)*x]=[dx*(1 ∨(D(x,dx)*x))]∨(D(dx,dx*x)*x)=[dx*1]∨(D(dx,dx*x)*x)=1∨(D(dx,dx*x)*x)=1.

The second equality is proved given d(x*dx)=d1=1.

In [5] the notion of the kernel was introduced for a derivation, where a characteri- zation was given. Although, the trace of a bi-derivation *D* : *A A A*, say *d*, is not necessarily a derivation, nevertheless we can give a similar characterization for *d.* Take *d* as before and set: ×

Proposition 5

Ker d = {x ∈ A|dx =1}.
Ker d={dx∗x|x∈A} for A as a bitonic system with d representing the trace of a symmetric bi-derivation D on A.

Proof

According to Lemma 4(ii), {dx∗ x|x∈A}⊆ Ker d. Let y∈Ker d, then dy=1 and therefore y=dy∗y, i.e. Ker d⊆{dx∗x|x∈A}.

Remark

As mentioned above, the conclusion of the Proposition 5 holds for any derivation, but by taking d as the trace of the bi-derivation D, defined in example 5, we can get mapping d that is not a derivation but satisfies in the Ker d = {dx ∗ x|x ∈A}.

Theorem 3

D is a symmetric mapping that satisfies D(x,y∗z)=y∗D(x, z) for every x, y, z∈A if and only if the mapping D:A×A→A is a symmetric bi-derivation.

Proof

Let a symmetric mapping D : A × A → A satisfy D(x, y ∗ z) = y ∗ D(x, z) for all x, y, z ∈ A . We show that D is a bi-derivation on A × A. First of all, we prove that x ∗ D(x, y) = 1. We have, D(1, y) = D(D(1, y) ∗ 1, y) = D(y, D(1, y) ∗ 1) =

D(1, y) ∗ D(y, 1) = D(1, y) ∗ D(1, y) = 1. Thus, x ∗ D(x, y) = x ∗ D(y, x) = D(y, x ∗ x) =

D(y, 1) = 1. For every x, y ∈ A , y ≤ D(y, x) = D(x, y) and z ≤ D(z, x) = D(x, z)

we have,

$D(x, y) * z \leq y * z \leq y * D(x, z)$.

Hence $D(x, y * z) = y * D(x, z) = 1 * (y * D(x, z)) = ((D(x, y) * z) * (y * D(x, z)) *$

$(y*D(x, z))=(D(x,y)*z)\vee(y*D(x,z))$, suggesting that D is considered a bi-derivation.

Moreover, $D : A \times A \to A$ is a symmetric bi-derivation. For all $x, y, z \in A$, $y \leq D(x, y)$ and $z \leq D(x, z)$ and so,

$D(x, y) * z \leq y * z \leq y * D(x, z)$.

Hence $D(x, y * z) = (D(x, y) * z) \vee (y * D(x, z)) = y * D(x, z)$.

Remark

Note that, by symmetry, we can institute the identity in Theorem 3 with,

$D(x*y,z)=x*D(y, z)$ for every $x, y, z\in A$

Corollary

For every $x,y\in A$, if $D:A\times A\to A$ constitutes a symmetric bi-derivation with d representing the trace of D:

(i) $D(x,x*y)=1$
(ii) $d(x*y)=x*(x*dy)$

Proof

Given D as a symmetric bi-derivation,

$D(x,x*y)=D(x*y,x)=x*D(y,x)=x*D(x,y)=D(x*x, y)=D(1, x)=1$

For every $x,y\in A$,
$d(x*y)=D(x*y,x*y)=(x*D(y,x*y))$
$=(x*D(x*y,y))$
$=(x*(x*D(y, y)))$
$=x*(x*dy)$.

Corollary

For A as a bitonic system and $D:A\times A\to A$ as a symmetric bi-derivation with trace d and $x*(y*z)=(x*y)*(x*z)$ for every $x,y,z\in A$:

$d(x *y)=x*dy$

Note that this property holds in Example 1.

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