Research Article

Deskins's conjecture on Lie algebras

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Abstract: We investigate relationships between the properties of maximal sub- algebras of L and the members of P(M) and solvability and supersolv- ability in Lie algebras. that corresponds to similar relationships in the group-theory. Further, we show that if L be a Lie algebra and algebri- caly closed field of zero characteristic, there exists a θ-subalgebra C such that L=M+C and $\frac{C}{Core(C\cup M)}$ is abelian for all maximal subalgebras M and L, L is solvable.

Keywords: Lie algebra, Completion, Maximal subalgebra

1. Introduction

Let M be a maximal subalgebra of the Lie algebra L. A subalgebra C of L is said to be a completion for M if C is not contained in M but every proper subalgebra of C that is an ideal of L is contained in M. The set I(M) of all completions of M is called the index complex of M in L. This is analogous to the concept of the index complex of a maximal subgroup of a finite group as introduced by Deskins in (Deskins, 1954); this concept has since been further studied by a number of authors, including Ballester-Bolinches and Ezquerro (1992), Bei- dleman and Spencer (1972), Deskins (1990), Mukherjee (1975), and Mukherjee and Bhattacharya (1988). The objective of this paper is to investigate relationships between the properties of maximal subalgebras of L and the members of P(M) and solvability and supersolvability in Lie algebras. that corresponds to simi- lar relationships in the group-theory.

It is easy to see that the sum of all ideals of L that are proper subalgebras of C is itself a proper subalgebras of L. We define the strict core (resp.core) of a subalgebra $B\, f\!= 0$ to be the sum of all ideals of L that are proper subalge- bras (resp.subalgebras) of B, and denote it by k(B) or $k_L(B)$ (resp.B_L). The subalgebra C is then a completion of the maximal subalgebra M of L (*that is,* $C \in I(M)$) if $L =< M, C >$ and $k(C) \subseteq M$.

In section two, we study completions that are members of P(M) and show that if M is a maximal subalgebra of L and N is a maximal ideal of L, such that If

$C \in P\,(M)$ and $N \leq K(C)$, then $\frac{C}{N} \in I(\frac{M}{N})$; and $K(\frac{C}{N}) = \frac{K(C)}{N}$.

In the last section, maximal completions that are *θ-subLiealgebras* are shown that over an algebrically closed field, with characteristic zero, a Lie algebra is supersolvable if for each maximal subalgebra M of composite index in L there exists a maximal θ-subalgebra C for M that $L = C + M$ and $\frac{C}{Core\ M\cap C(L)}$ is cyclic. This is analogous to that of Deskins for groups.

2. Maximal subalgebras and maximal comple- tions

Definition 2.1. *Let M be a maximal subalgebra of L, then set P(M)=* $\{C \in I(M) \mid C$ *is maximal in* $I(M)$ *and* $L = C + M\}$.

Lemma 2.2. *Let M be a maximal subalgebra of L and N be a maximal ideal of L. If* $C \in P(M)$ *and* $N \leq K(C)$, *then*

(i) $\frac{C}{N} \in I\left(\frac{M}{N}\right)$

(ii) $K\left(\frac{C}{N}\right) \in \frac{K(C)}{N}$.

Proof. It is clear that $\frac{C}{N} \in I\left(\frac{M}{N}\right)$. We only proof (ii). Since $C \in P\,(M)$, We have $K(C) < C$ and $K(C) \leq Core(M)$. Hence $\frac{K(C)}{N} < \frac{C}{N}$, and $\frac{K(C)}{M} \leq \frac{M}{N}$, therefore, $K\left(\frac{C}{N}\right) \geq \frac{K(C)}{N}$. In addition, $\frac{C}{N} \in I\left(\frac{M}{N}\right)$ implies that $K\left(\frac{C}{N}\right) < \frac{C}{N}$. If $\left(\frac{C}{N}\right) < \frac{H}{N}$, then $H \lhd L$ and $H < C$. From the definition of $K(C)$, it is deduced that $H \leq K(C)$. Therefore, $\left(\frac{C}{N}\right) < \frac{C}{N}$.

Lemma 2.3. *Let M be a maximal subalgebra of L and* $C \in P\,(M)$. *If* $CoreM \nleq C$, *then there exists an ideal completion H of M in L such that* $\frac{H}{K(H)}$*isomorph with a subalgebra of a quotient algebra of* $\frac{C}{K(C)}$.

Proof. Since $CoreM \nleq C$, we have $C < C + CoreM$. The maximality of C in I(M) leads to the conclusion that $C + CoreM$ is not in I(M). Hence, the collection $S = \{T \lhd L \mid T \nleq CoreM,\ T < C + CoreM\}$ is nonempty. Let H be minimal in this partially ordered set S. Then $H \in I(M)$, $H + CoreM > CoreM$ and $H + CoreM \leq C + CoreM$. $(CoreM) \cap H < H$ and $(CoreM \cap H) \lhd L$ imply that $(CoreM \cap H) \leq K(H)$. On the other hand, it is clear that $K(H) \leq (CoreM \cap H)$. Hence, $K(H) = CoreM \cap H$. Therefore,

$$\frac{H}{K(H)} = \frac{H}{CoreM \cap H} \cong \frac{H + CoreM}{CoreM} \leq \frac{C + CoreM}{CoreM}$$

From $K(C) \leq (C \cap CoreM)$, we can conclude that $\frac{C+CoreM}{CoreM} \cong \frac{C}{CoreM \cap C}$ is a quotient algebra of $\frac{C}{K(C)}$. This completes the proof.

Theorem 2.4. *Let M be a maximal subalgebra of L, N be a minimal ideal subalgebra of L, and $N \leq M$. If there exists a C in P(M) with $\frac{C}{K(C)}$ cyclic, then there exists a $\frac{C_1}{N}$ in $P(\frac{M}{N})$ such that $\frac{\frac{C_1}{N}}{K(\frac{C_1}{N})}$ is cyclic.*

Proof. Assume that $N \leq K(C)$; then it follows from Lemma 2.2 that $\frac{C}{N} \in I(\frac{M}{N})$ *and* $K\left(\frac{C}{N}\right) = \frac{K(C)}{N}$. If $\frac{C}{N}$ is a maximal element of $I(\frac{M}{N})$, then $\frac{C}{N} \in P(\frac{M}{N})$ and $\frac{\frac{C}{N}}{K(\frac{C}{N})} \cong \frac{C}{K(\frac{C}{N})}$ is cyclic. So we can assume that $\frac{C}{N}$ is not maximal in I(M). According to this assumption, there exists $\frac{C_1}{N} \in I(\frac{M}{N})$ such that $\frac{C}{N} < \frac{C_1}{N}$. We have $< C_1$. The maximality of C in I(M) leads to the conclusion that C_1 is not in I(M). Hence, the set S $=\{T \lhd L \mid T \nleq M. T < C_1\}$, is nonempty. Choose H as the minimal element in S. This means that $H \in I(M)$. If $N + H < C_1$, then $\frac{N+H}{N} < \frac{C_1}{N}$ and $\frac{N+H}{N} \nleq \frac{M}{N}$, which is in contradiction to $\frac{C_1}{N} \in I(M)$. Hence, $C_1 = N + H \lhd L$, consequently $C_1 \in P(\frac{M}{N})$.

The minimality of N leads to the conclusion $N \cap H = 1$ (if $N \cap H = N$, then $C_1 = N + H = H$, in contradiction to that $H < C_1$) and $C_1 = N + H$ If there exists a subalgebra C_2 of C_1 such that $C < C_2 < C_1$, then, for any proper subalgebra H_1 of C_2, which is ideal in L, $N + H_1 \lhd L$ and $N + H_1 \leq C_2 < C_1$. Since $N + H$=$C_1 \cap N + H_1 = N(H \cap N + H_1$, $H \cap N + H_1 < H$ and $H \cap N + H_1 \lhd L$, we have $H \cap N + H_1 \leq M$, and $H_1 \leq N + H_1 = N + \big(H \cap (N + H_1)\big) \leq M$. Hence, $C_2 \in I(M)$, in contradiction to the maximality of C in I(M). It follows that C is a maximal subalgebra of C_1. If $K(H) \leq C$, then $K(H) < C$ (if not, $C = K(H) \leq M$). So $K(H) \leq K(C)$, and $K(C) \geq (N + K(H))$. Noticing that $K(C) = N + (K(C) \cap H)$, $(K(C) \cap H) < H$ and $K(C) \cap H \lhd L$, we have $K(C) \cap H \leq K(H)$ and $K(C) = N + K(H)$. It follows that $\frac{C_1}{K(C)}$ is a minimal ideal of $\frac{L}{K(C)}$ and $\frac{C}{K(C)}$ is maximal in $\frac{C_1}{K(C)}$. We have $\frac{C_1}{K(C)}$ is a solvable algebra, and therefore $\frac{C_1}{K(C)}$ is an elementary abelian Lie algebra. $(\frac{C_1}{K(C)}) + \left(\frac{M}{K(C)}\right) = \frac{L}{K(C)}$ and $(\frac{C_1}{K(C)}) \cap \left(\frac{M}{K(C)}\right) = 1$ imply that [L:M]= dim $(\frac{C_1}{K(C)})$. On the other hand, from $C \in P(M)$ we have . $(\frac{M}{K(C)}) + \left(\frac{C}{K(C)}\right) = \frac{L}{K(C)}$. It follows that dim $(\frac{C}{K(C)}) \geq$ [L: M] $=$ dim $(\frac{C_1}{K(C)})$. which is a contradiction. So we can assume that $K(H) \nleq C$, and therefore $C + K(H) = C_1$. Since $K(C) \cap H \lhd L$ and $K(C) \cap H < H$, we have $K(C) \cap H \leq K(H)$ and $K(C) \cap H \leq C \cap K(H)$ It follows that

$K(C) = N + (K(C) \cap H) \leq N + (C \cap K(H)) = C \cap (N + K(H))$.

Hence,

$$\frac{(\frac{C_1}{N})}{(\frac{N + K(H)}{N})} \simeq \frac{C_1}{N + K(H)} = \frac{C + K(H)}{N + K(H)} \simeq \frac{C}{(C \cap (N + K(H)))}$$

is cyclic. Noticing $K(\frac{C_1}{N}) \geq \frac{N+K(H)}{N}$, we have $\frac{(\frac{C_1}{N})}{K(\frac{C_1}{N})} N \nleq K(C)$, which results in CoreM $\nleq C$.

It follows from Lemma 2.3 that there exists an ideal completion H of M such that $\frac{H}{K(H)}$ is cyclic. If N $\leq K(H)$, then clearly $\frac{H}{N} \in P(\frac{M}{N})$, and by using Lemma 2.2, it is deduced that $\frac{(\frac{H}{N})}{K(\frac{H}{N})} \simeq \frac{H}{K(H)}$ is cyclic. If $N \nleq K(H)$, then $N \nleq$

H. The minimality of N leads to the conclusion that N ∩ H = 1. Clearly, $\frac{N+H}{N} \in P(\frac{M}{N})$ and $\frac{(\frac{N+H}{N})}{(\frac{K(H)+N}{N}) \simeq \frac{H}{K(H)}}$ is cyclic. Noticing that $K(\frac{N+H}{N}) \geq \frac{K(H)+N}{N}$, we have $\frac{(\frac{N+H}{N})}{K(\frac{H+N}{N})}$ is cyclic. Now the proof is complete.

3. Solvability and supersolvability in Lie algebras

Defition 3.1. For a maximal subalgebra M of a fnite subalgebra L, a $\theta -$ subalgebra for M is any subalgebra C of L such that C ⋬ M and $Core_L$(M∩C) is maximal among proper ideal subalgebras of L contained in C.

Lemma 3.2. Assume that N ⊴L and that $\frac{U}{N}$ is the unique minimal ideal of $\frac{L}{N}$. Let M be a maximal subalgebra of L containing N but not containing U, and let C be a maximal member of I(M). Furthermore, suppose that $\frac{U}{N}$ is not involved in $\frac{C}{K(C)}$. Then,

(i) N=K(C)

(ii) C is a maximal subalgebra of U + C

Proof. Due to $K(C) \leq Core_L(M)$, and the hypothesis implies that $N = Core_L(M)$, we see that $K(C) \leq N$. If $K(C) < N$, then $N \nsubseteq C$ and $C + N > C$. $C + N$ is not in I(M) because C is maximal in I(M), and consequently $Core_L(C + N) \nsubseteq M$. Since $N \subseteq M$ and $N \subseteq Core_L(C + N)$, it follows that $N < Core_L(C + N)$, and consequently $U \subseteq C + N$ as $\frac{U}{N}$ is the unique minimal ideal of $\frac{L}{N}$. It follows that $\frac{U}{N}$ is involved in $\frac{C}{K(C)}$. contrary to the hypothesis.

This proves (i).

If $C \supseteq U$, then $\frac{U}{N}$ is a subalgebra of $\frac{C}{N} = \frac{C}{K(C)}$. This means that $\frac{U}{N}$ is involved in $\frac{C}{K(C)}$, contrary to the hypothesis. Thus, $C < U + C$. Let B be a subalgebra of $U + C$ such that $C < B \leq U + C$. As in the proof of (i), we have B is not in I(M), and hence $N \subseteq Core_L(B) \nsubseteq M$ and it follows that $U \subseteq Core_L(B)$. We thus conclude that $B = C + U$, which proves (ii).

Theorem 3.3. *Let L be a lie algebra. Assume that for each maximal sub- algebra M of composite index in L, there exists a maximal member C in I(M) such that* $\frac{C}{K(C)}$ *is cyclic of order more than or equal to the index of M in L. Then, L is solvable and every maximal subalgebra of L either has prime index of 4.*

Proof. By $\frac{C}{K(C)}$, then L is solvable. Suppose that L is nonsupersolvable and let M be a maximal subalgebra of L composite index. We must show that $[L : M] = 4$. Let $N = Core_M(L)$ and let $\frac{U}{N}$ be a chief factor of L. Then $U + M = L$ and $U \cap M = N$ because $\frac{U}{N}$ is abelian. Also, $C_{\frac{U}{N}}(M) \lhd L$, and thus $C_{\frac{U}{N}}(M) = N$. It follows that $C_{\frac{U}{N}}(L) = U$, and this implies that $\frac{U}{N}$ is the unique minimal ideal of $\frac{L}{N}$. Since $[L : M] = [U : N]$, we only need to show that $dim(\frac{U}{N}) = 4$.

By hypothesis, we may assume that C be maximal in I(M), where $\frac{C}{K(C)}$ is cyclic and $dim(\frac{C}{K(C)}) \geq [L : M]$. Since $\frac{U}{N}$ is noncyclic, it cannot be involved in the cyclic Lie algebra $\frac{C}{K(C)}$. Applying Lemma 3.2, we see that $K(C) = N$ and C is maximal in $E = U + C$. Also, $dim(\frac{C}{N}) \geq [L : M] = dim(\frac{U}{N})$, and consequently $dim(C) \geq dim(U)$. We claim that $C \lhd E$. Thus, $dim(\frac{C}{N}) \neq dim(\frac{U}{N})$, and we conclude that $dim(C) \geq dim(U)$. Let B be a conjugate of C in E and $B \neq C$. Then $dim(B> dim(C))$ and

$$\frac{dim(B) + dim(C)}{dim(B \cap C)} = dim(B + C) \leq dim(E) = \frac{dim(U) + dim(C)}{dim(U \cap C)}$$

so $dim(B \cap C) > dim(U \cap C)$. It follows that $B \cap C$ is not contained in U, and thus this intersection does not centralize $\frac{U}{N}$ because $C_L(\frac{U}{N}) = U$. Let $\frac{X}{N} = C_{\frac{L}{N}}(B \cap C)$. Then $U \nsubseteq X$, and since $\frac{U}{N}$ is the unique minimal ideal of $\frac{L}{N}$, $\frac{X}{N}$ is also core free, we deduce that $X \in I(M)$. But $\frac{B}{N}$ and $\frac{C}{N}$ are abelian, and thus X contains both C and B. By the maximality of C, we have $T = X \supseteq B$, which is not the case. Thus $C \lhd E$, as claimed.

Let $T = U \cap C$. Now C is maximal and ideal in E, so $dim[U : T] = dim[E : C]$ is prime. Since $\frac{T}{N}$ is cyclic and is contained in $\frac{U}{N}$, its order divides p, and we conclude that $|\frac{U}{N}| = p^2$. What remains is to show that $p = 2$.

We have $dim(C) \geq dim(U) > dim(T)$, and thus $E > U$. Let V be a subalgebra of E containing U such that $[V : U] = p$. Then $V \cap C > T$ and $\frac{(V \cap C)}{N}$ is cyclic.

Thus, $\frac{V}{N}$ is an algebra of order p^3 and exponent p^2. Let $Q = V \cap M$, so that $\frac{Q}{N}$ is a subalgebra of order p in $\frac{V}{N}$, and thus $\frac{V}{N}$ has more than p^2 elements of order dividing p. If $p > 2$, only Lie algebras of dim p^3 having this property have exponent p, and thus we deduce that $p = 2$.

Finally

$[L : M] = dim(\frac{U}{N}) = p^2 = 4$.

Lemma 3.4. *If C is a maximal θ-subalgebra for a maximal subalgebra M of L and* $N \trianglelefteq L$, $N \leq Core_L(M \cap C)$, *then* $\frac{C}{N}$ *is a maximal θ-subalgebra for* $\frac{M}{N}$. *Conversely, if* $\frac{C}{N}$ *is a maximal θ-subalgebra for* $\frac{M}{N}$, *then C is a maximal θ-subalgebra for M.*

Proof. Suppose that C is a maximal θ- subalgebra for M. It follows that $\frac{C}{N} \in \theta(\frac{M}{N})$. If $\frac{C}{N}$ is not a maximal θ-subalgebra in $\theta(\frac{M}{N})$, then $\frac{C}{N} < \frac{H}{N}$, $\frac{H}{N} \in \theta(\frac{M}{N})$, implies that $C < H$. Now we see that H is a θ-algebra for M, violating the maximality of C in $\theta(M)$.

Conversely, it is easy to see that if $\frac{C}{N}$ is a maximal θ-subalgebra for $\frac{M}{N}$, then C is a θ-subalgebra for M. If C is not a maximal θ-subalgebra, suppose that

$C < H$, $H \in \theta(M)$. This implies that $\frac{C}{N} < \frac{H}{N}$. Since $N \leq Core_L(M \cap C) \leq Core_L(M \cap H)$, we have $\frac{H}{N} \in \theta(\frac{M}{N})$, violating the maximality of $\frac{C}{N} \in \theta(\frac{M}{N})$.

Theorem 3.5. *Let L be a finite Lie algebra over a field F, where F has characteristic zero, suppose that for each maximal subalgebra M of composite index in L, there exists a maximal θ-subalgebra C for M such that* $L = C + M$ *and* $\frac{C}{Core_{M \cap C}(L)}$ *is cyclic. Then L is supersolvable.*

Proof. Assume that L is not supersolvable, and N is a minimal ideal of L.

(i) $\frac{L}{N}$ is supersolvable by induction.

First of all, we note that if M is a maximal subalgebra of L, $H = Core_L(M)$

and $\frac{K}{H}$ is a chief factor of L, then it is easy to see that K is a maximal element of $\theta(M)$.

To show that $\frac{L}{N}$ satisfies the hypothesis and consequently is supersolvable, let $\frac{M}{N}$ be a maximal subalgebra of composite index. From Lemma 3.4, we must find a maximal element A of $\theta(M)$ such that A contains N, $A + M = L$ and $\frac{A}{Core_L(A \cap M)}$ is cyclic. To do this, let C be a maximal element of $\theta(M)$ and suppose that $C + M = L$ and $\frac{C}{Core_L(C \cap M)}$ is cyclic. If C contain N, we are done by taking $A = C$. Otherwise, write $H = Core_L(M)$ and note that L is not contained in C so that $C < H + C$ and hence $H + C$ is not in θ(M). Also, note that $H = Core_L(H + C \cap M)$ and consequently there exists a subalgebra A, which is ideal in L with $H < A < H + C$. We may choose A such that $\frac{A}{H}$ is a chief factor of L. So, A is a maximal element of $\theta(M)$ and certainly A contains N. Since M is maximal and does not contain the ideal A, wehave $A + M = L$. Finally, $H = Core_L(A \cap M)$ and we need only to show that $\frac{A}{H}$ is cyclic. This follows because $\frac{C+H}{H}$ is cyclic, because $\frac{C}{(C \cap H)}$ is a homomorphic image of $\frac{C}{Core_L(C \cap H)}$, which is cyclic.

(ii) N is solvable.

We may assume that N is the unique minimal ideal of L. Since L is not supersolvable and $\frac{L}{N}$

is supersolvable, there exists a maximal subalgebra M of composite index and we know that it does not contain N. It follows that

$\theta(M) = \{N\} \cup \{X \subseteq L \mid X \nsubseteq M and N \nsubseteq X\}$.

Since $Core_L(C \cap M) = 1$, by hypothesis, there exists a maximal θ-subalgebra C of this set such that $C + M = L$ and C is cyclic. If $C = N$, then certainly N is solvable. So we can assume that C does not contain N. By the maximality of C as an element of $\theta(M)$, we know that every subalgebra of L is strictly larger than C containing in N. Suppose Y that is any subalgebra of N that is ideal in C but not contained in $C \cap N$. Then $C < Y + C$ and it follows that $N \subseteq Y + C$ and $N = Y + (N \cap C)$. Thus Y is ideal in N and $\frac{N}{Y}$ is cyclic and consequently $N' \subseteq Y$. But $N' = N$, or else $N' = 1$ and we are done, and thus $Y = N$. C is cyclic and $Y \trianglelefteq C$ then Y is cyclic. Where is $Y = N$ and N is abelian, then there is nilpotent. Thus N is solvable. This is a contradiction.

Theorem 3.6. *Let L be a finite Lie algebra over a field F, where F has characteristic zero. Suppose that for each maximal subalgebra M in L, there exists a maximal θ-subalgebra C for M such that L = C + M and* $\frac{C}{Core_{M\cap C}(L)}$ *is cyclic. Then L is solvable.*

Proof. Suppose that for each maximal subalgebra M in L, there exists a maximal θ-subalgebra C for M such that $L = C + M$ and $\frac{C}{Core_{M\cap C}(L)}$ is cyclic.

Now, it is revealed that C is an ideal in L.

$\forall c \in C$, then $c + Core_L(M \cap C) \in \frac{C}{Core_L(M\cap C)} \cdot \frac{C}{Core_L(M\cap C)}$ is abelian, then $[c+Core_L(M \cap C)] = 0_{\frac{C}{Core_L(M\cap C)}} = Core_L(M \cap C)$. Therefore $[c, l]+[Core_L(M \cap C), l] = Core_L(M \cap C)$. Since $Core_L(M \cap C)$ is ideal, and $[Core_L(M \cap C), l] \in Core_L(M \cap C)$ then $[c, l] + Core_L(M \cap C) = Core_L(M \cap C)$. Further $[c, l] \in Core_L(M\cap C) \leq C$. Finally $[c, l] \in C$. Hence, it was revealed that C is an ideal in L. Therefore

$$[C, M \cap C] \subseteq [C, C] \subseteq C^2 \subseteq Core_L(M \cap C) \cap C \subseteq M \cap C \subseteq M_L.$$

$M \cap C$ is an ideal in L thus M a $c-ideal$ of L, then L is solvable.

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