Research Article

Fractional Differential Equation of LC and LR Electrical Circuit.

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Abstract:

The fractional derivative isused to calculate the current and charge at any time t in LC,RC and LR circuits by using Laplace Transform of fractional derivatives.

Keywords: Caputo derivative, Fractional derivative,Laplace Transform, LC, RC and LR circuits, Mitag-Leffler function.

Introduction:

The fractional derivative is an overall instance of common subordinate to partial number. The fragmentary math administrators execute integrals and subordinates of the discretionary request through clarification[1].Different types of meanings have been introduced by different creators to assess a legitimate derivative or integral request during writing research.Currently, Fractional Calculus has been commonly used in various fields, such as discourse signals, ultrasonic wave proliferation, control framework display, heat transition, entropy age and dissemination, cardiovascular tissue cathode interface display, visco versatility, self-governing vehicle lateral and longitudinal control, temperature, food science etc[2].

Sequential improvement of fragmentary subsidiary was made in this conditions by Weyl in 1917, Krug in 1890, Nekrassov in 1888, Laurent in 1884, Sonin in 1869, Letnikov in 1868, Grunwald in 1867, Holmgren in 1865, Greer in 1859, Riemann in 1847, Liouville in 1832, Fourier in 1822, Lacroix in 1819, Laplace in 1812, Lagrange in 1772 and Euler in 1730 [3].

French mathematician S. F. Lacrix was the first author who used derivative of non integer order correctly [4]. In 1819 hepresented the derivative of non integer order $\frac{1}{2}$ in terms of Legendre's fractional symbol gamma function properly.

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

considering the function $y = x^m$, Lacroixit expressedas

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma\,(m+1)}{\Gamma\,(m-n+1)} x^{m-n}$$

replacing with n by $\frac{1}{2}$ and putting m=1, he got the proper result for the derivative of fractional order of the function of x.

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}}\sqrt{x}$$

In any case, the main use of Fractional Calculus was given by N. In 1823, H. Abel. He applied Fractional Calculus to the organization of an important condition for tautochrone problem management [5].

Basic Concepts:

Fractional calculus have been developed by various mathematicians as discussed in the introduction. Many mathematicians have given different types of definitions of fractional derivative and integration.

a) Riemann-Liouville partial subordinate:

For a function g given on interval [a,b], the Riemann-Liouville fractional order derivative of order $\propto (n-1 < \propto \leq n)$ of f is given by

$$D_a^{\propto} = \frac{1}{\Gamma\,(n-\alpha\,)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{-\propto+n-1} g(s)\,ds \qquad (4.1)$$

Consider that the integration on the right hand side should be defined.

b) Caputo Fractional Derivative[6]:

The Caputo fractional derivative of order $\propto > 0$ is introduced by Caputo in the form $(if\ m-1 < \propto \leq m, Re(\propto) > 0, m \in N)$

$${}_a^cD_t^{\propto} f(t) = I^{m-\propto} D^m f(t)$$

It can be expressed in other ways as

$${}_a^cD_t^{\propto} f(t) = \frac{1}{\Gamma\,(m-\alpha\,)} \int_0^t (t-\tau)^{m-\propto-1} f^m(\tau)\,d\tau,\ \ t > 0 \qquad (4.2)$$

$$= \frac{d^m f(t)}{dt^m}\ \ if\ \propto = m$$

where $\frac{d^m f(t)}{dt^m}$ is the m[th]derivative of order m of the function f(t) with respect to t.

or

$${}_a^cD_t^{\propto} f(x) = \frac{1}{\Gamma\,(1-\alpha\,)} \int_0^x \frac{f'(t)}{(x-t)^{\propto}}\,dt \qquad (4.3)$$

$$(where\ 0 < \propto < 1)$$

According to this definition

$${}_a^cD_t^{\propto} A = 0, f(t) = A = constant$$

that is Caputo's partial subordinate of a consistent is zero.

c) The Mittag-Leffler function:

The Mittag-Leffler function introduced by Mittag-Leffler[7] in 1903 is defined as,

$$E_{\propto}(x) = \sum_{k=0}^{\infty}\left(\frac{x^k}{\Gamma(\alpha k+1)}\right) \quad (4.4)$$

$$(\propto \in c, Re(\propto) > 0)$$

A speculation of the Mittag-Leffler work is given by Wiman [8] in 1905 characterized as,

$$E_{\propto,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)} \quad (4.5)$$

$$(\propto, \beta \in c, Re(\alpha) > 0, Re(\beta) > 0)$$

Prabhakar [8] presented a speculation of (2) in 1971 in the structure

$$E^{\gamma}{}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{(\gamma)_k x^k}{\Gamma(\alpha k+\beta)\, k!} \quad (4.6)$$

$$(\alpha, \beta, \gamma \in c, Re(\alpha) > 0, Re(\beta) > 0)$$

where $(\gamma)_k$ is the Pochammer symbol.

The Laplace transform of the fractional integrals and Caputo fractional derivatives are given in the following lemma.

Lemma 1:[9-11]

a) Let $R(\propto) > 0$ and $f \in L(0,b)$ for any $b > 0$. Also let the estimate $|f(t)| < Ae^{p_0 t}, t > b > 0$ holds for the constants A, $P_0 > 0$ then

$$L\{I_{0^+}^{\propto} f\}(s) = s^{-\propto}L\{f\} \quad (4.7)$$

b) Let $\propto > 0, n-1 < \propto \leq n, (n \in N)$be such that $f \in C^n(R^+)$, $f^{(n)} \in L(0,b)$ that for any $b > 0$,$|f(t)| < A\, e^{p_0 t}$, the Laplace transform of f and $f^{(n)}$ exist and $\lim_{t\to\infty}(f)^k = 0$, for k = 0,1,2,......,n. Then

$$L\{cD_{0^+}^{\propto} f\}(s) = s^{\propto}L\{f\}(s) - \sum_{k=0}^{\infty} s^{\propto-k-1}f(k)(0) \quad (4.8)$$

Lemma 2: [12, 13]

Let $\propto > 0$, $n = \Gamma(\alpha)$ and $\lambda \in R$.The solution of the initial value problem

$$c_{D^{\propto}}\, y(t) = \lambda\, y\,(t) + q(t)$$

$$y^{(k)}(0)y_k\,, k = 0,1,\ldots,n-1$$

where $q \in C\,[0,b]$ is a given function and can be expressed in the form

$$y(t) = \sum_{k=0}^{n-1} y_k u_k(t) + y_*(t) \quad (4.9)$$

with

$$y_*(t) = I_0^{\propto} q(t), if\ ג = 0$$

$$= \frac{1}{ג} \int_0^t (q(t) - s)\acute{u}_0(s)ds, if\ ג \neq 0$$

where

$u_k(t) = I_0^k e_{\propto}(t), k = 0,1, \dots, n-1$ and $E_{\propto}(t) = E_{\propto}(גt^{\propto})$

5. Formulation of RC circuit in the form of fractional ODE:

In this research work we express electrical circuit with a capacitor and resistance in series. The resistance and the capacitance are considered positive constants and E is the applied emf with zero voltage. According to the Kirchoff's voltage law RC circuit can be formulated in differential equation. The voltage drop across the resistor is

$$V_R(t) = RI(t)$$

Also the voltage drop across the capacitance is

$V_c(t) = \frac{1}{c}\int_0^t I(\delta)d\delta$ (5.1)

The differential equation for RC circuit is

$$R\frac{dq}{dt} + \frac{q}{c} = 0 \quad (5.2)$$

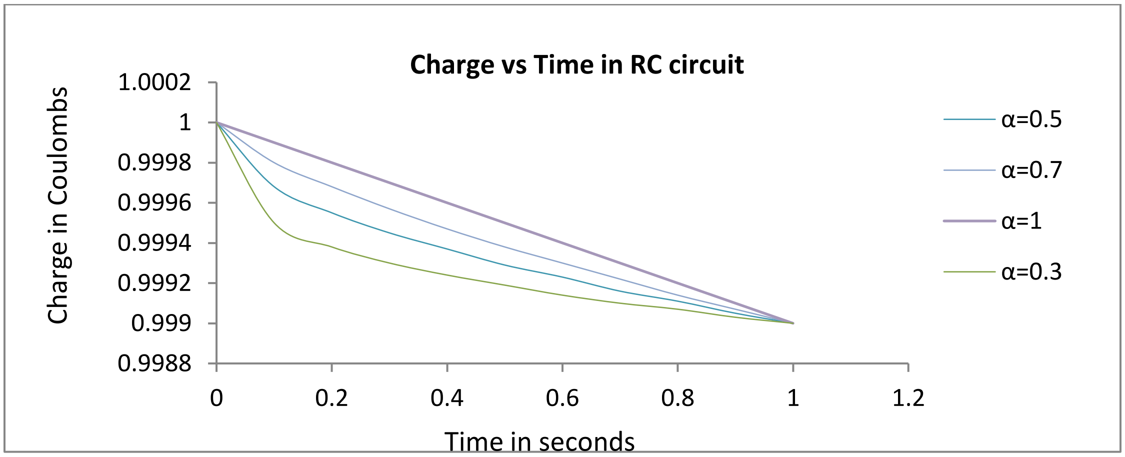
with the initial condition $q(0) = 1$

$\frac{d^{\propto}q}{dt^{\propto}} + \frac{q}{RC} = 0$ (5.3)

Applying Laplace transform defined in Lemma 1 to Equation (5.3)

$q(t) = E_{\alpha}(-at^{\alpha})$(5.4)

where a = 1/RC



Graph-1: **Charge *vs* Time in RC circuit.**

LC circuit :

The ordinary differential equation for LC circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{c} = 0 \qquad (6.1)$$

with initial conditions q(0) = 1 and i(0) = 2

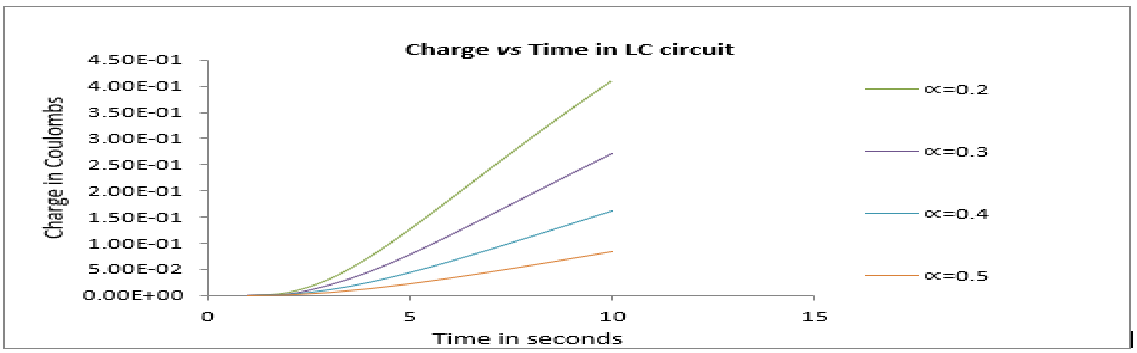
The fractional form of above differential equation is

$$\frac{d^{2\alpha}q}{dt^{2\alpha}} + \frac{q}{LC} = 0 \qquad (6.2)$$

Applying Laplace transform defined in Lemma 1 to Equation (6.2)

$$q(t) = E_\alpha(-at^{a\alpha}) + 2\,E_\alpha(-at^{a(\alpha-1)}) (6.3)$$

where $a = \frac{1}{LC}$



Graph-2 :Charge *vs* Time in LR circuit

LR circuit:

The ordinary differential equation for LR circuit is

$$L\frac{di}{dt} + Ri = 0 \qquad (7.1)$$

with initial conditioni(0) = 1

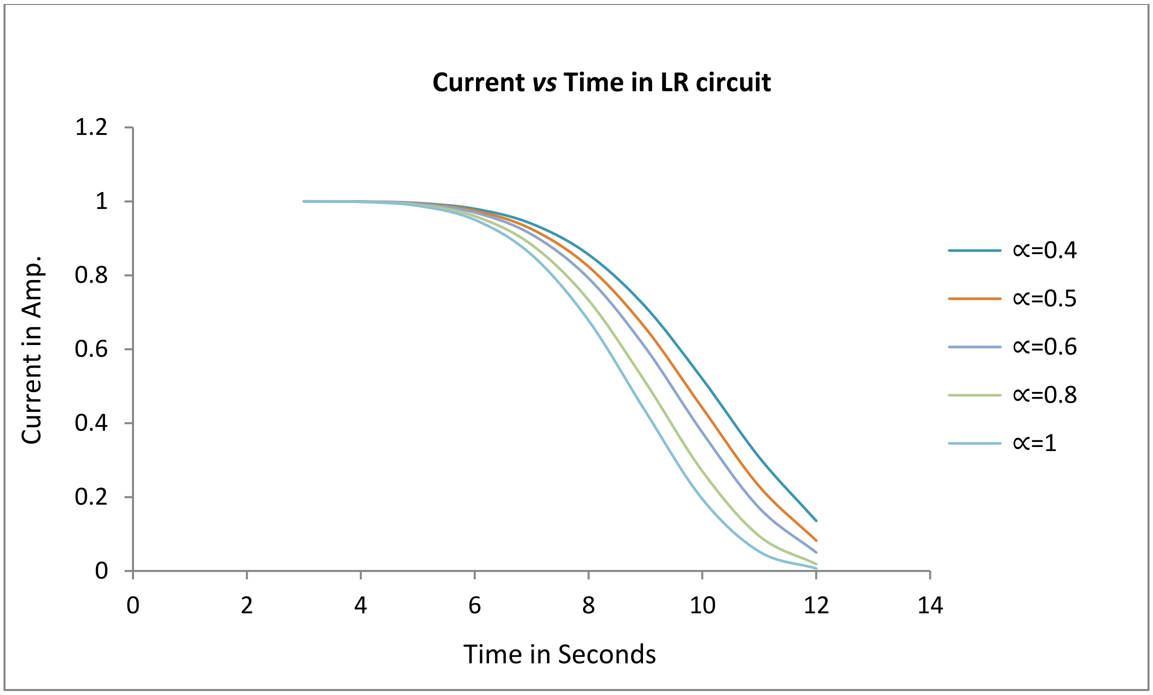
The fractional form of above differential equation is

$$\frac{d^\alpha i}{dt^\alpha} + \frac{R}{L}i = 0 \qquad (7.2)$$

Applying Laplace transform defined in Lemma 1 to Equation (7.2)

$$i(t) = E_\alpha(-At^{A\alpha}) \qquad (7.3)$$

where $A = \frac{R}{L}$



Graph-3: Current *vs* Time in LR circuit

Conclusion:

In RC circuit as charge is dependent on the time and as time is increasing, charge is decreasing shown in graph-1.As time increases the magnitude of the current decreases since potential difference across the resistor, which is the negative of the capacitor voltage decreases by loop rule.

Also fractional order of fractional differential equation gives the value of charge at an arbitrary value of alpha which is power of time.

It seems to be assumed from condition (7.3) that when R is large, current in the L-R circuit will decrease quickly as shown in diagram 3 and there is a possibility of sparkler formation.To stay away from the present situation; L is kept sufficiently enormous to produce enormous L/R with the objective of gradually decreasing the current. For enormous time consistent the rot is moderate and for modest steady the rot is quick.

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