Exponential stability and Exponential synchronization results for fractional order impulsive Neural Networks

Devija Shaji [a], Megha S [b]

[a,b1]Department of Mathematics, Amrita School of Arts and Science,Kochi, Kerala-682024, INDIA:
[a1]devijashaji@gmail.com
[b1]meghas174@gmail.com

Article History: Received: 24 January 2021; Revised: 25 February 2021; Accepted: 28 March 2021; Published: 4 June 2021

Abstract: In this paper we focus on exponential stability analysis of Caputo fractional order impulsive neural network using convex Lyapunov function and obtain the suitable LMI conditions. We have also included exponential synchronization of error system that is derived from the drive-response system. The obtained results are verified using examples.

Keywords: Fractional Order Caputo Derivative, Impulsive Control, Neural Network Lyapunov Function, Exponential Synchronization, Exponential Stability , Linear Matrix Inequality.

1. INTRODUCTION

Fractional calculus is a mathematical field which studies about integrals and derivatives of arbitrary order. It has applications in science and engineering fields like mathematical biology, analytical science, electrochemistry, electromagnetics, physics, economics, fluid mechanics, signal processing, viscoelasticity, image processing, Robotics, mechanic and dynamic systems, telecommunication etc. (Debnath, 2003), (Yang Q. C., 2016), (Manoj Kumar, 2016), (Sun, 2018), (Matlob, 2019).

Mathematical models that are based on fractional calculus has the ability to describe the real-world systems more accurately than models based on integer order calculus. (Wajdi M Ahmad, 2004), (Hammouch, 2015), (Rivero, 2013) outlines some of the major research works that were carried out in the area of stability analysis of fractional order systems. The stability of nonlinear systems can be studied by using Lyapunov direct method, which is an efficient tool to analyze the system stability without solving the system. In (Liu K. a., 2016) and (Weisheng Chen, 2017) nonlinear Caputo type fractional order dynamic systems are looked upon to analyze the stability by using fractional Lyapunov method.

Neural networks are parallel computing devices which are basically an at- tempt to make a computer model of brain. Neural networks have widespread applications in different fields such as cybersecurity, optimization problems, system identification & control, signal and image processing, data mining, pattern recognition etc. These broad areas of applications make it an active research area.

Over the past few years, some researchers incorporated fractional calculus to neural networks to frame fractional order neural network models. Proper- ties of fractional calculus like long-term memory, nonlocality, weak singularity characteristics and its potential to depict the memory and hereditary properties of the neural network enables fractional order neural models describe numerous phenomena more accurately. A great deal of literatures on exponential stability and synchronization of neural networks are on integer order networks than fractional order. A dynamic analysis of fractional-order neural networks is given in (Chen, 2013).

In majority of systems impulsive effects are common phenomenon due to instantaneous perturbations at certain moments. Impulsive control is used for stabilization and synchronization of systems that cannot be controlled usingcontinuous control. Lyapunov stability of impulsive fractional-order nonlinear systems is investigated in (Song X. Y., 2017) . Methods such as active control (Khan, 2018), global synchronization , adaptive control (Jajarmi, 2017) , linear and nonlinear control etc. are used forsynchronization. The impulsive synchronization is explored on fractional-order neural works in (Yang, 2018) and on fractional-order discrete-time chaotic systems (i.e., systems that are sensitive to initial conditions), in (Megherbi, 2017). Exponential synchronization of chaotic system along with its application in the area of secure communication is examined in (Naderi, 2016). Exponential synchronization is being employed in domains such as associative memory, image encryption and combinational optimization also.

LMI Conditions are formulated for global stability of fractional order neural networks in (Shuo Zhang, 2017) and a generalized projective synchronization method is also drawn from it. In (Stamova, 2014) global Mittag-Leffler stability of an impulsive Caputo fractional-order cellular neural networks with time-varying delays is studied by applying fractional Lyapunov method. The synchronization of fractional chaotic networks by employing non-impulsive linear controller was also considered. (Wu, 2016) has investigated global Mittag-Leffler stability for fractional-order Hopfield neural networks with impulse effects in terms of LMIs . Global exponential stability of complex-valued neural networks is analyzed in (Song Q. e., 2016) via Lyapunov functional method by adopting matrix inequality method. Fixed-time Synchronization of Neural Networks

with discrete delay is studied in (Liu S. C., 2020).
Motivated by above discussion , in this paper we consider a fractional order impulsive Neural Networks and analyze its exponential stability using Lyapunov function .Further the exponential synchronization of this system is also discussed.
This paper is structured as follows: Section 2 states some definitions and Lemmas that are fundamental for our research. A model for fractional order impulsive Neural Networks is also proposed . In section 3 conditions under which system achieves exponential stability and synchronization are discussed. Section 4 consists of examples.

Notations: In this manuscript $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^n$ denotes n- dimensional Euclidean space $\mathbb{R}^{n\,x\,n}$ denotes the set of real $n \times n$ matrices, $||.||$ denotes Euclidean norm.

2.PRELIMINARIES

Definition 2.1. The Caputo fractional order derivative of order $\alpha \in \mathbb{R}^+$ on the half axis R^+ is defined as follows

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

for $t > t_0$ with $n = \min\{k \epsilon N \mid k > \alpha > 0\}$, where $f^{(n)}(t)$ is the n-order derivative of $f(t)$, and $\Gamma(.)$ is the Gamma function.

Definition 2.2. . The Reimann-Liouville fractional derivative of order α of function $f(t)$ is defined as

$$ {}_{t_0}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

where $n-1 \le \alpha < n, n \epsilon \mathbb{Z}^+$, $\Gamma(.)$ denotes the Gamma function.

Definition 2.3. A function f defined on $D \subseteq \mathbb{R}^+$ is said to satisfy the Lipschitz condition if there is a constant L such that

$$ \|f(y) - f(y_n)\| \le L\|y - y_n\| \quad \forall\, y, y_n \,\epsilon\, D $$

Model description
Let us consider the Caputo fractional order impulsive neural networks of the following form:

$$ {}_{t_0}^{C}D_t^{\alpha}\boldsymbol{y}(t) = -A\boldsymbol{y}(t) + B\boldsymbol{f}\big(\boldsymbol{y}(t)\big) + I\,;\, t \ne t_k $$

$$ \mathbf{y}(t_k^+) = B_k\, \mathbf{y}(t_k^-)\,;\, t = t_k $$

$$ \mathbf{y}(t_0) = \mathbf{y}_0, \qquad (1) $$

where: $\alpha \epsilon (0,1), \boldsymbol{y}(t) = (y_1(t), y_2(t), \dots, y_n(t)^T \epsilon\, \mathbb{R}^n, A = dia(a_1, a_2, \dots, a_n)$ and $B = (b_{ij})_{n \times n}$. For $i, j = 1,2, \dots, n, y_i(t)$ is the state of the i^{th} neuron , $f_i(y_i(t))$ is the activation function of the i^{th} neuron, $a_i > 0$ is the charging rate for the i^{th} neuron. $I = (I_1, I_2, \dots, I_n)^T$, a constant vector, is the external input. $B_k \,\epsilon\, \mathbb{R}^{n \times n}$ is impulsive gain matrix, $t_1 < t_2 < t_3 \dots < t_k$ with $\lim_{k \to +\infty} t_k = +\infty$. Assume that $\boldsymbol{y}(t)$ is right continuous at $t = t_k$ and $\boldsymbol{y}(t_k) = \boldsymbol{y}(t_k^+)$.

Assumption 1. The function f is continuous on $\mathbb{R}$ and satisfy the Lipschitz condition in $\mathbb{R}$, there exist a linear matrix $L = dia\{l_1, l_2, \dots, l_n\} > 0$, such that:

$$ \|f(x) - f(y)\|_2 \le L\,\|x - y\|_2 \quad \forall\, y, x \epsilon\, \mathbb{R}^n $$

Lemma 2.1. For the given vectors $\boldsymbol{y}, \boldsymbol{x} \,\epsilon\, \mathbb{R}^n$ and any positive constant $\varepsilon > 0$, the following inequality holds

$$ 2\boldsymbol{y}^T\boldsymbol{x} \le \varepsilon\, \boldsymbol{y}^T\boldsymbol{y} + \varepsilon^{-1}\boldsymbol{x}^T\boldsymbol{x} $$

Lemma 2.2. Let $\Omega \epsilon \mathbb{R}^n$. If $V\big(y(t)\big): \Omega \to \mathbb{R}$ and $y(t): [0, \infty) \to \Omega$ are two continuous and differentiable functions and $V(y(t))$ is convex over Ω, then

$$ {}_{t_0}^{C}D_t^{\alpha}V(y(t)) \leq \left(\frac{\partial V}{\partial y}\right)^T {}_{t_0}^{C}D_t^{\alpha}\big(y(t)\big), \forall\ \alpha \in (0,1), \forall\ t \geq 0 $$

Specially, for any $P > 0$ when $V\big(y(t)\big) = y^T(t)Py(t)$, then the following well known holds:

$$ {}_{t_0}^{C}D_t^{\alpha}(y^{\boldsymbol{T}}(t)Py(t) \leq 2y^T(t)P\ {}_{t_0}^{C}(y(t)) $$

Lemma 2.3. For all $\alpha \in \mathbb{R}$ and a real valued continuous function $G(t)$ on $[a, \infty)$, if there exist a constant θ such that

$$ {}_{t_0}^{C}D_t^{\alpha}G(t)) \leq \theta G(t);\ \alpha \in (0,1] $$

Then

$$ G(t) \leq G(a)e^{\int_a^t \frac{\theta}{\Gamma\alpha}(t-\tau)^{\alpha-1}}d\tau $$
$$ = G(a)e^{\frac{\theta}{\Gamma(\alpha+1)}(t-a)^{\alpha}} $$

3.MAIN RESULTS

In this section we examine exponential stability and synchronization results for fractional order impulsive Neural Networks via convex Lyapunov function.

3.1 Exponential Stability

Theorem 3.1. *Let P be a positive definition matrix. If there exist constants $\gamma, \mu > 0$ and $\zeta^k > 1$ such that the following conditions*

(i) $-PA - A^TP + \varepsilon PBB^TP + \varepsilon^{-1}L^2 \leq -\gamma P$

(ii) $B_k^TPB_k < e^{-\mu}P$

(iii) $\zeta^k e^{-\mu k} e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t_k - t_{k-1})^{\alpha}} < 1$

are satisfied. Then the system (1) is exponentially stable.

Proof. Let us assume the solution of equation (1) is piece-wise right continuous function. Consider the convex Lyapunov function:

$$ W(t) = \boldsymbol{y}^T(t)P\boldsymbol{y}(t), $$

Taking Caputo derivative and using Lemma 2.2 when $t\epsilon(t_k, t_{k+1})$ for $k\epsilon\mathbb{Z}^+$

$$ {}_{t_0}^{C}D_t^{\alpha}W(t) \leq 2\boldsymbol{y}^T(t)P\{{}_{t_0}^{C}D_t^{\alpha}(\boldsymbol{y}(t))\} $$
$$ = 2\boldsymbol{y}^T(t)P\big[-A\boldsymbol{y}(t) + Bf\big(\boldsymbol{y}(t)\big)\big] $$
$$ = -2\boldsymbol{y}^{\boldsymbol{T}}(t)PA\boldsymbol{y}(t) + 2\boldsymbol{y}^T(t)PBf\big(\boldsymbol{y}(t)\big) $$
$$ \leq -\boldsymbol{y}^T(t)PA\boldsymbol{y}(t) - \boldsymbol{y}^T(t)A^TP\boldsymbol{y}(t) + 2\boldsymbol{y}^T(t)PBB^Tf\big(\boldsymbol{y}(t)\big) $$

By Lemma 2.1

$$ {}_{t_0}^{C}D_t^{\alpha}W(t) \leq -\boldsymbol{y}^T(t)PA\boldsymbol{y}(t) - \boldsymbol{y}^TA^TP\boldsymbol{y}(t) + \varepsilon\boldsymbol{y}^T(t)PBB^T\boldsymbol{y}(t) + \varepsilon^{-1}f^T\big(\boldsymbol{y}(t)\big)f(\boldsymbol{y}(t)) $$

By assumption (1), if follows that

$$ {}_{t_0}^{C}D_t^{\alpha}W(t) \leq -\boldsymbol{y}^T(t)PA\boldsymbol{y}(t) - \boldsymbol{y}^T(t)A^TP\boldsymbol{y}(t) + \varepsilon\boldsymbol{y}^T(t)PBB^TP\boldsymbol{y}(t) + \varepsilon^{-1}L\boldsymbol{y}^T(t)L\boldsymbol{y}(t) $$
$$ = -\boldsymbol{y}^T(t)PA\boldsymbol{y}(t) - \boldsymbol{y}^T(t)A^TP\boldsymbol{y}(t) + \varepsilon\boldsymbol{y}^T(t)PBB^TP\boldsymbol{y}(t) + \varepsilon^{-1}L^2\boldsymbol{y}(t) $$
$$ = \boldsymbol{y}^T(t)[-PA - A^TP + \varepsilon PBB^TP + \varepsilon^{-1}L^2]\boldsymbol{y}(t) $$

Let $-PA - A^TP + \varepsilon PBB^TP + \varepsilon^{-1}L^2 \leq -\gamma P$

Thus,

$$ {}_{t_0}^{C}D_t^{\alpha}W(t) \leq \boldsymbol{y}^T(t)(-\gamma P)\boldsymbol{y}(t) $$
$$ \leq -\gamma\boldsymbol{y}^TP\boldsymbol{y}(t) $$

That is $${}_{t_0}^{C}D_t^{\alpha}W(t) \leq -\gamma W(t) \text{ for } t \neq t_k.$$

When $t = t_k$, it follows from second equation of (1) that

$$\begin{aligned} W(t_k) &= \boldsymbol{y}^T(t_k)P\boldsymbol{y}(t_k) \\ &= (B_k\boldsymbol{y}(t_k^-))^T P(B_k\boldsymbol{y}(t_k^-)) \\ &= \boldsymbol{y}^T(t_k^-)(B_k)^T P(B_k)\boldsymbol{y}(t_k^-), \end{aligned}$$

Now take $B_k^T P B_k < e^{-\mu}P$ Then

$$W(t_k) \leq \boldsymbol{y}^T(t_k^-)e^{-\mu}P\boldsymbol{y}(t_k^-)$$

Thus $W(t_k) \leq e^{-\mu}W(t_k^-)$

By using Lemma 2.3, we can write for any $t\epsilon(t_0, t_1)$

$$W(t) \leq W(t_0)e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}}$$

Similarly, for any $t\epsilon(t_1, t_2)$

$$W(t) \leq W(t_0)e^{-\mu}e^{\frac{-\gamma}{\Gamma(\alpha+1)}[(t-t_1)^{\alpha}+(t_1-t_0)^{\alpha}]}$$

Similarly, for any $t\ \epsilon(t_k, t_{k+1})$

$$W(t) \leq W(t_0)\prod_{i=1}^{k} e^{-\mu_i}e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t_i-t_{i-1})^{\alpha}} \times e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}}$$

From condition *(iii)* , we get

$$W(t) \leq W(t_0)\frac{1}{\zeta^k}e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_k)^{\alpha}}$$

Where $\frac{1}{\zeta^k} \to 0$ as $k \to \infty$, then $W(t) \leq 0$. Then system (1) is exponentially stable.

3.2 Exponential Synchronization

Consider the drive system:

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\alpha}\,\boldsymbol{y}(t) = -A\,\boldsymbol{y}(t) + B\boldsymbol{f}(\boldsymbol{y}(t))\ ;t \neq t_k, t \geq t_0 \\ &\mathbf{y}(t_k^+) = C\,\mathbf{y}(t_k^-)\ ; t = t_k\ , k\epsilon\mathbb{Z}^+, \\ &\boldsymbol{y}(t_0) = y_0 \end{aligned} \quad (4)$$

The corresponding Response system can be described as follows :

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\alpha}\boldsymbol{z}(t) = -A\,\boldsymbol{z}(t) + B\boldsymbol{f}(\boldsymbol{z}(t))\ ;t \neq t_k, t \geq t_0 \\ &\boldsymbol{z}(t_k^+) = C\,\boldsymbol{z}(t_k^-)\ ; t = t_k\ , k\epsilon\mathbb{Z}^+ \\ &\boldsymbol{z}(t_0) = \boldsymbol{z_0} \end{aligned} \quad (5)$$

Define error variable as $\boldsymbol{\theta}(t) = \boldsymbol{z}(t) - \boldsymbol{y}(t)$. Then we obtain error system from (5)-(4) , it is defined as :

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\alpha}\boldsymbol{\theta}(t) = -A\,\boldsymbol{\theta}(t) + B\boldsymbol{f}(\boldsymbol{\theta}(t))\ ;t \neq t_k, t \geq t_0 \\ &\boldsymbol{\theta}(t_k^+) = C\,\boldsymbol{\theta}(t_k^-)\ ; t = t_k\ , k\epsilon\mathbb{Z}^+ \\ &\boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0 \end{aligned} \quad (6)$$

where $\boldsymbol{f}(\boldsymbol{\theta}(t)) = \boldsymbol{f}((\boldsymbol{z}(t) + \boldsymbol{y}(t)) - \boldsymbol{f}(\boldsymbol{y}(t))$

Theorem 3.2. *Let P be a positive definition matrix. If there exist constants* $\gamma, \mu > 0$ *and* $\zeta^k > 1$ *such that the following conditions*

$(iv) -PA - A^TP + \varepsilon PBB^TP + \varepsilon^{-1}L^2 \leq -\gamma P$
$(v)\ B_k^T P B_k < e^{-\mu}P$
$(vi) \zeta^k e^{-\mu k} e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t_k - t_{k-1})^\alpha} < 1$

are satisfied. Then the system (6) is exponentially synchronized

Proof. The proof is similar to Theorem 3.1, so we omit it.

4.EXAMPLES

In this section, we give two examples to verify the effectiveness of exponential stability and exponential synchronization results that we obtained.

Example 1: *In system (1) consider the following impulsive neural network with α=0.98,* $, \boldsymbol{y}(t) = (y_1, y_2, y_3)^T$ $\boldsymbol{f}(\boldsymbol{y}) = (tanh(y1), tanh(y2), tanh(y3))^T, I = (0,0,0)^T, \boldsymbol{A} = diag(1,1,1),$ $\boldsymbol{B} = \begin{pmatrix} 2 & -1.2 & 0 \\ 1.8 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}$ $\boldsymbol{B_k} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}$.

Under parameters ε =0.1,γ=0.1, μ_k =0.1 and Lipschitz constant $L = diag(0.1,0.1,0.1)$, with P = $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the LMI conditions of Theorem (3.1) are satisfied. Thus, by Theorem (3.1) this neural network is exponentially stable.

Example 2 : *Consider the Drive system:*

$${}_{t_0}^{C}D_t^\alpha\, \boldsymbol{y}(t) = -A\, \boldsymbol{y}(t) + B\boldsymbol{f}(\boldsymbol{y}(t))\ ; t \neq t_k, t \geq t_0$$
$$\boldsymbol{y}(t_k^+) = C\ \boldsymbol{y}(t_k^-)\ ; t = t_k\ , k \epsilon \mathbb{Z}^+$$
$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0\ ; t = t_0$$

The corresponding Response system can be described as :

$${}_{t_0}^{C}D_t^\alpha \boldsymbol{z}(t) = -A\boldsymbol{z}(t) + Bf(\boldsymbol{z}(t))\ ; t \neq t_k, t \geq t_0$$
$$\boldsymbol{z}(t_k^+) = C\,\boldsymbol{z}(t_k^-)\ ; t = t_k\ , k \epsilon \mathbb{Z}^+$$
$$\boldsymbol{z}(t_0) = \boldsymbol{z}_0\ ; t = t_0$$

Where α=0.98 $, \boldsymbol{y}(t) = (y_1, y_2, y_3)^T$, $\boldsymbol{f}(\boldsymbol{y}) = (tanh(y_1), tanh(y_2), tanh(y_3))^T$ $, \boldsymbol{z}(t) = (z_1, z_2, z_3)^T$, $\boldsymbol{f}(\boldsymbol{z}) = (tanh(z_1), tanh(z_2), tanh(z_3))^T$, $\boldsymbol{A}$*=diag(1,1,1),* $\boldsymbol{B} = \begin{pmatrix} 2 & -1.2 & 0 \\ 1.8 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}$ $\boldsymbol{C} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}$.

Under parameters $\varepsilon = 0.1, \gamma = 0.1$, $\mu k = 0.1$ and Lipschitz constant $L = diag(0.1,0.1,0.1)$, with $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the LMI conditions of Theorem (3.2) are satisfied. Thus, by Theorem (3.2) the given system achieves exponential synchronization.

5.CONCLUSION

In this paper we have considered a Caputo fractional order impulsive Neural Networks. By using convex Lyapunov function, the exponential stability conditions for the fractional order impulsive neural networks are derived and the results are formulated in terms of linear matrix inequalities (LMIs).

References

[1] Chen, L. e. (2013). Dynamic analysis of a class of fractional-order neural networks with delay. *Neurocomputing 111*, 190-194.

[2] Debnath, L. (2003). Recent applications of fractional calculus to science and engineering. *International Journal of Mathematics and Mathematical Sciences*, 30. Retrieved from https://doi.org/10.1155/S0161171203301486

[3] Hammouch, Z. a. (2015). Control of a new chaotic fractional- order system using Mittag-Leffler stability. *Nonlinear Studies 22.4*, 565-577.

[4] Jajarmi, A. M. (2017). New aspects of the adaptive synchronization and hyperchaos suppression of a financial model. *Chaos, Solitons & Fractals 99* , 285-296.

[5] Khan, A. K. (2018). Synchronization of a new fractional order chaotic system. *Int. J. Dynamic Control6*, 1585-1591. doi: https://doi.org/10.1007/s40435-017-0389-4

[6] Liu, K. a. (2016). Stability of nonlinear Caputo fractional differential equations. *Applied Mathematical Modelling 40.5-6*, 3919-3924.

[7] Liu, S. C. (2020). Fixed-Time Synchronization of Neural Networks with Discrete Delay. *Mathematical Problems in Engineering*.

[8] Manoj Kumar, A. S. (2016). RECENT ADVANCEMENT IN FRACTIONAL CALCULUS. *International Journal of Advanced Technology in Engineering and Sciences 4(4)*, 177-186.

[9] Matlob, M. A. (2019). The concepts and ap- plications of fractional order differential calculus in modeling of viscoelastic systems: a primer. *Critical Reviews™ in Biomedical Engineering 47.4*.

[10] Megherbi, O. (2017). A new contribution for the impulsive synchronizaton of fractional-order discrete-time chaotic system. *Nonlinear Dynamics 90.3*, 1519-1533.

[11] Naderi, B. ,. (2016). Exponential synchronization of chaotic system and application in secure communication. *Optik 127.5*, 2407-2412.

[12] Rivero, M. ,. (2013). Stability of fractional order systems. *Mathematical Problems in Engineering*.

[13] Shuo Zhang, Y. Y. (2017, october). LMI Conditions for Global Stability of Fractional-Order Neural Net-works,. *IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS, 28*, 2423-2433.

[14] Song, Q. e. (2016). Global exponential stability of complex-valued neural networks with both time-varying delays and impulsive effects . *Neural Networks 79*, 108-116.

[15] Song, X. Y. (2017). Mittag Leffler stability analysis of nonlinear fractional-order systems with impulses. *Applied Mathematics and Computation, 293*, 416-422. doi: https://doi.org/10.1016/j.amc.20

[16] Stamova, I. (2014). Global Mittag-Leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays. *Nonlinear Dynamics 77.4*, 1251-1260.

[17] Sun, H. ,. (2018). A new collection of real-world applications of fractional calculus in science and engineering . *Communications in Nonlinear Science and Numerical Simulation 64*, 213-231.

[18] Wajdi M Ahmad, R. E.-K.-A. (2004). Stabilization of generalized fractional order chaotic systems using state feedback control, Chaos. *Solitons & Fractals, 22*(1), 141-150. doi: https://doi.org/10

[19] Weisheng Chen, H. D. (2017). Convex Lyapunov functions for stability analysis of fractional order systems. *,IET Control Theory Appl, 7*(7), 1070-1074.

[20] Wu, H. (2016). LMI conditions to global Mittag-Leffler stability of fractional-order neural networks with impulses. *Neurocomputing 193* , 148-154.

[21] Yang, Q, Chen, D., Zhao, T., & Chen, Y. (2016). Fractional calculus in image processing: a review. *Fractional Calculus and Applied Analysis,*, 1222-1249.

[22] Yang, X. (2018). Global Mittag-Leffler synchronization of fractional- order neural networks via impulsive control. *Neural Processing Letters 48.1*, 459-479.