Algebra at the Meta and the Object Level

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Abstract: Two different interpretations of algebra that differ in the ontological status assigned to variables are distinguished. Variables may either be viewed as meta-mathematical tools to express generality or as objects similar to numbers and other members of the mathematical ontology. Both interpretations are detailed and linked with the literature and the use of variables in computer programming. Furthermore, it is analyzed how these two conceptualizations lead to two different understandings of the process of change of values. Some evidence from algebra assessment on the understanding of change by students is given that that illustrate that the theory is useful in analyzing students work.

Keywords: Algebra, variables, logic, change

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1. Introduction

Variables in Algebra may be conceptualized and used in different ways (Küchemann 1978, Usiskin 1988). In this paper we concentrate on the ontological status of variables and distinguish two possible conceptualizations that define variables either at the object or at the meta level. In the first case, variables are mathematical objects just like numbers or sets. In the second case they are part of the meta language that we use to talk about the mathematical objects, but they are, e.g. never elements of a set. This distinction is not only of academic interest. Different programming languages can be categorized into using one or the other conception of variable. Moreover, understanding the concept of change is influenced by the conception of variable used.

This paper is mainly theoretical in nature: We expose a theory that describes the two conceptions MLA (meta level algebra) and OLA (object level algebra). will be defined. As the topic is quite unusual, the reader is first presented our view, and only afterwards a broad literature review is made. This has the advantage that the reader can judge our conceptions against ideas from the cited literature. Finally, we present some empirical observations that can be interpreted in the theoretical framework.

2. Variables

Variables are not only used in algebra but exist in natural language as well (hence it is not too surprising that language and algebraic understanding correlate as found in MacGregor and Price (1999). Whenever we speak about generalities we use linguistic tools to refer to all objects of a certain set or to a specific unknown of these objects. E.g. in rules for games one often sys “The player who…has to…”. Here ‘player’ is a linguistic variable that can refer to any of the actual players. This example shows that variables may

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almost be invisible. Variables here are linguistic tools. In algebra we may either take over this framework or we may alternatively consider variables to be objects under discourse.

A more detailed look at variables as conceptualized in the literature will follow in a later section when the two kinds of algebra have been introduced.

3. Variables at the Object- and Metal-language level

In the mathematical theory of arithmetic, the basic set of objects is a set of numbers, usually the set of integer. True sentences of arithmetic are 2+3=5, 99-1=100-2, 5|45 or 1<2. Although the objects of arithmetic are only numbers, any typical book on arithmetic is full of variables used to talk about numbers and assuming that they stand for numbers (e.g. stating that $x+y=y+x$). They come into play, when we wish to make general statements, e.g. when we want to be able to describe the whole theory by few general statements (axioms) from which all true statements can be deduced, or when we want to express general (deducible) facts, such as $a|b \wedge b|c \rightarrow a|c$ (and quantifiers may be used to fix the semantics of this). These variables are not members of the set (the numbers) under discourse but are part of the meta language we use to talk about numbers and arithmetic (see (Li 2010) for the distinction of meta and object language), they are just a linguistic trick for the sake of abstraction. The algebra of these variables is then the theory of logical statements about the domain of numbers. We refer to this view of algebra as the meta-level algebra (MLA).

This is the standard point of view taken in most mathematical works. It is based on mathematical logic where semantics is defined using interpretations (cf. Tourlakis 2003, p. 53; Mendelson 1997, p. 48). An interpretation of a set of formulae of predicate calculus is given by a set S (the domain of the interpretation) and an assignment that gives an element of S for every occurrence of an unbound variable in the formula, and functions and predicates over S for every function and predicate symbol in the formulae. After applying all these assignments, the formulae reduce to statements in the domain with no unbound variables remaining. A model is an interpretation that makes all formulae true. If a set of formulae is true for (respectively) every, some or none interpretations then it is called true (tautology), satisfyable, contradictory. Variables here again are meta-linguistic concepts and not part of the domain of e.g. integer numbers. This logical framework can be applied to any domain S, not just to sets of numbers, but this may blur the distinction to be made. So we'll stick to S being a set of numbers for now and consider consequences of more general domains later on.

From a didactical perspective this view correlates with the approach to algebra as generalized arithmetic (cf. Bednorz, Kieran & Lee, 1996). In this approach students typically investigate number sequences (defined e.g. by geometric configurations like figurate numbers). Variables then have the role to enable expressing the general form of a sequence of numbers.

This completes the discussion of MLA. The other possible point of view is to consider variables as objects in the universe under discourse. This means that the variable itself is an object in the domain of the theory and can be manipulated by processes in the theory. Then, for example, it is possible to refer with one variable to another variable (rather than to a number. Note according to the theory of ontological commitment one can reinterpret the predicate 'exists' as 'is possible as the value of a variable' – in this sense this position gives variables the status of ontological objects.). This point of view shall be called the object level algebra (OLA).

In OLA the meaning of operational signs like + has then to be extended to take as operands not only numbers but also variables and results from operations, i.e. one is automatically led to consider the set of all expressions that can be build up from numbers and variables. The operators thus build up structures according to the well-known syntax of arithmetic, i.e. if e.g. x,y are objects than $x+y$ is necessarily on object as well, namely an expression. The semantic of this extension is suggested but not fixed by the underlying arithmetic. This results in an important distinction between OLA and MLA when the justification of transformation laws is considered. In MLA $x \cdot y = y \cdot x$ is merely the same statement as "multiplication is commutative for all objects from the domain, usually numbers", especially $x \cdot y$ is just a number. In OLA, $x \cdot y$ is a compound object with – besides its structure – undefined properties. It can be manipulated in many ways. It has no semantics determined by some domain –one may define arbitrary rules for its manipulation– although one might wish to impose rules such that one is consistent when replacing variables by numbers from some domain, but one needs not. One has the freedom to impose e.g. $x \cdot y = -y \cdot x$. Thus, the view of OLA opens up the view e.g. for non-commutative objects as used in group presentations, e.g. $qp=pq+h$ as imposed in the operator structures of quantum mechanics.

This, however does not mean, that MLA cannot deal with non-commutative objects or with letters as objects of calculations. This is merely the question of the domain considered in MLA and one may define non-commutative operations for this domain or include letters in this domain. This leads to the distinction between letters in the domain of the theory and meta-linguistic symbols that are used in formulas about this domain. This is also discussed very clearly in (Freudenthal 1973, p.338). So, in a sense, every OLA activity can be modeled in MLA by extending the domain to include the variable and expression objects one would like to have. However, the setup of MLA makes it necessary to even then distinguish between meta level variables and object level variables. We will explain this with an example.

The case of polynomials $Z[x]$shall make this situation a bit more transparent. In OLA x is an object in its own right and + has to operate on it. With the assumption that an object x (of type variable) exists one must assume that even more objects exists such as x+1. Among these expressions certain of a special form can be identified and called polynomials. In standard mathematics one usually assumes that the symbol x in $Z[x]$is not meant to represent a number (and care is taken about this by explicitly considering the insertion morphism $\phi_a: Z[x] \to Z, x \mapsto a$ that replaces x by an element a from Z).

The polynomial ring $Z[x]$can be handled as well in MLA. This means that one constructs (just like number sets are constructed) a domain for the theory. In the case of polynomials this may be e.g. the finite sequences of integers (and thus eliminating the need to use *x* explicitly). Nicaud, Bouhineau and Gélis, (2001) show how to construct further algebraic domains from this point of view.

If MLA is carried out with a domain that contains symbols (variables; such as the set of polynomials) then nevertheless there is a clear distinction between the variables from meta language and the variables that are elements of the domain, e.g. over the domain $Z[x]$one may state (x-1)|p where *x* is an object from the domain (thus $x-1 \in Z[x]$) and *p* is a meta-level variable referring to an element of $Z[x]$. In OLA this distinction is not made and this leads to the following shift necessary in the meaning of the equal sign: The interpretation of "Let in the following *x*=*y*…" in MLA is that either both variables are meta variables, then one simply assumes that in all interpretations to be considered they refer to the same objects, and if one is a meta variable and the other an object variable then this means that in all interpretations the meta variable (say *x*) refers to the object y. In OLA, on the other hand, this means that the objects *x* and *y* are to be identified in the sense that they can be freely exchanged. Thus, the equal sign – in this use – does not mean identity of referred objects but declares a certain use of objects.

As made plausible by the above explanations all algebraic problems can be addressed both by MLA and OLA and hence it is merely a question of choosing the best thinking tool for a problem at hand. When a domain has been fixed, however, both modes of algebraic thinking can be clearly distinguished. Allowing a certain lack of rigor and letting the domain considered open we may obscure the distinction. The same is achieved by freely switching domains under consideration. Experts can thus navigate between the views perhaps without even noting that they differ. Learners however, as we suppose, usually stick to one domain (especially if they feel "at home" in it). Thus for them the distinction is sharp.

4. Object- and Meta-language algebra from the perspectives of semiotics and computer science

From a semiotic point of view (Filloy, Rojano & Puig, 2008), variables should be signs (labels) and refer to something and this something can be understood from both viewpoints but with a minor difference. If we take *x* to refer to 5, from the point of view of MLA *x* and 5 are identical, while for OLA they are not. In MLA we may say "*x* is 5" in OLA we may say "the value of *x* is 5". The second formulation is according to our observations preferred by a vast majority of students.

Consider the problem "*x*=5, what is 2+*x*". Interpreted in MLA *x* is 5, so one really can read this as "what is 2+5?". This is also the interpretation taken by most programming languages: Consider a simple small (Basic) program like the following:

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Input a : Input b : c=a+b : Print c
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The variables occurring here are not objects themselves, but they stand for numbers. The compiler takes this and turns it into machine code that reads two numbers and outputs their sum. At runtime, the computer's memory is only occupied by numbers. The variables have only been needed on the meta level of programming to describe the operation to be carried out. In fact, the names a,b,c don't show up in the created executable program (unless they are included as debugging information).

If "x=5, what is 2+x" is to be brought to computers on the OLA one must represent x as an object in the computer's memory and 2+x is then to be represented as a compound data structure. The computer can then distinguish between the symbol x and the value of x and go from one to the other is a process usually called evaluation. This is the point of view that originated in the Lisp programing language and that has been taken over to computer algebra systems (CAS). So from this point of view we can summarize: MLA is the algebra used by ordinary programming languages, OLA is the algebra used by CAS. This has important consequences. The point of view of Frege (Drouhard & Teppo, 2004) that x^2-x and x·(x-1) have the same denotation (but a different sense) is aligned to MLA over $Z[x]$. For OLA as used in computer algebra systems these two expressions denote different objects and there are explicit conversion functions (expand, factor) in most CAS that are used to convert one to the other. Thus the meaning of the equal sign depends on the OLA-MLA distinction and is more complicated in the former form. This point is elaborated in more depth in (Oldenburg, 2015).

5. Literature Review

This section investigates which theories of algebra relate to the MLA-OLA distinction.

The understanding and use of variables has been investigated in many studies. Küchemann (1978) identified six ways in which students use variables (he speaks of letters and reserves variable for the case of actually varying quantities). Three of them are assigned to Piaget's formal operational thinking: letter as a specific unknown, as a generalized number and as a varying quantity. Although the use and the ontological status of variables are not independent, this classification is concerned with a different dimension of variables than the OLA-MLA distinction. Especially, despite the name, the use of letters as objects in Küchemann's classification has nothing to do with OLA.

Usiskin (1988) has classified variable use according to the conception of algebra used. He relates the approach of algebra as generalized arithmetic with the use of variables as patterns generalizers. While it is particularly convenient to view placeholders in the light of MLA one should note that the same can be achieved in OLA by allowing the variables to refer to some other objects.

Euler's classical textbook on algebra may be seen to reason mainly on the level of MLA. He repeatedly states that letters represent numbers (i.e. they are not objects that refer to numbers), e.g. on p.4. he writes "In Algebra, in order to generalize numbers, we represent them by letters, as *a, b, c, d,* & *e*. Thus the expression $a+b$, signifies the sum of two numbers." Interestingly, Euler links such expressions with the occurrence of new mathematical objects when he writes with regard to the fraction $b=c/a$ (Euler 1810, p. 88) "Now, as it frequently happens that the number *c* cannot be really divided by the number *a*, while the letter *b* must however have a determinate value, another new kind of numbers presents itself, which are fractions." We'll come back to this in the section on reification.

An even more radical understanding of MLA is presented by Linchevski (2001). She states (p. 143) "Operating on and with the unknown implies understanding that the letter is a number. It does not only symbolize a number, stand for a number, and it does not only tag/label/sign for an unknown number." It is not clear what problems this interpretation solves that more modest forms of understanding may have. Moreover, it is not held by teacher students. In two university courses for teacher students we asked (questionnaire) taken together114 students to decide, which sentence represents better the way they think. The options were either "A variable is a number" or "A variable stands for a number". 100% opted for the latter version. The first version is incompatible with OLA, the second can be interpreted both in MLA and OLA. So we cannot not deduce what kind of algebra is preferred by students but we can state that the radical MLA version if Linchevski is rejected by students (which, of course, does not mean that it is a false or useless position).

OLA, on the other hand, can also be detected in the history of algebra. Peacock (see the enlightening discussion in (Menghini, 1994) distinguishes between arithmetical algebra and symbolic algebra. (Chiappini, 2011) has characterized them in the following way:

"According to Peacock, Arithmetical Algebra differs from Arithmetic for the use of letters that allow to operate on indeterminate quantities, namely on quantities whose value is not specified. In this algebra, however, the operations are those of Arithmetic, with the same natural limitations that they have in this knowledge domain, so that an expression like a-b has a sense only for $b<a$. With the Symbolical Algebra, the meaning of symbols becomes operational, namely defined according to the operation (and its properties). In Symbolical Algebra, symbols can represent any kind of quantity that is incorporated into them through specific operations."

It is clear that Peacocks arithmetical algebra is close to MLA and his symbolic algebra is close to OLA. However, both pairs of concepts are not identical. Peacock is mainly concerned with the problem of expressions that make sense in the domain only under certain conditions. His domain consists only of positive numbers, thus he states (Peacock 2004, p. viii) that in arithmetical algebra "a-($a+b$) (…) obviously express an impossible operation (..); but if $a+b$ was replaced by a single symbol c, the expression a-c, though equally impossible with a-($a+b$), would cease to express it." This shows an entanglement between syntax and semantics that leads to restrictions on substitutions that are not present in MLA. In modern mathematical language one would simply find out that no model over a certain domain exists.

The resolution to this problem of Peacock may be either to invent new numbers (i.e. negative numbers) or to revise the definition of the operations. Peacock does the latter: “The assumption however of the independent existence of the signs + and – removes this limitation, and renders the performance of the operation denoted by – equally possible in all cases.” (p. ix). Thus these modified operators do not operate on numbers but on symbolic expression and they build up new structures just as we have postulated for OLA. The quote from Euler above has shown that he too links operations and the emergence of new objects. This will be considered in more detail in the next section.

6. Reification

The theory of reification (Sfard, 1991) is useful for the understanding of algebra. We won’t review this well-known theory here. Instead, we point out that two different strands of reification are relevant to our subject. A graphical illustration is given below.

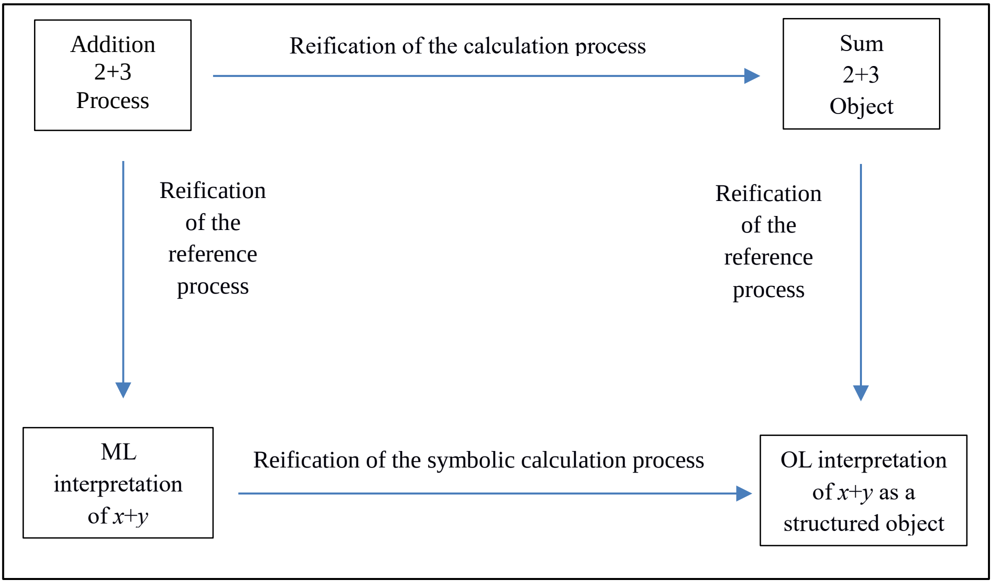

Figure 1. An illustration for reification

A calculation processes may be reified into unevaluated structures (i.e. objects on hold) which are then considered as expressions. This is the horizontal reification direction in the diagram above. On the lower row it means that the linguistic sign (with its associated processes of interpretation and manipulation) gets recognized as an object in the domain of investigation and is operated on according to the rules of formal calculus.

The top-down-direction may be viewed as a reification step as well. The process reified is that of “giving a number”. Names of numbers such as 8, VIII or ‘eight’ directly give a number. Using a letter to denote a known number is a more indirect version of this process. Next, letters can refer to any (possible varying) number of a certain domain and finally this reference process is completely reified and the letter becomes a symbol. The unevaluated operator + in the example and the symbol reified in this sense are the parts of the symbolic structure $x+y$.

7. Learning Trajectories

For Peacock it is clear that symbolic algebra is more advanced than arithmetical algebra and this may indicate a natural learning trajectory. However, neither MLA nor OLA fit exactly Peacocks definitions and the question is not that easily answered.

The approach to algebra as generalized arithmetic starts with MLA and OLA may follow later. Approaching algebra as in the tradition of Davydov, see e.g. Dougherty (2008), may lead directly to OLA with mastering arithmetic and MLA later on.

In this situation we do not attribute higher value to either of these thinking modes but try to understand what consequences it has if one of the modes is dominant.

One should consider the possibility that a preference for either OLA or MLA is linked with the learner’s thinking style according to Schwank (1999). According to this theory predicative thinking is thinking in terms of relations and judgments; functional thinking is thinking in terms of available actions and achievable effects. This it seems to be plausible that relational thinkers should show a preference for MLA while functional thinkers should prefer OLA. This questions have, however, not been considered in the present study.

One may speculate that OLA has a special role in the creation of mathematics. We’ll take up the polynomial example above to explain this point: Assume that the domain of integer numbers has already been successfully constructed and one is using MLA style variables in working with it. When the interest in expressions like x^2+1 shifts from the operations they describe to their structural properties, one may take these expressions as objects of thought, thus arriving on OLA. Later on, when the precise structure of these objects and their operations is clarified, one arrives at the ring of polynomials Z[x] which form then the domain of a new, higher MLA.

As computer algebra systems operate in a way that is best described by OLA the instrumentation process of these systems is likely to shift the users understanding of algebra towards this mode as well. We therefore suspect that the further investigation of computer algebra use in learning may profit from the distinction described here.

8. Change

The description of change in continuous processes is at the core of differential calculus. Discrete changes may even be more important as they may result from discretization of continuous problems and from the description of immanent discrete processes. Nemirowski, Tierney and Ogonowski (1993) have investigated this in detail and found that even young children can gain a good understanding of processes involving change. They didn't however investigate the algebraic formulation of change.

Nie, Cai and Moyer (2009) have compared standard based and traditional curricula and found that the newer standard based approach does give more weight to the concept of variable and change. However, there is no detailed analysis on how the change of the value of a variable is to be understood.

Usiskin (1988) has remarked that understanding the change of values is one of the most difficult aspects of algebra. His example is the question "What happens to the value of 1/x as x gets larger and larger?" Later on we will report on students' responses to similar tasks which are, however, not functional but relational in nature.

Success in calculus teaching is often limited according to the findings of (White & Mitchelmore 1996) by a symbolic centered view which prevents the students to see variables as related, changing quantities. White and Mitchelmore (1996) conclude "Detailed analysis revealed three main categories of error, in all of which variables are treated as symbols to be manipulated rather than as quantities to be related."

Quine (1960) has clearly described how change should be understood in mathematical logic, i.e. under our view of MLA: "As x increases, we are told, $2/x$ decreases. Since numbers never increase or decrease, such talk of variables must be taken metaphorically. The meaning of this example is of course simply the general statement that if $x>y$ then $\frac{2}{x} < \frac{2}{y}$."

The distinction between a variable (which is an object in its own right) and the (current, if any) value of this variable is fundamental to understand how algebra deals with change. When we say that x increases, in OLA the variable (as an object) is unchanged but its current value is changed. In MLA a precise description is far more difficult: Either one doubles every variable (i.e. instead of x one considers x_1 and x_2 to refer to two values of the same quantity at different stages of the change process) or one investigates at least two interpretations (in the technical sense given above) or – for continuous change – families of interpretations that assign different values to the variable.

In the established versions of OLA such as those in CAS the current value of a variable is not a part of the variable itself. The current value of say x is contained in a separate table of substitutions (frame of bindings or environment in computer language).

The difference between OLA an MLA views may be made clearer by defining two possible understandings of a statement like "x increases by 1":

- Internal Change: The symbol x is unchanged but it refers to another number which is one more than it referred to before.

- External change: x itself is not changed. The change is an operation denoted exterior to x: x is replaced by $x+1$.

Internal change can only be understood from an OLA point of view and external change is the natural way from the MLA point of view but can also be understood from OLA. But note that there is no wrong or right interpretation as there are situations that show advantages of both versions:

- The derivative analyses how $f(x)$ changes when one moves away from x by h, i.e. it considers the difference $f(x+h)-f(x)$ as the numerator of the difference quotient. This shows an external representation of a change of x.

- In physics all kinds of measurable quantities are denoted by variables, e.g. current I, voltage U, resistance R and relations between them like $U=RI$ are considered. Physical laws stay the same over time even while the quantities change. Typical mathematical statements in physics thus reflect an OLA understanding. To reformulate this in MLA one would have to treat all physical quantities as functions (of time, and maybe position in space). This is certainly formally a perfect approach. For the learner, however, it has the disadvantage that functions are the most common object to deal with and operations on these functions are performed. Even in math courses the internal view is applied when graphing a function like $y=x^2$. Here one changes x to see how y behaves and this change is considered internal.

Note that the difficulties in understanding change has led many computer scientists to recognize purely functional programming languages as a good tool both for teaching and for writing efficient and correct programs (e.g. Felleisen, Findler, Flatt & Krishnamurthi, 2001).

9. Change in students' test answers

We now look into students understanding of change processes and try to interpret them in the light of the distinction between OLA and MLA. The data was collected in a written test on algebra for 16 year old students that has been conducted by 329 high school students from various schools in the Frankfurt urban region. The schools varied much in socio-economic background of students and there were both low and high achieving schools in the set. For our present analysis we concentrate on the following test items:

Item A

a) Assume that $a=b+3$ always holds, what happens to a if b is increased by 2?

b) Assume that $a=b+3$ always holds, what happens to b if a is increased by 2?

c) Assume that $a=2b+3$ always holds, what happens to b if a is increased by 2?

Item B

a and b are positive numbers. Assume that $\frac{1}{a}+\frac{1}{b}=2$holds, how does b change when a increases?

The answers of students to these questions are striking as they show a much greater variety than expected. A number of wrong answers can be traced down to the following sources that are not related to the understanding of change:

- Some students only gave a solution to item Aa but explicitly wrote that Ab and on have no solution or that *b* cannot change. This may be explained as the result of a dominating functional understanding: If the equation is given in the form of a functional equation $y=f(x)$ as in item Aa the problem can be solved but for these students the concept of change is in clash with more general relationships.
- Others stated in all items that the change leads to a false equation.
- Some students explain that in item B $a=b=1$ is the only solution. This indicates that they limit variables to refer to natural numbers.
- Some students seem to mistake addition and multiplication when they come up with expressions involving $2a$.

Among the wrong answers that do not fall into any of these cases there are approximately 10% of wrong answers that can be attributed to an external understanding of change. Typical answers have been:

Aa:

- $a+2=b+5$
- $a+2=(b+2)+3$
- $a+2=b+2+3$ *a* increases by 2
- $\mathrm{a}+2=b+2+3 \quad = \quad a+2=b+5$

Ab:

- $a+2=b+5$
- $a+2=b+2+3$ and then *b* must get greater as well
- Then either *b* or 3 get larger by 2 or both get larger by 1.

Ac:

- $a+2=2b+5$
- *a* gets larger by the same amount that *b* gets smaller, i.e. $a+3=b-3$

Interestingly, as the last answer to Ab shows, some students even considered that numbers might change. Others explicitly explained the perceived impossibility of the change (“a number can’t change”). Another answer of this kind has been “Maybe *b* increases by 2 or the other number (2+2, 3+2)“. Of course one may take this as speaking metaphorically to mean that the number is replaced by another number in the formula.

Although not asked for, several students explicitly justified their internal understanding of change, e.g. in response to Aa: “$a=b+3$ the calculation stays the same, but for *b* another number is inserted”

The answers listed above that indicate an external understanding of change were in originally considered to be wrong, as they indicate insufficient understanding of covariation. However, it turned out that these students scored better than the other students on the whole test. In fact, a t-test confirmed this difference to be significant (p=2,5%, effect size d=0.76, i.e. a strong effect). This may be explained by assuming that the external understanding of change indicated a thinking according to MLA and this is appropriate for most of algebra and in a sense typical for finished learning processes. MLA is also simpler in a sense because the laws from arithmetic carry over more directly to algebra, hence, one may speculate that it puts less load on the working memory when doing tasks where is adequate.

However, there is also an apparent difficulty with MLA: As the semantics of MLA is given by interpretations, it seems plausible that students who hold a dominating MLA view are especially affected by the lack-of-closure obstacle (the inability to accept expressions containing free variables as valid answers). This was tested using the following item: “At the bakery I buy *c* croissants and *r* rolls. How many parts are bought altogether?” Only two students who showed an external understanding of change (and thus are supposed to hold an MLA view) answered this item correctly giving the open expression $c+r$.

10. Didactical conclusions

As mentioned above, introducing algebra to describe number sequences (i.e. generalized arithmetic) can be seen as a common didactical pathway to MLA. Similarly, introducing variables as boxes that contain a certain number of smaller things (e.g. beans) is a pathway to OLA. Both approaches work but each has its own shortcomings. When working with real boxes one has to obey the additional rule that boxes labelled by the same letter have to contain the same number. Moreover, the process of substitution is not obvious: If *x* is substituted for y, does this mean that we put a box in a box? It seems that keeping the current value of *x* external to *x* and using a referential notation as $x+5$ is less confusing. Nevertheless, this approach has the advantage of starting at the level of tangible objects and thus seems to foster learning.

We suppose that teaching algebra cannot be simplified by concentrating only on one mode, MLA or OLA. While formal mathematics requires only MLA it is much more convenient to interpret change processes in OLA and neglecting OLA may reduce the heuristic power of the algebraic language. Moreover, we argued that OLA has a special role in the creation of mathematics and it reflects the way computer algebra systems deal with algebra. Students may encounter obstacles if they tackle a question in one mode which is more suited for the other mode.

This paper thus identified the research desiderata to investigate how MLA and OLA can both be incorporated into the teaching of algebra. Moreover, it opens up the possibility to use this distinction in the explanation of students learning obstacles.

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