The New Integral Transform "SEE Transform" of Bessel's Functions

Eman A. Mansour[1], Emad A. Kuffi[2], Sadiq A. Mehdi[3]

[1,3]**Department of Mathematics, Mustansiriyah University, Iraq, Baghdad**

[2]**Collage of Engineering, Al − Qadisiyah University**

Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published Online: 4 June 2021

Abstract

In this modern time, Bessel's functions appear in solving many problems of engineering and science together with many equations such as wave equation, heat equation, Laplace equation, Helmholtz equation, Schrodinger equation in spherical or cylindrical coordinates. in this paper, we present SEE integral transform of Bessel's functions. Some problems of SEE transform of Bessel's functions for calculating the integral, which contain Bessel's functions, are given.

Keywords: SEE integral transform, Convolution theorem, Bessel's function.

1. Introduction

Bessel's functions have many applications [4,6] to solve the mathematical physics, engineering, acoustics, and natural sciences such as heat transfer, hydrodynamics, flux distribution in a nuclear reactor etc.

Consider Bessel's function of order n, where n is given by [2-5,8]:

$$J_n(t) = \frac{t^n}{2^n n!}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)} + \cdots\right] \quad \ldots (1)$$

when $n = 0$, Bessel's function of zero order and it denoted by $J_0(t)$ and it is given by:

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots \qquad (2)$$

And when $n = 1$, Bessel's function of order one, it is given by:

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.4} + \frac{t^5}{2^2.4^2.6} - \frac{t^7}{2^2.4^2.6^2.8} + \cdots \qquad (3)$$

Equation (3) can be written as:

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + \cdots \qquad (4)$$

For $n = 2$, we have Bessel's function of order two:

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2.4^2.6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + \cdots \qquad (5)$$

The SEE integral transform of the function $f(t)$ is defined as [1]:

$$S[f(t)] = \frac{1}{v^n} \int_{t=0}^{\infty} f(t) e^{-vt} dt \ = T(v) \ , \qquad t \geq 0 \ , \ \ n \in \mathbb{Z} \ , \ \ l_1 \leq v \leq l_2 \quad \ldots (6)$$

The$S[.]$ is called SEE integral transform operator.

The SEE transform of the function $f(x)$ exist if $f(t)$ is a piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of SEE integral transform of the function $f(t)$.

The object of the present study is to determine SEE transform of Bessel's functions and explain the advantage of SEE transform of Bessel's functions for calculating the integral which contain Bessel's functions.

2. Convolution Theorem for SEE Integral Transform

If $S[f(t)] = T_1(v)$and $S[H(t)] = T_2(v)$, then $S[f(t) * H(t)] = v^n S[f(t)] * S[H(t)] = v^n T_1(v).T_2(v)$.

3. Linearity Propriety of SEE Integral Transform

$$S[af(t) + bg(t)] = aS[f(t)] + bS[g(t)]$$

Where a, b are constants.

4. SEE Integral Transform of Some Elementary Functions, [1]

S.N.	$f(t)$	$S[f(t)] = T(v)$
1.	$K \equiv Constant$	$\frac{K}{v^{n+1}}$
2.	t^m	$\frac{m!}{v^{n+m+1}}$, m is a positive integer number

3.	e^{at}	$\frac{1}{v^n(v-a)}$, a is a constant
4.	$\sin(at)$	$\frac{a}{v^n(v^2+a^2)}$
5.	$\cos(at)$	$\frac{v}{v^n(v^2+a^2)}$
6.	$\sinh(at)$	$\frac{a}{v^n(v^2-a^2)}$
7.	$\cosh(at)$	$\frac{v}{v^n(v^2-a^2)}$

5. Change of Scale Property of SEE Transform

If $S[f(t)] = T(v)$ and $S[f(t)] = \frac{1}{v^n}\int_{t=0}^{\infty} f(t)e^{-vt}dt$

Put $at = p$, $adt = pdp$, $thendt = \frac{p}{a}dpandt = \frac{p}{a}$.

So

$$S[f(at)] = \frac{1}{a^n v^n}\int_{t=0}^{\infty} f(p)e^{-v\frac{p}{a}}\frac{1}{a}dp$$

$= \frac{1}{a^{n+1}}T\left(\frac{v}{a}\right)$.

Thus if $S[f(t)] = T(v)$, then $S[f(at)] = \frac{1}{a^{n+1}}T\left(\frac{v}{a}\right)$.

6. The SEE Integral Transform of The Derivatives of The Function f(t), [1]

If $S[f(t)] = T(v)$, then

(a) $S[f'(t)] = \frac{-1}{v^n}f(0) + vT(v)$.

(b) $S[f''(t)] = \frac{-f'(0)}{v^n} - \frac{f(0)}{v^{n-1}} - v^2T(v)$.

(c) In general case:

$$S\left[f^{(m)}(t)\right] = \frac{-f^{(m-1)}(0)}{v^n} - \frac{f^{(m-2)}(0)}{v^n-1} - \cdots - \frac{f(0)}{v^{n-m+1}} + v^mT(v).$$

7. Relation Between $J_0(t)$ and $J_1(t)$, [5,8]

$$\frac{d}{dt}[J_0(t)] = -J_1(t) \qquad \ldots(6)$$

8. Relation Between $J_0(t)$ and $J_2(t)$, [8]

$$J_2(t) = J_0(t) + 2J''_0(t) \qquad \ldots(7)$$

9. The SEE Transform of Bessel's Functions

(9.a) The SEE Transform of $J_0(t)$

Taking SEE transform of equation (2), both sides, we get:

$$S[J_0(t)] = S[1] - \frac{1}{2^2}S[t^2] + \frac{1}{2^2.4^2}S[t^4] - \frac{1}{2^2.4^2.6^2}S[t^6] + \cdots$$

$$= \frac{1}{v^{n+1}} - \frac{1}{2^2}\frac{2}{v^{n+3}} + \frac{1}{2^2.4^2}\cdot\frac{4!}{v^{n+5}} - \frac{1}{2^2.4^2.6^2}\frac{6!}{v^{n+7}} + \cdots$$

$$= \frac{1}{v^{n+1}}\left[1 - \frac{1}{2^2}.\frac{2}{v^2} + \frac{1}{2^2.4^2}\frac{4!}{v^4} - \frac{1}{2^2.4^2.6^2}.\frac{6!}{v^6} + \cdots\right]$$

$$= \frac{1}{v^{n+1}}\left[1 - \frac{1}{2}\left(\frac{1}{v^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{v^2}\right)^2 - \frac{5.3.1}{2.4.6}\left(\frac{1}{v^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{v^{n+1}}\left(1 + \frac{1}{v^2}\right)^{\frac{-1}{2}} = \frac{1}{v^{n+1}\sqrt{\left(1+\frac{1}{v^2}\right)}}.$$

$$= \frac{1}{v^n\sqrt{1+v^2}}.$$

(9.b) The SEE Transform of $J_1(t)$

Since $\frac{d}{dt}[J_0(t)] = -J_1(t)$,

Then $S[J_1(t)] = -S\left[\frac{d}{dt}J_0(t)\right]$

$$= -\left[\frac{-1}{v^n}J_0(0) + vS[J_0(t)]\right]$$

$$= \frac{1}{v^n}J_0(0) - vS[J_0(t)]$$

$$= \frac{1}{v^n}(1) - \frac{v}{v^n\sqrt{(1+v^2)}}$$

$$= \frac{1}{v^n}\left[1 - \frac{v}{\sqrt{(1+v^2)}}\right]$$

(9.c) The SEE Transform of $J_2(t)$

Since $J_2(t) = J_0(t) + 2J_0''(t)$, then:

$S[J_2(t)] = S[J_0(t)] + 2S[J_0''(t)]$.

$$= \frac{1}{v^n\sqrt{1+v^2}} + 2\left[\frac{-J_0'(0)}{v^n} - \frac{vJ_0(0)}{v^n} + v^2 S[J_0(t)]\right]$$

$$= \frac{1}{v^n\sqrt{(1+v^2)}} + 2\left[\frac{J_1(0)}{v^n} - v\frac{J_0(0)}{v^n} + v^2.\frac{1}{v^n\sqrt{(1+v^2)}}\right]$$

$$= \frac{1}{v^n\sqrt{(1+v^2)}} + \left[\frac{-2v}{v^n} + \frac{2v^2}{v^n\sqrt{(1+v^2)}}\right]$$

$$= \frac{1}{v^n\sqrt{(1+v^2)}} + \frac{2v^2}{v^n\sqrt{(1+v^2)}} - \frac{2v}{v^n} = \frac{1+2v^2-2v\sqrt{1+v^2}}{v^n\sqrt{(1+v^2)}}$$

(9.d) The SEE Transform of $J_0(at)$

Since $S[J_0(t)] = \frac{1}{v^n\sqrt{(1+v^2)}}$.

Now, applying change of scale property of SEE transform of scale property of SEE transform, we get:

$S[J_0(at)] = \frac{1}{a^{n+1}}\left[\frac{1}{\frac{v^n}{a^n}\sqrt{1+\left(\frac{v}{a}\right)^2}}\right]$,

$= \frac{1}{v^n} \cdot \frac{1}{\sqrt{a^2+v^2}}$.

(9.e) The SEE Transform of $J_1(at)$

Since $S[J_1(t)] = \frac{1}{v^n}\left[1 - \frac{v}{\sqrt{(1+v^2)}}\right]$

Now, applying change of scale property of SEE transform, we have:

$S[J_1(at)] = \frac{1}{a}.\frac{1}{v^n}\left[1 - \frac{v}{\sqrt{a^2+v^2}}\right]$.

(9.f) The SEE Transform of $J_2(at)$

Since $S[J_2(t)] = \frac{1+2v^2-2v\sqrt{1+v^2}}{v^n\sqrt{1+v^2}}$.

Then $S[J_2(at)] = \frac{1}{a^{n+1}}\left[\frac{1+\frac{2v^2}{a^2}-\frac{2v}{a}\sqrt{1+\left(\frac{v}{a}\right)^2}}{\frac{v^n}{a^{n+1}}\sqrt{a^2+v^2}}\right]$,

$= \frac{1}{a^2v^n}\left[\frac{a^2+2v^2-2v\sqrt{(a^2+v^2)}}{\sqrt{(a^2+v^2)}}\right]$.

10.Applications

Application (1) Evaluate the integral:

$I(t) = \int_{u=0}^{t} J_0(u).J_0(t-u)du.$

Applying the SEE transform to both sides, we have:

$S[I(t)] = S\left[\int_{u=0}^{t} J_0(u)\,.J_0(t-u)du\right].$

Using convolution theorem of SEE transform, we have

$$S[I(t)] = v^n S[J_0(t)].S[J_0(t)]$$

$$= v^n.\left[\frac{1}{v^n\sqrt{(1+v^2)}}.\frac{1}{v^n\sqrt{(1+v^2)}}\right]$$

So $S[I(t)] = \frac{1}{v^n(1+v^2)}$,

Take inverse to both sides, we get

$I(t) = \sin(t).$

Which is the required exact solution of equation.

Application (2): Evaluate the integral

$I(t) = \int_{u=0}^{t} J_0(u)J_1(t-u)du.$

Applying the SEE integral transfer transform to both sides of equation, we have:

$S[I(t)] = S\left[\int_{u=0}^{t} J_0(u)\,.J_1(t-u)du\right].$

Using convolution theorem of SEE transform, we get:

$$S[I(t)] = v^n S[J_0(t)].S[J_1(t)]$$

$$= v^n \left[\frac{1}{v^n\sqrt{(1+v^2)}}\right].\left[\frac{1}{v^n}\left(1-\frac{v}{\sqrt{(1+v^2)}}\right)\right]$$

$$= \frac{1}{v^n\sqrt{(1+v^2)}} - \frac{1}{v^n}.\frac{v}{(1+v^2)}$$

$$I(t) = S^{-1}\left[\frac{1}{v^n\sqrt{(1+v^2)}}\right] - S^{-1}\left[\frac{v}{v^n(1+v^2)}\right]$$

$I(t) = J_0(t) - \cos(t)$.

Which is the required exact solution of equation.

Application (3): Evaluate the integral:

$I(t) = \int_{u=0}^{t} J_1(t-u)du$.

Applying SEE integral transform, we have

$S[I(t)] = S\left[\int_{u=0}^{t} J_1(t-u)du\right]$.

Using convolution theorem of SEE integral transform, we have:

$$S[I(t)] = v^n S[1].S[J_1(t)]$$

$$= v^n \left[\frac{1}{v^{n+1}}\right].\left[\frac{1}{v^n}\left(1-\frac{v}{\sqrt{(1+v^2)}}\right)\right]$$

$= \frac{1}{v^{n+1}} - \frac{1}{v^n\sqrt{1+v^2}}$.

Then

$$I(t) = S^{-1}\left[\frac{1}{v^{n+1}}\right] - S^{-1}\left[\frac{1}{v^n\sqrt{(1+v^2)}}\right]$$

So $I(t) = 1 - J_0(t)$.

Which is the required exact solution of equation.

11.Conclusions

In this paper, we discussed the SEE integral transform of Bessel’s functions. Also, the given applications show that the advantage of SEE (Sadiq-Emad-Eman) integral transform of Bessel’s functions to calculate the integral which contain Bessel’s functions.

References

[1] Eman A. Mansour, Sadiq A. Mehdi, Emad A. Kuffi, “The New Integral Transform “SEE Transform” and its Applications”, Periodical of Engineering and Natural Sciences (PEN), Vol.9, No.2 (2021).

[2] Bell, W.W. Special Functions for Scientists and Engineer, D.Van Nostrand Company LTD London.

[3] Mclachlan, N.W. Bessel Functions for Engineers, Longman. Oxford (1955).

[4] Korenev. B.G. Bessel Functions and Their Applications, Chapman and Hall/CRC.

[5] Watson. G.N. A Treatise on the Theory of Bessel Functions. Cambridge University Press.

[6] Farrell, O.J. and Ross, B. Solved Problems in Analysis: As Applied to Gamma, Beta, Legendre and Bessel Functions, Dover Publications, Inc. Mineola, New York.

[7] Lokenath Debnath and Bhatta, D. Integral Transform and Their Applications, 2nd edition, Chapman and Hall/CRC (2006).

[8] Raisinghania, M.D. Advanced Differential Equations, S. Chand and Company PVT LTD Ramnagar, New-Delhi.