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Research Article

New existence results on neutral fractional differential equations in the concept of Atangana-Baleanu derivative with impulsive conditions

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Abstract

In this article, We study an existence and uniqueness of solutions for an impulsive fractional neutral differential equations via Atangana-Baleanu fractional derivative with dependence on the lipschitz first derivative conditions in Banach space. This results are based on fixed point theorems. An example is given to illustrate the main results.

Keywords: Fractional Calculus; Neutral differential equations; impulsive; AB-derivative; Lipschitz first derivatives; Fixed point techniques.

MSC: 34A08; 34K37; 34K40; 58C30.

1 Introduction

Recently, fractional differential equations involving the Atangana-Baleanu fractional derivative have been paid more and more attentions like the RiemannLiouville fractional derivative and the Caputo fractional derivative. Fractional differential equations have applied in numerous fields in the past few decades such as chemistry, physics, engineering, control theory, aerodynamics, electrodynamics of complex medium and control of dynamical systems and so on. In consequence, fractional differential equations is obtaining much significance and attention. For details, we refer readers to [19, 25, 26, 30]

Impulsive fractional differential equations are used to describe many practical dynamical systems including evolutionary processes characterized by abrupt changes of the state at certain instants. Nowadays, the theory of impulsive fractional differential equations has received great attention, devoted to many applications in mechanical, engineering, medicine, biology, ecology and etc [11-16]. This paper is motivated from some recent papers treating the problem of the existence of solutions for impulsive differential equations with fractional derivative. By directly computation it is easy to see that the concepts of piecewise continuous solutions used in many papers are not appropriate.

We investigate the existence and uniqueness of solutions of the Atangana-Baleanu fractional neutral differential equation in the sense of Caputo to the following abstract form

$$\begin{aligned} ({}^{ABC}_{0}D^{\alpha})(u(t)-g(t,u(t),u'(t,u(t))) &= f(t,u(t),u'(t,u(t))), \quad 1<\alpha\le 2, && (1)\\ u|_{[-\tau^*,0]} &= u_0. && (2)\\ \Delta u(t_k) &= u(t_k^+)-u(t_k^-) \\ &= I_k^* u(t_k^-),\ t\in J^* \end{aligned}$$

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with $u(t)$, $({}_{a}^{ABC}D^{\alpha})(u-g) \in C[0,1], f(t,u(t),\mathfrak{D}u(t)) : J \times PC^1 \times J \to \Re^n$ is continuous on $PC^1([-\tau^*,0],\Re^n)$ is the piecewise continuous functions $\rho : [-\tau^*,0],\Re^n$.

The $u_t(s) = u(t+s)$ for $-\tau^* \le s \le 0$ where $u(t_k^+) = \lim_{\delta\to 0^+} u(u_k+\delta)$ and $u(t_k^-) = \lim_{\delta\to 0^-} u(u_k+\delta)$. $I^* : \Re^n \to \Re^n$, $0 = t_0 < t_1 < .... < t_m < t_{m+1} = T$, $J^* = \{J - t_1, t_2, ..., t_m : J = [0,1]\}$. Consider $\mathfrak{D}u(t) = u'(t,u(t))$. Then (1) becomes

$$\begin{aligned} ({}_{0}^{ABC}D^{\alpha})(u(t) - g(t,u(t),\mathfrak{D}u(t)) &= f(t,u(t),\mathfrak{D}u(t)), \qquad 1 < \alpha \le 2, & (3)\\ u|_{[-\tau^*,0]} &= u_0. & (4)\\ \Delta u(t_k) &= I_k^* u(t_k^-),\ t \in J^* \end{aligned}$$

The rest of this paper is organized as follows: In Section 2, we review some useful properties, definitions, propositions and lemmas of fractional calculus. The existence and uniqueness of solutions for AB-fractional neutral derivative results are proved in Section 3. In section 4, we investigte the fractional derivative with non-local condition. In the final sectionm is devoted to illustrate an example numerically solved.

2 Preliminaries

In this section, we presents some definitions, lemmas and proposotions of fractonal calculus, which will be used throughout this paper.

The definition of Riemann-Liouville fractional integral and derivatives are given as follows:

- For $\alpha > 0$, the left R-L fractional integral of order α is given as [27]

$$({}_{0}I^{\alpha}u)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds. \tag{5}$$

- For $0 < \alpha < 1$, the left R-L fractional derivative of order α is given as [27]

$$({}_{0}D^{\alpha}u)(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(s)ds\right) \tag{6}$$

- For $0 \le \alpha \le 1$, the Caputo fractional derivative of order α is given as [27]

$$({}_{0}^{C}D^{\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds. \tag{7}$$

Definition 2.1 *[7] Let $u \in H^1(0,1)$ and α in [0,1]. The Caputo Atangana-Baleanu fractional derivative of u of order α is defined by*

$$({}_{0}^{ABC}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\int_0^t u'(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]ds. \tag{8}$$

where E_α is the Mittag-Leffler function defined by $E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}$ [34, 41] and $B(\alpha) > 0$ is a normalizing function satisfying $B(0) = B(1) = 1$. The Riemann Atangana-Baleanu fractional derivative of u of order α is defined by

$$({}_{0}^{ABC}D^{\alpha}u)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]ds. \tag{9}$$

The associative fractional integral is defined by

$$({}_0^{AB}I^\alpha u)(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)}({}_0I^\alpha u)(t) \tag{10}$$

where ${}_0I^\alpha$ is the left Riemann-Liouville fractional integral given in (13).

Lemma 2.2 *[7] Let $u \in H^1(a,b)$ and $\alpha \in [0,1]$. Then the following relation holds.*

$$({}_0^{ABC}D^\alpha u)(t) = ({}_0^{ABR}D^\alpha u)(t) - \frac{B(\alpha)}{1-\alpha}u(0)E_\alpha\left(-\frac{\alpha}{1-\alpha}t^\alpha\right). \tag{11}$$

Theorem 2.3 *(Ascoli-Arzela Theorem)([23]) Let S be a compact metric spaces.Then $M \subset C(\Omega)$ is relatively compact iff M is uniformly bounded and uniformly equicontinuous.*

Theorem 2.4 *(Krasnoselskii Fixed Point Theorem)([23]) Let S be a closed, bounded and convex subset of a real Banach space X and let T_1 and T_2 be operators on S satisfying the following conditions*

- $T_1(s) + T_2(s) \subset S$
- *T_1 is a strict contraction on S, i.e., there exist a $k \in [a,b)$ such that* $\|T_1(u) - T_1(v)\| \leq k\|u-v\| \quad \forall\, u,v \in S$
- *T_2 is continuous on S and $T_2(s)$ is a relatively compact subset of X.*

Then, there exist a $u \in S$ such that $T_1u + T_2u = u$

Proposition 2.6 ([4]) For $0 \leq \alpha \leq 1$, we conclude that

$$\begin{aligned} ({}_0^{AB}I^\alpha({}_0^{ABC}D^\alpha u))(t) &= u(t) - u(0)E_\alpha(\lambda t^\alpha) - \frac{\alpha}{1-\alpha}u(0)E_{\alpha,\alpha+1}(\lambda t^\alpha) \\ &= u(t) - u(0). \end{aligned}$$

Proposition 2.7 ([29, 37]) $f'(u) \in D$ satisfy the Lipschitz condition.
i.e.,There exist a constant $k > 0$ such that

$$\|f'(u) - f'(v)\| \leq k\ (\|u-v\|), \quad u,v \in D. \tag{12}$$

3 Existence and Uniqueness

In this section, we prove the existence and uniqueness of (3) and (4).

We need the following assumptions to prove the existence and uniqueness results for the problem (3) and (4) by using the Banach contraction principle.

A₁ Let $u \in C[0,1]$ and $g \in (J \times PC^1 \times J, J)$ is piecewise continuous function and there exist a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}$ such that

$$\|g(t,u_1,v_1) - g(t,u_2,v_2)\| \leq \mathfrak{M}_1(\|u_1-u_2\| + \|v_1-v_2\|)$$

for all u_1, v_1, u_2, v_2 in Y, $\mathfrak{M}_2 = \max_{t\in J}\|g(t,0,0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[J,X]$ be the set continuous functions on J with values in the Banach spaces X .

A_2 Let $u \in C[0,1]$ and $f \in (J \times PC^1 \times J, J)$ is piecewise continuous function and there exist a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq \mathfrak{N}_1(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for all u_1, v_1, u_2, v_2 in Y, $\mathfrak{N}_2 = \max_{t\in J} \|f(t,0,0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

A_3 Let $u' \in C[0,1]$ satisfy the Lipschitz condition. i.e.,There exist a positive constants $\mathfrak{L}_1, \mathfrak{L}_2$ and $\mathfrak{L}$ such that

$$\|\mathfrak{D}(t,u) - \mathfrak{D}(t,v)\| \leq \mathfrak{L}_1(\|u - v\|),$$

for all u, v in Y. $\mathfrak{L}_2 = \max_{t\in D} \|\mathfrak{D}(t,0)\|$ and $\mathfrak{L} = \max\{\mathfrak{L}_1, \mathfrak{L}_2\}$.

A_4 The impulses $I_k^* \in (\Re^n, \Re^n)$ be bounded and for $\alpha^* > 0$ we have $\|I^* y_1(t_k^-) - I^* y_2(t_k^-)\| \leq \alpha^*(t)\|y_1 - y_2\|_\infty$.

A_5 For each $\lambda > 0$, Let $B_\lambda \in \{u \in Y : \|u\| \leq \lambda\} \subset Y$, them B_λ is clearly a bounded closed and convex set in $(C[0,1], J)$ where $\lambda = ((1 - 2\mathfrak{C})^{-1}(\|u_0\|) + \mathfrak{C})$ and take $\mathfrak{C} = \max\{\mathfrak{M}, \mathfrak{N}\}$ and $\mathfrak{C} < \frac{1}{2}$.

Lemma 3.1 *If* A_3 *are satisfied, then the estimate* $\|\mathfrak{D}u(t)\| \leq t(\mathfrak{L}_1\|u\| + \mathfrak{L}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{L}t\|u - v\|$, *are satisfied for any* $t \in \Re$, *and* $u, v \in Y$.

Definition 3.2 *If* $u(0) = u_0$ *and* $u \in C[0,1]$ *is a solution of (3) and (4) then there is an* $f \in (J \times PC^1 \times J, J)$ *where* $t \in [0,t_1] \cup (t_m, T], m = 1, 2, .$ *and*

$$v(t) = \begin{cases} v_0 & t \in [\tau^*, 0] \\ v_0 - g(0,u_0,\mathfrak{D}u(0)) +_a^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in [0,t_1] \\ v_0 - g(0,u_0,\mathfrak{D}u(0)) + I_1^* u(t_1^-) +_a^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_1,t_2] \\ v_0 - g(0,u_0,\mathfrak{D}u(0)) + \sum_{k=1}^{2} I_1^* u(t_1^-) +_a^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_2,t_3] \\ . \\ . \\ . \\ . \\ . \\ v_0 - g(0,u_0,\mathfrak{D}u(0)) + \sum_{k=1}^{m} I_1^* u(t_1^-) +_a^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_m,T] \end{cases} \tag{13}$$

is satisfied

Theorem 3.3 *Let* $u(t) \in C[0,1]$ *such that* $({}_0^{ABC}D^\alpha u)(t) \in C[0,1]$. *Suppose that* $f \in C([0,1]\times J\times J, J)$ *satisfies* $A_1 - A_5$. *Then, if* $g(a,u(a),\mathfrak{D}u(a)) = f(a,u(a),\mathfrak{D}u(a)) = 0$ *and* $\Big(\mathfrak{L}t + (1 + \mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big) \leq 1$ *the problem (3) and (4) has an unique solution.*

Proof. Suppose $u(t)$ satisfy (3) and (4), then by use (13), for $t \in [0,t_1]$ we get the integral equation

$$({}_0^{AB}I^\alpha)({}_0^{AB}D^\alpha)[u(t) - g(t,u(t),\mathfrak{D}u(t))] =_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) \tag{14}$$

Now, by using Proposition 2.6, for $t \in [0,t_1]$ we obtain

$$u(t) - g(t,u(t),\mathfrak{D}u(t)) = u_0 - g(a,u(a),\mathfrak{D}u(a)) +_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t))$$

.

Therfore,

$$v(t)=\begin{cases} v_0 & t\in[\tau^*,0]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in[0,t_1]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_1,t_2]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+\sum_{k=1}^{2}I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_2,t_3]\\ .\\ .\\ .\\ .\\ .\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+\sum_{k=1}^{m}I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_m,T]\end{cases}$$

Since $u(0)=u_0$ from (4)and taking $f(0,u_0,0)=g(0,u_0,0)=0$, then (3) is satisfied. Next, take u(t) satisfy (3), then by using $f(0,u_0,0)=0$ which implies $u(0)=u_0$.
By using R-L sense with AB-derivative in (4)and subsitute $({}_0^{AB}D^{\alpha}({}_0^{AB}D^{\alpha}))(t)=v(t)$ for $t\in[0,t_1]$ we obtain

$$\begin{aligned}({}_0^{ABR}D^{\alpha}u)(t) &= u_0({}_0^{ABR}D^{\alpha}1)(t)-g(0,u_0,\mathfrak{D}u(0))({}_0^{ABR}D^{\alpha}1)+({}_0^{ABR}D^{\alpha})g(t,u(t),\mathfrak{D}u(t))\\ &\quad+({}_0^{ABR}D^{\alpha}({}_0^{AB}I^{\alpha}))f(t,u(t),\mathfrak{D}u(t))\end{aligned}$$

Thus we have

$$({}_0^{ABR}D^{\alpha})(u(t)-g(t,u(t),\mathfrak{D}u(t))) = (u_0-g(0,u_0,\mathfrak{D}u(0)))E_{\alpha}\left(\frac{-\alpha}{1-\alpha}t^{\alpha}\right)+f(t,u(t),\mathfrak{D}u(t))$$

By Theorem 1 in [5], the result will be obtained. Now, define the operator

$$Tv(t)=\begin{cases} v_0 & t\in[\tau^*,0]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in[0,t_1]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_1,t_2]\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+\sum_{k=1}^{2}I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_2,t_3]\\ .\\ .\\ .\\ .\\ .\\ v_0-g(0,u_0,\mathfrak{D}u(0))+g(t,u(t),\mathfrak{D}u(t))+\sum_{k=1}^{m}I_1^*u(t_1^-)+{}_0^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)) & t\in(t_m,T]\end{cases}$$

Then by **A5**, $\|u\|\le\lambda$ and by the Lemma 2.3, for $t\in[0,t_1]$ we have

$$\begin{aligned}\|Tu(t)\| &\le \|u_0\|+\mathfrak{C}\|u\|+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u\|\\ &\quad+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\\ &\le \lambda\end{aligned}$$

For $t\in(t_1,t_2]$

$$\begin{aligned}\|Tu(t)\| &\le \|u_0\|+\|I_1^*v(t_1^-)\|+\mathfrak{C}\|u\|+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u\|\\ &\quad+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\\ &\le \lambda\end{aligned}$$

For $t \in (t_2, t_3]$

$$\begin{aligned} \|Tu(t)\| &\leq \|u_0\| + \|\sum_{k=1}^{2} I_1^* v(t_1^-)\| + \mathfrak{C}\|u\| + \mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u\| \\ &\quad + \mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big) \\ &\leq \lambda \end{aligned}$$

Similarly, for $t \in (t_m, T]$ $\|Tu(t)\| \leq \lambda$ where m=1,2,3,..

Now, to prove uniqueness for $t \in [0, t_1]$, we have

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| &\leq \mathfrak{M}(1+\mathfrak{L}t)\|u_1 - u_2\| + \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}(1+\mathfrak{L}t))\|u_1 - u_2\|] \\ &\quad + \frac{\alpha}{B(\alpha)}(\mathfrak{M}(1+\mathfrak{L}t))\|u_1 - u_2\|(({}_0^{AB}I^\alpha)(t) \\ &\leq \mathfrak{C}\|u - v\| + \mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u_1 - u_2\| \\ &\leq \|u_1 - u_2\| \end{aligned}$$

For $t \in (t_2, t_3]$

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| &\leq +\alpha^*\|u_1 - u_2\| + \mathfrak{M}(1+\mathfrak{L}t)\|u_1 - u_2\| + \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}(1+\mathfrak{L}t))\|u_1 - u_2\|] \\ &\quad + \frac{\alpha}{B(\alpha)}(\mathfrak{M}(1+\mathfrak{L}t))\|u - v\|(({}_0^{AB}I^\alpha)(t) \\ &\leq \mathfrak{C}\|u_1 - u_2\| + \mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u_1 - u_2\| \\ &\leq \|u_1 - u_2\| \end{aligned}$$

Similarly, for $t \in (t_m, T]$we have $\|Tu_1 - Tu_2\| \leq \|u_1 - u_2\|$ where m=1,2,3,.. Therefore $Tu(t)$ has an unique solution.

Hence, the operator $Tu(t), t \in B_\lambda$ proved the existence and uniqueness conditions and has a fixed point by Banach contraction principle in Banach spaces X.

Next, we investigate the problem (3) and (4) has a fixed point by using another fixed point technique, namely Krasnoselskii's fixed point theroem.

Theorem 3.4 *If* $\mathbf{A_1} - \mathbf{A_5}$ *are satisfied and* $q(t_2 - t_1) = [\mathfrak{N}(\|u(t_2) - u(t_1)\| + \mathfrak{L}t\|u(t_2) - u(t_1)\|)]$*, then the problem (3) and (4) has a solution.*

Proof. For any constant $\lambda_0 > 0$ and $u \in B_{\lambda_0}$, defined two operator T_1 and T_2 on B_{λ_0} as follows

$$(T_1u)(t) = u_0 + \|\sum_{k=1}^{m} I_1^* v(t_1^-)\| - g(0, u(0), 0) + g(t, u(t), \mathfrak{D}u(t)) \tag{15}$$

$$(T_2u)(t) = {}_a^{AB}I^\alpha f(t, u(t), \mathfrak{D}u(t)). \tag{16}$$

Obviously, u is a solution of (3) and (4) iff the operator $T_1u + T_2u = u$ has a solution $u \in B_{\lambda_0}$
Our proof will be divided into three steps.

Step 1. $\|T_1u+T_2u\|\leq\lambda_0$ whenever $u\in B_{\lambda_0}$. For every $u\in B_{\lambda_0}$ and $t\in[0,t_1]$, we have

$$\begin{aligned}\|(T_1u)(t)+(T_2u)(t)\| &\leq \|u_0\|+\mathfrak{C}\|u\|+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u\| \\ &\quad +\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big) \\ &\leq \lambda_0\end{aligned}$$

For $t\in(t_1,t_2]$, we have

$$\begin{aligned}\|(T_1u)(t)+(T_2u)(t)\| &\leq \|u_0\|+\|I_1^*v(t_1^-)\|+\mathfrak{C}\|u\|+\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u\| \\ &\quad +\mathfrak{C}\Big(\mathfrak{L}t+(1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)}+\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big) \\ &\leq \lambda_0\end{aligned}$$

Similarly, for $t\in(t_m,T]$ we have where m=1,2,3,..

Hence, $\|T_1u+T_2u\|\leq\lambda_0$ for every $u\in B_{\lambda_0}$.

Step 2. T_1 is a contraction on B_{λ_0} for any $u,v\in B_{\lambda_0}$, according to $\mathbf{A_4}$ and $t\in[0,t_1]$, we have

$$\begin{aligned}\|(T_1u)(t)-(T_1v)(t\| &\leq \|u_0-v_0\|+\mathfrak{M}\|u-v\|+\mathfrak{M}\mathfrak{L}t\|u-v\| \\ &\leq \|u_0-v_0\|[1+(\mathfrak{M}+\mathfrak{M}\mathfrak{L}t)\|u-v\|] \\ &\leq R\|u_0-v_0\|\end{aligned}$$

For $t\in(t_1,t_2]$, we have

$$\begin{aligned}\|(T_1u)(t)-(T_1v)(t\| &\leq \|u_0-v_0\|+\alpha^*\|u-v\|+\mathfrak{M}\|u-v\|+\mathfrak{M}\mathfrak{L}t\|u-v\| \\ &\leq \|u_0-v_0\|[1+(\alpha^*+\mathfrak{M}+\mathfrak{M}\mathfrak{L}t)\|u-v\|] \\ &\leq R\|u_0-v_0\|\end{aligned}$$

Similarly, for $t\in(t_m,T]$ we have where m=1,2,3,.. which implies that $\|T_1u-T_1v\|\leq R\|u_0-v_0\|$, since $R=1$, where $R=1+(\alpha^*+\mathfrak{M}+\mathfrak{M}\mathfrak{L}t)\|u-v\|$. i.e., T_1 is a contraction.

Step 3. T_2 is completely continuous operator.

First we have to prove that T_2 is continuous on B_{λ_0}. For any $u_n,u\subset B_{\lambda_0}$, $n=1,2,3....$ with $\lim_{n\to u}\|u_n-u\|=0$, we get $\lim_{n\to u}u_n(t)=u(t)$, for $t\in[0,t_1]\cup(t_m,T]$.

Thus by $\mathbf{A_1}$, we have $\lim_{n\to\infty}f(t,u_n(t),\mathfrak{D}u_n(t))=f(t,u(t),\mathfrak{D}u(t))$ for $t\in[0,t_1]\cup(t_m,T]$.

We can conclude that

$$\sup_{s\in[0,1]}\|f(t,u_n(t),\mathfrak{D}u_n(t))-f(t,u(t),\mathfrak{D}u(t))\|\to 0 \text{ as } n\to\infty$$

On other hand, for $t\in[0,t_1]\cup(t_m,T]$

$$\begin{aligned}\|(T_2u_n)(t)-(T_2u)(t)\| &\leq \Big(\frac{1-\alpha}{B(\alpha)}-\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\|f(t,u_n(t),\mathfrak{D}u_n(t))-f(t,u(t),\mathfrak{D}u(t))\| \\ &\leq \Big(\frac{1-\alpha}{B(\alpha)}-\frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\sup_{s\in[t_m,T]}\|f(t,u_n(t),\mathfrak{D}u_n(t))-f(t,u(t),\mathfrak{D}u(t))\|\end{aligned}$$

Hence $\|(T_2u_n)(t)-(T_2u)(t)\|\to 0$ as $n\to\infty$. Therefore T_2 is continuous on B_{λ_0}.

Now, we have to show that $T_2u,u\in B_{\lambda_0}$ is relatively compact which is sufficient to prove that the function $T_2u,u\in B_{\lambda_0}$ uniformly bounded and equicontinuous, and $\forall\ t\in[0,t_1]\cup(t_m,T]$

$\|T_2u\| \leq \lambda_0$, for any $u \in B_{\lambda_0}$, therefore $(T_2u)(t), u \in B_{\lambda_0}$ is bounded uniformly.
Now, we prove that $(T_2u)(t), u \in B_{\lambda_0}$ is a equicontinuous.
For any $u \in B_{\lambda_0}$ and $0 \leq t_1 \leq t_2 \leq t$, we get

$$\begin{aligned} \|(T_2u)(t_2) - (T_2u)(t_1)\| &\leq \frac{1-\alpha}{B(\alpha)} q(t_2-t_1) + \frac{\alpha}{B(\alpha)} q(t_2-t_1)\frac{(t_2-t_1)^\alpha}{\alpha\Gamma(\alpha)} \\ &\leq q\Big(\frac{1-\alpha}{B(\alpha)} - \frac{(t_2-t_1)^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)(t_2-t_1) \end{aligned}$$

$\|(T_2u)(t_2) - (T_2u)(t_1)\| \to 0$ as $t_2 \to t_1$. Therefore, the operator T_2 is a equicontinuous on B_{λ_0}. Hence, which implies T_2 is relatively compact on B_{λ_0}.

Therefore T_2 is relatively compact subset of X by theorem 2.4. And, by theorem 2.5 we can conclude that T_2 has atleast one fixed point. Therefore the operator T has a fixed point u which is the solution of (3) and (4).

4 Nonlocal Conditions

The existence results of (3) with nonlocal condition in the form

$u|_{[-\tau^*,0]} = u_0 + p^*(0)$

The $p^*(t)$ be

$p^*(t) = \sum_{i=1}^m \lambda_1 u_i(t)$

where $u_i \in PC^1, \sum_{i=1}^m \lambda_1 < 1$ for i=1,2,3,...m and $p : C([0,1], X) \to X$ is a given function

A7 thereexist a constant $C_1 > 0$ such that $\|p^*(u) - p^*(v)\| \leq C_1\|u - v\|$
A8 Consider $\|p^*(0)\| + \sum_{k=1}^m \|I_1^* u(t_1^-)\| \leq \lambda$

Definition 4.1 *If $u(0) = u_0 + p^*(0)$ and $u \in C[0,1]$ is a solution of (3) and (4) then there is an $f \in (J \times PC^1 \times J, J)$ where $t \in [0,t_1] \cup (t_m,T], m = 1,2,.$ and*

$$v(t) = \begin{cases} v_0 + p^*(0) & t \in [\tau^*, 0] \\ v_0 + p^*(0) - g(0,u_0,\mathfrak{D}u(0)) +_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in [0,t_1] \\ v_0 + p^*(0) - g(0,u_0,\mathfrak{D}u(0)) + I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_1,t_2] \\ v_0 + p^*(0) - g(0,u_0,\mathfrak{D}u(0)) + \sum_{k=1}^2 I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_2,t_3] \\ . \\ . \\ . \\ . \\ . \\ v_0 + p^*(0) - g(0,u_0,\mathfrak{D}u(0)) + \sum_{k=1}^m I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t,u(t),\mathfrak{D}u(t)) & t \in (t_m,T] \end{cases}$$

is satisfied

Theorem 4.2 *If $\mathbf{A_1} - \mathbf{A_6}$ are satisfied and $\mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{t^\alpha}{B(\alpha)\Gamma(\alpha)}\Big)\Big)$, then the problem (3) and (4) has a solution.*

Proof. Using the hypothesis $\mathbf{A_7}$ and $\mathbf{A_8}$, consider the problem

$$v(t) = \begin{cases} v_0 + p^*(0) & t \in [\tau^*, 0] \\ v_0 + p^*(0) - g(0, u_0, \mathfrak{D}u(0)) +_0^{AB} I^\alpha f(t, u(t), \mathfrak{D}u(t)) & t \in [0, t_1] \\ v_0 + p^*(0) - g(0, u_0, \mathfrak{D}u(0)) + I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t, u(t), \mathfrak{D}u(t)) & t \in (t_1, t_2] \\ v_0 + p^*(0) - g(0, u_0, \mathfrak{D}u(0)) + \sum_{k=1}^{2} I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t, u(t), \mathfrak{D}u(t)) & t \in (t_2, t_3] \\ . \\ . \\ . \\ . \\ . \\ v_0 + p^*(0) - g(0, u_0, \mathfrak{D}u(0)) + \sum_{k=1}^{m} I_1^* u(t_1^-) +_0^{AB} I^\alpha f(t, u(t), \mathfrak{D}u(t)) & t \in (t_m, T] \end{cases}$$

By using the technique in theroem 3.2, we can easily prove that $\|\mathfrak{D}u(t))\| \leq r$ and $\|\mathfrak{D}u_1(t)) - \mathfrak{D}u_2(t))\| \leq \|u_1(t) - u_2(t)\|$ are fixed point and obtain a unique solution. Then the system () with nonlocal conditions () is relatively compact by theorem 3.3. This proof is similar to theorem 3.2 and 3.3. therefore its omitted

5 Example

Consider the following problem

$$({}_0^{ABC}D^{\frac{3}{2}})(u(t) - \frac{t}{3\sqrt{(\pi)}} sin(u(t) + u'(t)))) = \frac{t}{3\sqrt{(\pi)}} cos(u(t) + u'(t)), \tag{17}$$

$$u(t) = 1, \quad t \in [1, 2], \; B(\alpha) = 1 \tag{18}$$

Notice that $g(0, u(0), \mathfrak{D}u(0)) = f(0, u(0), \mathfrak{D}u(0)) = 0$ and $u'(t) \in C[1, 2]$ satisfy the Lipschitz conditions.
Let $g(t, u, v) = \frac{t}{3\sqrt{(\pi)}} sin(u + v), \quad f(t, u, v) = \frac{t}{3\sqrt{(\pi)}} cos(u + v), \; t \in [-\tau^*,]$.
It is easy to see that

$$({}_0^{ABC}D^{\frac{3}{2}})(u(t) - g(t, u, v)) = f(t, u, v), \tag{19}$$

$$u(0) = 1, \quad t \in [1, 2], \; B(\alpha) = 1 \tag{20}$$

Therefore, by Banach contraction principle theorem (19) and (20) has a unique solution, it can be written as
$u(t) = \lim_{n\to\infty} u_n(t)$, where

$$u_n(t) \;\; = \;\; 1 + \frac{1}{3\sqrt{\pi}} g_{n-1}(t) + \frac{1-\alpha}{3\sqrt{\pi}} f_{n-1}(t) + \frac{\alpha}{3\sqrt{\pi}\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; f_{n-1}(s) ds,$$

where $n = 1, 2, 3, ...$
Solving (19) and (20), we apply the method proposed by Mekkaoui and Atangana in [38], utilizing from the two-step Lagrange polynomial interpolation.

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