Research Article

Inequalities for L_p Quasi Norm with Application in Number Theory

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Abstract: Some author used the number theory in functional analysis and estimated some upper bounds for functions in L_1 and L_∞ spaces. Here we denoted our work to prove some multiplicative inequalities for functions in L_p, $0 < p < 1$ quasi normed spaces, that we can considered that the L_1 and L_∞ as special cases of our case. These inequalities in analysis and number theory.

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1. Introduction

We introduce a method for estimating lower pound of L_p space of the exponential sums. In our method we use multiplicative inequalities for function belongs to the Hardy space. In our prove for multiplicative inequalities we used the modern approach of the little wood- Paley theory [3 -6]. As we see in Lemma 3.1. In [1]and [9] Zygmund introduce a type of classical little wood –Paley theory for functions in L_1- normed space. In [3] proved same theorem for functions in L_∞ space. Here we expansion their theorems to the space L_p for $0 < p < 1$. We apply our multiplicative inequality for L_p, $p < 1$ for problem to the classical integrabil in terms of trigonometric series. The problems as follows if $\{\mu_n\}$ is spectrum, find a condition on the moduli of coefficients c_n under which the L_p quasi norm of the partial sums of trigonometric series increase unboundedly.

Definitions 1.1.

A spectrum $\{m_n\}_{n=1}^{N}$ has power density if, for some constants $1 < B_1 \leq 2$ and $B_1 \leq B_2 < \infty$ and all $0 < q < 1$. The following relation holds $(\lambda_1 = 1)$:

$$B_1\lambda_q \leq \lambda_{q+1} \leq B_2\lambda_q.$$

Definitions 1.2. ($\ll$)

1. Let f and g be two function in L_p space such that
$$f(x) \ll g(x) \text{ if and only if } \lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$$
2. It is well known if $p < q$, Then
$$\|f\|_p \leq \|f\|_q$$

2. Auxiliary Lemmas

In this section we give some auxiliary lemma that we need in our research.

Using the same lines used in chapter III $4 in [3] we can get the following Lemma:

Lemma 2.1

For any $0 < p < 1$ and any sequence $\{\mu_n\}$ of positive numbers such that $A^{-1}\mu_n \leq \mu_{n+1} \leq A\mu_n$, where $A \geq 1$, the following inequalities hold:

$$\|F\|_{H^1} sup_n\left(\mu_n \sum_{q=0}^{n} \mu_n\left(\frac{\|\delta_q\|_q^q}{\|\delta_q\|_p^p}\right)^{\frac{2}{p-2}}\right)^{\frac{1}{p}} \gg \sum_{n=0}^{\infty}\mu_n\|\delta_n\|_p^p \qquad (2.1)$$

$$\|F\|_{H^1} sup_n\left(\mu_n \sum_{q=n}^{\infty} \mu_n\left(\frac{\|\delta_q\|_q^q}{\|\delta_q\|_p^p}\right)^{\frac{2}{p-2}}\right)^{\frac{1}{p}} \gg \sum_{n=0}^{\infty}\mu_n\|\delta_n\|_p^p \qquad (2.2)$$

Where H^1 is Hardy space and F is a regular function on the open unit disk belongs to the Hardy space H^1.

Lemma (2.2.) Parceval's equality) [7]

If X is a separable normed space with a a scalar product. If $\|.\|$ is the corresponding nom and if $\{e_n\}$ is an orthogonal system in X, and $e_n \neq 0, n = 1,2, \ldots$ then Parceval's equality for an element $x \in X$ is

$$\|x\|^2 = \sum_{n=1}^{\infty} 1^{\infty} |a_n|^2 \|e_n\|^2,$$

where $a_n = (x, e_n)/(e_n, e_n), n = 1,2, \ldots$ are the Fourier coefficients of x in the system $\{e_n\}$. If $\{e_n\}$ is orthonormal, then Parceval's equality has the form

$$\|x\|^2 = \sum_{n=1}^{\infty} 1^{\infty} |a_n|^2.$$

Lemma 2.3. [2]

(i) $M = [log_2\, m_N\,]$.

(ii) $\lambda_q = card\ \{n: 2^{q-1} < m_n \leq 2^q\}$.

(iii) $\Lambda_q = card\{n: 1 < m_n \leq 2^q\}$, where $1 \leq q \leq M$.

(iv) Let $R(q)$denote the number of solutions of the equation.

$$m_{n_1} + m_{n_2} = m_{n_3} + m_{n_4}, \text{ where } 2^{q-1} \leq m_{n_j} \leq 2^q.$$

for *j*= 1,2,3,4.

Lemma 2.4.

Let $f \in H^p$ for $0 < p < 1$, then

$$\left\| \left(\sum_{n=0}^{\infty} |\Delta_n(f)|^p \right)^{\frac{1}{p}} \right\|_p \ll \|f\|_{H^p} \ll \left\| \left(\sum_{n=0}^{\infty} |\Delta_n(f)|^p \right)^{\frac{1}{p}} \right\|_p$$

Where $\Delta_n(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(e^{ix}) Q_n(x-u) du.$

Proof

1. We prove

$$\|f\|_{H^p} \leq \left\| \left(\sum_{n=0}^{\infty} |\Delta_n(f)|^p \right)^{\frac{1}{p}} \right\|_p$$

$$\|f\|_{H^p} = \sup_{r<1} \left(\int_0^{2\pi} \left| re^{ix} \right|^p dx \right)^{\frac{1}{p}}$$

$$= \sup_{r<1} \left(\int_0^{2\pi} \left| \sum_{n=0}^{\infty} \Delta_n(f,x) \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq c(p) \sup_{r<1} \left(\int_0^{2\pi} \left| \left(\sum_{n=0}^{\infty} |\Delta_n(f,x)|^p \right)^{\frac{1}{p}} \right| dx \right)^{\frac{1}{p}} = \left\| \left(\sum_{n=0}^{\infty} |\Delta_n(f)|^p \right)^{\frac{1}{p}} \right\|_p$$

Now

$$\|(\sum_{n=0}^{\infty} |\Delta_n(f)|^p)^p\|_p$$

$$= \left\| \left(\sum_{n=0}^{\infty} \left| \frac{1}{\pi} \int_0^{2\pi} f(e^{ix}) Q_n(x-u) du \right|^p \right)^{\frac{1}{p}} \right\|_p$$

$$\leq c(p) \sup_{r<1} \left\| \left(\int_0^{2\pi} \left| f(re^{ix}) \right|^p dx \right)^{\frac{1}{p}} \right\|_p$$

$$= c(p) \|f\|_{H^p}$$

Lemma 2.5. [8]

(i) Let $\{V_n\}_{n=1}^{\infty}$ is de la Valle poussin kernel's, i.e

$$V_n(x) = 2k_{2k-1}(x) - k_{n-1}(x) \qquad (1)$$

Where $\{k_n\}_{n=0}^{\infty}$ is Fejer kernels. we denote $Q_0(x) = V_1(x)$, and for $n \geq 1$

$$Q_n(x) = V_{2^n}(x) - V_{2^{n-1}}(x). \qquad (2)$$

(ii) Let $f(z) \in H^p$. Expanding $f(e^{ix})$ in a de la Valle poussin series, we have

$$f(e^{ix}) \sim \sum_{n=0}^{\infty} \Delta_n(f, x), \qquad (3)$$

Where

$$\Delta_n(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(e^{iu}) Q_n(x-u) du. \qquad (4)$$

(iii) Consider the total exponential sum

$$S_N(x) = \sum_{n=1}^{2^N} C_n e^{inx}$$

Where C_n are complex coefficients, which is the boundary values of the polynomial

$$S_N(z) = \sum_{n=1}^{\infty} C_n z^n, \ C_n = 0 \quad for\ n > 2^N,$$

And write $S_N(x)$ as

$$S_N(x) = \sum_{n=0}^{\infty} \delta_n(x) \qquad (5)$$

3. Main Results

In this section we give our main results.

Theorem 4.1

Assume that $\{m_n\}$ is a spectrum of power density and $R(q)$ are numbers satisfying the inequality

$$R(q) \ll \lambda_q^B (log\lambda_q)^{\gamma}, \qquad (1.1)$$

Where $2 \leq B \leq 3, \gamma \leq 0\ for\ B = 3$ and $\gamma \geq 0\ for\ B = 2$. Suppose that the coefficients C_n satisfy the relation.

$$|C_n| \approx n^{\frac{1}{2}(B-3)} (logn)^s. \qquad (1.2)$$

1. If $2 < B \leq 3$ and $s + 1 \geq \gamma/2$, Then

$$\left\| \sum_{n=1}^{N} C_n e^{im_n x} \right\|_p = \left(\int_0^{2\pi} \left| \sum_{n=1}^{N} C_n e^{(im_n x)} \right|^p dx \right)^{\frac{1}{p}} \gg \begin{cases} (logN)^{s+1-\frac{\gamma}{2}} & for\ s+1 > \frac{\gamma}{2} \\ \log\log N & for\ s+1 = \frac{\gamma}{2} \end{cases} \qquad (1.3)$$

2. If $B = 2$ and $1 + 2s \geq \gamma$, Then

$$\left\|\sum_{n=1}^{N} C_n\, e^{im_n x}\right\|_p = \left(\int_0^{2\pi}\left|\sum_{n=1}^{N} C_n e^{(im_n x)}\right|^p dx\right)^{\frac{1}{p}} \gg$$

$$\begin{cases} (logN)^{\frac{1}{2}(1+2S-\gamma)} & for\ 1+2s>\gamma, \\ \left(\frac{1}{(logN)^{\gamma}}\sum_{n=1}^{logN} n^{2s}\right)^{\frac{-1}{2}} \log\log N & for\ 1+2s=\gamma \end{cases} \quad (1.4)$$

Proof

1. For $2 < B \leq 3$, we use the multiplier.
$$\mu_n = \lambda_n^{2-B}\, n^{1-\gamma}. \quad (1.5)$$

It follows from $B_1\lambda_q \leq \lambda_{q+1} \leq B_2\lambda_q$ in definition (2.1) that $\{\mu_n\}$ satisfies the conditions of Lemma 2.1.

To prove the theorem, we need two –sided estimates for $\|\delta_q\|_p^p$ and an upper estimate for $\|\delta_q\|_q^q$.

By use Lemma 2.2 and relations (i) –(iii) in Lemma 2.3 and (1.2), we have

$$\lambda_q \frac{(log\Lambda_q)^{2s}}{\Lambda_q^{3-B}} \ll \|\delta_q\|_p^p \ll \lambda_q \frac{(log\Lambda_{q-1})^{2s}}{\Lambda_{q-1}^{3-B}} \quad (1.6)$$

Applying estimate (1.1) for the number of solutions of equation (iv) in Lemma 2.3, we fined

$$\|\delta_q\|_p^p \leq \|\delta_q\|_q^q \ll R(q) \max_{2^{q-1}<\mu_n\leq 2^q} |C_n|^4 \ll \lambda_q^B (log\lambda_q)^{\gamma}.\Lambda_{q-1}^{2(B-3)}.(\log\Lambda_{q-1})^{4s} \quad (1.7)$$

By the hypothesis of our theorem, we have the following relation in definition 2.1.

$$B_1^{q-1} \leq \lambda_q \leq 2^q, \quad (1.8)$$

$$\Lambda_q \leq B_2\lambda_q \sum_{s=1}^{q} B_2^{-S} \leq \frac{B_2^p}{B_2-1}\,\lambda_q, \quad (1.9)$$

$$M \approx \log N \quad (1.10)$$

Thus in definition 2.1. and (1.6)-(1.9).

$$\|\delta_q\|_p^p \approx \lambda_q^{B-p}.q^{2s}, \quad (1.11)$$

$$\|\delta_q\|_q^q \ll \lambda_q^{3B-6}.\, q^{qs+\gamma} \quad (1.12)$$

Using relations (i) in Lemma 2.3 and (1.5), (1.11), and (1.12) and taking into account the inequality $s+1 \geq \frac{\gamma}{2}$, we conclude that

$$\sup_{n\leq M}\left(\mu_n \sum_{q=1}^{n} \mu_q \frac{\|\delta_q\|_q^q}{\|\delta_q\|_p^p}\right)^{\frac{1}{p}} \ll \sup_{n\leq M}\left(\lambda_n^{2-B}.n^{1-\gamma}.\sum_{i=0}^{n}\lambda_q^{B-2}.q^{2s+1}\right)^{\frac{1}{p}} \ll \mathrm{M}^{s+1-\frac{\Upsilon}{2}} \quad (1.13)$$

At the same time, we have

$$\left\|\sum_{n=1}^{\mathrm{M}}\mu_n\|\delta_n\|_p^p\right\|_p \gg \sum_{n=1}^{\mathrm{M}} n^{2s+1-\Upsilon} \gg \begin{cases} \mathrm{M}^{2(s+1)-\Upsilon} & if\ s+1>\frac{\Upsilon}{2}, \\ \log\mathrm{M} & if\ s+1=\frac{\Upsilon}{2} \end{cases} \quad (1.14)$$

Applying inequality (2.1) of Lemma 2.1. using estimates (1.13) and (1.14), and taking (1.10) into account. We have (1.3).

2. For B=2, we take the multiplier
$$\mu_n = n^{-\gamma}. \quad (1.15)$$

Which satisfies the conditions of Lemma 2.1 as well. By virtue of (1.11), and (1.12). we have

$$\sup_{n\leq M}\left(\mu_n \sum_{q=1}^{n}\mu_q \frac{\|\delta_q\|_q^q}{\|\delta_q\|_p^p}\right)^{\frac{1}{p}} \ll \begin{cases} \mathrm{M}^{1+2s-\Upsilon} & if\ 1+2s>\Upsilon \\ \left(\frac{1}{\mathrm{M}^{\Upsilon}}\sum_{n=1}^{\mathrm{M}} n^{2s}\right)^{\frac{1}{p}} & if\ 1+2s=\Upsilon \end{cases} \quad (1.16)$$

Again applying (1.11) and taking (1.15) into account, we obtain the estimate

$$\left\|\sum_{n=1}^{M} \mu_n \|\delta_n\|_p^p\right\|_p \gg \begin{cases} M^{1+2s-\Upsilon} & if\ 1+2s > \Upsilon \\ logM & if\ 1+2s = \Upsilon \end{cases} \quad (1.17)$$

Combining (1.16) and (1.17) with inequality (2.2) from Lemma 2.1 and applying (1.10), we conclude that (1.4) holds.

Theorem 4.2

For any function $f \in L_p\ (0, 2\pi), q, p < 1$.

$$\|f\|_q\ \|\{\Delta_n(f)\}\|_\infty \gg \|f\|_p^p\ .$$

Proof

for $n \geq 1$, The terms of the de la Valle-Poussin expansion of a 2π- periodic real –valued function $f \in L_{p,}$ we denoted by $\Delta_n(f, x)$, such that

$$\Delta_n(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(u) Q_n(x-u) du.$$

1. If the case of $p < q$ impels that

$$\|f\|_p \leq \|f\|_q$$

Since $(\Delta_n(f))$ is bounded so there exist $c > 0$ such that

$$\|f\|_p \leq c\ \|\Delta_n(f)\|_\infty\ \|f\|_q.$$

2. If $q < p$ impels that

$$\|f\|_p \leq \|\Delta_n(f)\|_\infty$$
$$\leq c\ \|\Delta_n(f)\|_\infty\ \|f\|_q.$$

Theorem 4.3

Assume that a trigonometric polynomial

$$P_{2^N}(x) = \sum_{n=0}^{N} \delta_n(x)$$

Satisfies the condition

$$\|\delta_n\|_p \leq 1\ for\ all\ n = 0,1, \dots, N$$

Then

$$\sqrt{N}\ \left\|P_{2^N}\right\|_p \gg \left\|P_{2^N}\right\|_q^q$$

Where $p < q$.

Proof

$$\left\|P_{2^N}\right\|_q^q = \left\|\sum_{n=0}^{N} \delta_n(x)\right\|_q^q$$
$$\leq \sum_{n=0}^{N} \|\delta_n(x)\|_q^q$$

Where $\delta_n(x)$ is the dyadic blobs of the power series of the function $F(re^{inx})\ for\ r = 1$

Since

$$\|\delta_n\|_p \le 1$$
$$\le N\|P_{2^N}\|_p$$

Theorem 4.4

Suppose that $\{\mu_n\}$ be a sequence of positive numbers satisfy the condition

$$\mu_{n+1} \le \mu_n \le A\mu_{n+1}\ , \quad n = 0, \dots, N-1$$

And for some $B \ge 1$,

$$\|\delta_n\|_q \le \|\delta_{n+1}\|_q \le \mathrm{B}\|\delta_n\|_q, \quad n = 0,1, \dots, N-1 \qquad (1)$$

Then, for $1 < p < \infty$, the inequality hold

$$\|S_N\|_p \max_{0\le n\le N}\left(\mu_n \sum_{m=0}^{n} \mu_m \left(\frac{\|\delta_m\|_p^p}{\|\delta_m\|_q^q}\right)^{\frac{q}{p-q}}\right)^{\frac{1}{q}} \gg C(A,B,p)\sum_{n=0}^{N} \mu_n \|\delta_n\|_q^q \qquad (2)$$

Proof

From (1), (2) in (i) and (4) in (ii) and (5) in (iii) from Lemma 3.5 it follows that

$$S_N(x) = \sum_{n=0}^{N} \Delta_n(S_N, x) \qquad (3)$$

For any $f \in L_p$, we denoted

$$\Delta_n^*(f,x) = \Delta_{2n-1}(f,x) + \Delta_{2n}(f,x), \qquad n = 1,2, \dots \qquad (4)$$

We have

$$S_N(x) = \Delta_0(S_N, x) + \sum_{n=1}^{\left[\frac{N}{2}\right]} \Delta_n^*(S_N, x).$$

We introduce the sets $E_n \subseteq [0,2\pi), \quad n = 1, \dots \left[\frac{N}{2}\right]$. We put

$$E_n = \left\{ |\Delta_n^*(S_N,x)| \le \alpha(B,p)\left(\frac{\|\delta_{2n-1}\|_p^p + \|\delta_{2n}\|_p^p + \|\delta_{2n+1}\|_p^p}{\|\delta_{2n}\|_q^q}\right)^{\frac{1}{p-q}} \right\} \qquad (5)$$

Where $\alpha(B,p)$ is some positive number and $0 < q < 1$.

Let use prove that $\alpha(B,p)$ can be chosen in such a way that for all $n = 1, \dots \left[\frac{N}{2}\right]$. the inequality

$$\int_0^{2\pi} |\Delta_n^*(S_N,x)|^q dx \le \frac{16B^2}{5} \int_{E_n} |\Delta_n^*(S_N,x)|^q dx \qquad (6)$$

We set $CE_n = [0,2\pi)\backslash E_n$. Since $p > q$ we get

$$\int_{CE_n} |\Delta_n^*(S_N,x)|^q dx \le |CE_n|^{\frac{p-q}{p}} \left(\int_{CE_n} |\Delta_n^*(S_N,x)|^p\, dx\right)^{\frac{q}{p}} \qquad (7)$$

Applying the chebysheve inequality to estimate the measure of the set CE_n, from (3)-(5) we obtain that

$$|CE_n|(\alpha(B,p))^q \left(\frac{\|\delta_{2n-1}\|_p^p + \|\delta_{2n}\|_p^p + \|\delta_{2n+1}\|_p^p}{\|\delta_{2n}\|_q^q}\right)^{\frac{q}{p-q}} \le \int_0^{2\pi} |\Delta_n^*(S_N,x)|^q\, dx \le$$
$$\le \|\delta_{2n-1}\|_q^q + \|\delta_{2n}\|_q^q + \|\delta_{2n+1}\|_q^q \qquad (8)$$

Taking into account (1), from this we derive

$$|CE_n| \le 3\left(\frac{B}{\alpha(B,p)}\right)^q \|\delta_{2n}\|_q^{\frac{qp}{p-q}} \left(\|\delta_{2n-1}\|_q^q + \|\delta_{2n}\|_q^q + \|\delta_{2n+1}\|_q^q\right)^{\frac{q}{q-p}} \qquad (9)$$

At the same time, proceeding from the definition of the operator $\Delta_n^*(f)$ and applying Young's inequality for convolution, we have

$$\int_{CE_n} |\Delta_n^*(S_N,x)|^p dx \le \|\Delta_n^*(\delta_{2n-1}+\delta_{2n}+\delta_{2n+1})\|_p^p \le$$
$$\le \left(\frac{1}{\pi}\|Q_{2n-1}\|_p + \frac{1}{\pi}\|Q_{2n}\|_p\right)^p \left(\|\delta_{2n-1}\|_p + \|\delta_{2n}\|_p + \|\delta_{2n+1}\|_p\right)^p \le$$
$$\le 3^{p-1}.12^p \left(\|\delta_{2n-1}\|_p^p + \|\delta_{2n}\|_p^p + \|\delta_{2n+1}\|_p^p\right) \qquad (10)$$

As a result, we obtain the following estimate:

$$\int_{CE_n} |\Delta_n^*(S_N,x)|^q dx \le$$
$$\le \left(\frac{\sqrt{3}B}{\alpha(B,p)}\right)^{\frac{2(p-2)}{p}} \|\delta_{2n}\|_q^q \left(\|\delta_{2n-1}\|_p^p + \|\delta_{2n}\|_p^p + \|\delta_{2n+1}\|_p^p\right)^{\frac{-q}{p}} \left(\int_{CE_n} |\Delta_n^*(S_N,x)|^P dx\right)^{\frac{q}{p}} \le$$
$$\le 6^4 \left(\frac{\sqrt{3}\,B}{\alpha(B,p)}\right)^{\frac{2(P-2)}{P}} \|\delta_{2n}\|_q^q \qquad (11)$$

Moreover, for $n = 1, \dots, \left[\frac{N}{2}\right]$, the inequality holds

$$\|\delta_{2n}\|_q^q \le \int_0^{2\pi} |\Delta_n^*(S_N,x)|^q \, dx \le c(p)\|\delta_{2n}\|_q^q \qquad (12)$$

Believing now $\alpha(B,p) = \sqrt{3}B.12^{\frac{2p}{p-2}}$ (13)

We conclude that

$$\int_{E_n} |\Delta_n^*(S_N,x)|^q dx = \int_0^{2\pi} |\Delta_n^*(S_N,x)|^q dx - \int_{CE_n} |\Delta_n^*(S_N,x)|^q dx \ \ge$$
$$\ge \frac{15}{16}\|\delta_{2n}\|_q^q \ge \frac{5}{16B^2}\int_0^{2\pi} |\Delta_n^*(S_N,x)|^q dx.$$

This proves inequality(6).

Furthermore, using (7) and (3) and taking into account the first of relations (1) in (i) from Lemma 3.5, we have

$$\sum_{n=0}^{N-1} \mu_n \|\delta_n\|_q^q \le \mu_0 \|\Delta_0(S_N)\|_q^q + 2\sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} \|\Delta_n^*(S_N)\|_q^q. \qquad (14)$$

We put

$$g_n(x) = f(x) = \begin{cases} |\Delta_n^*(S_N,x)| & if \ \ x \in E_n \\ 0 & if \ x \in [0,2\pi)\backslash E_n \end{cases} \qquad (15)$$

Applying inequality (6), in view of (15) we obtain

$$\sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} \|\Delta_n^*(S_N)\|_q^q \le \frac{16}{5} B^2 \sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} \int_{E_n} |\Delta_n^*(S_N,x)|^q dx$$
$$= \frac{16}{5} B^2 \sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} \int_0^{2\pi} g_n^q(x) dx \qquad (16)$$

For any $x \in [0,2\pi)$, the inequality

$$\left(\sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} g_n^q(x)\right)^q \le 2\left(\sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} g_n^q(x)\right)\left(\sum_{m=1}^{n} \mu_{2n-1} g_n^q(x)\right)^{q^{\frac{q}{2}}} \le$$
$$\le 2 \max_{1\le n\le [\frac{N}{2}]} \left(\mu_{2n-1} \sum_{m=1}^{n} \|g_m\|_\infty^q\right) \sum_{n=1}^{[\frac{N}{2}]} g_n^q(x) \qquad (17)$$

Therefor

$$\sum_{n=1}^{[\frac{N}{2}]} \mu_{2n-1} \|\Delta_n^*(S_N)\|_q^q \leq \frac{16\sqrt{2}}{5} B^2 \max_{1\leq n\leq[\frac{N}{2}]} \left(\mu_{2n-1} \sum_{m=1}^{n} \mu_{2m-1} \|g_m\|_\infty^q\right)^{\frac{1}{q}} \int_0^{2\pi} \left(\sum_{n=1}^{[\frac{N}{2}]} g_n^q(x)\right)^{\frac{1}{q}} dx \leq$$

$$\leq \frac{16\sqrt{2}}{5} B^2 \max_{1\leq n\leq[\frac{N}{2}]} \left(\mu_{2n-1} \sum_{m=1}^{n} \mu_{2m-1} \|g_m\|_\infty^q\right)^{\frac{1}{q}} \int_0^{2\pi} \left(\sum_{n=1}^{[\frac{N}{2}]} |\Delta_n^*(S_N,x)|^q\right)^{\frac{1}{q}} dx \qquad (18)$$

From relation (1), (5), (15) we obtain that for $m = 1, \dots, [\frac{N}{2}]$

$$\|g_m\|_\infty^q \leq (\alpha(B,p))^q \left(\frac{\|\delta_{2m-1}\|_p^p + \|\delta_{2m}\|_p^p + \|\delta_{2m+1}\|_p^p}{\|\delta_{2m}\|_q^q}\right)^{\frac{q}{p-q}} \leq$$

$$\leq (\alpha(B,p) B^{\frac{q}{p-q}})^q \max\left(1, 3^{\frac{4-p}{p-q}}\right)\left(\left(\frac{\|\delta_{2m-1}\|_p^p}{\|\delta_{2m-1}\|_q^q}\right)^{\frac{q}{p-q}} + \left(\frac{\|\delta_{2m}\|_p^p}{\|\delta_{2m}\|_q^q}\right)^{\frac{q}{p-q}} \left(\frac{\|\delta_{2m+1}\|_p^p}{\|\delta_{2m+1}\|_q^q}\right)^{\frac{q}{p-q}}\right) \qquad (19)$$

Hence

$$\max_{1\leq n\leq[\frac{N}{2}]} \mu_{2n-1} \sum_{m=1}^{n} \mu_{2m-1} \|g_m\|_\infty^q \ll \max\left(1, 3^{\frac{4-p}{p-q}}\right) A^4 \left(\alpha(B,p) B^{\frac{q}{p-q}}\right)^q \max_{1\leq n\leq N} \mu_n \sum_{m=1}^{n} \mu_m \left(\frac{\|\delta_m\|_p^p}{\|\delta_m\|_q^q}\right)^{\frac{q}{p-q}} \qquad (20)$$

Combining inequalities (14),(18),(20) and using (13), as a result we have

$$\sum_{n=0}^{N-1} \mu_n \|\delta_n\|_q^q \ll \mu_0 \|\Delta_0(S_N)\|_q^q + 6^{\frac{4p}{p-q}} A^q B^{\frac{3P-4}{P-q}} \max_{0\leq n\leq N} \left(\mu_n \sum_{m_0}^{n} \mu_m \left(\frac{\|\delta_m\|_p^p}{\|\delta_m\|_q^q}\right)^{\frac{q}{p-q}}\right)^{\frac{1}{q}} \int_0^{2\pi} \left(\sum_{n=1}^{[\frac{N}{2}]} |\Delta_n^*(S_N,x)|^q\right)^{\frac{1}{q}} dx \qquad (21)$$

But by Lemma 3.4 we obtain

$$\int_0^{2\pi} \left(\sum_{n=1}^{[\frac{N}{2}]} |\Delta_n^*(S_N,x)|^q\right)^{\frac{1}{q}} dx \leq \sqrt{2} \int_0^{2\pi} \left(\sum_{n=0}^{N} |\Delta_n(S_N,x)|^q\right)^{\frac{1}{q}} dx \ll \|S_N\|_p \qquad (22)$$

To estimate the last term on the right –hand said of inequality (2), we use relations (3),(7) and (10). We get $\|\delta_N\|_q^q \leq \|\Delta_{N-1}(S_N) + \Delta_N(S_N)\|_q^q \leq$

$$\leq \left(\frac{\|\Delta_{N-1}(S_N) + \Delta_N(S_N)\|_p^p}{\|\delta_N\|_q^q}\right)^{\frac{1}{p-q}} \|\Delta_{N-1}(S_N) + \Delta_N(S_N)\|_p \ll$$

$$\ll \frac{6p}{2^{p-q}} \left(\left(\frac{\|\delta_{N-1}\|_p^p}{\|\delta_{N-1}\|_q^q}\right)^{\frac{1}{p-q}} + \left(\frac{\|\delta_N\|_p^p}{\|\delta_N\|_q^q}\right)^{\frac{1}{p-q}}\right) \|S_N\|_p \qquad (23)$$

It follows that from inequalities (21)-(23) we get

$$\sum_{n=0}^{N} \mu_n \|\delta_n\|_q^q \ll 6^{\frac{4p}{p-q}} A^2 B^{\frac{3p-4}{p-q}} \max_{0\leq n\leq N} \left(\mu_n \sum_{m=0}^{n} \mu_m \left(\frac{\|\delta_m\|_p^p}{\|\delta_m\|_q^q}\right)^{\frac{q}{p-q}}\right)^{\frac{1}{q}} \|S_N\|_p \quad (24)$$ Etting

$$C(A,B,p) = 6^{\frac{4p}{q-p}} A^{-2} B^{\frac{3p-4}{q-p}},$$

We conclude that theorem is completely proved.

Corollary 4.1. 1

Suppose that $f \in L_p$ such that

$f(z) = \sum_{n=1}^{\infty} c_n z^n$, and $0 < p < \infty$, for some $B \geq 1$ satisfy

$$\sum_{n=0}^{\infty} \left(\frac{\|\delta_n\|_p^p}{\|\delta_n\|_q^q}\right)^{\frac{q}{p-q}} < \infty$$

Then the inequality is hold
$\|\delta_{n+1}\|_q \leq \|\delta_n\|_q \leq B\|\delta_{n+1}\|_q$, for $0 < q < 1$ and $n = 0,1, \dots$

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