Research Article

Semi LocallS-LifitingMod.s

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Abstract: To consider $\mathcal{R}$ is a commutative ring with unity, $\mathcal{M}$be a nonzero unitary left $\mathcal{R}$-mod., $\mathcal{M}$is known semi lifitingmod. if for every subm. N from$\mathcal{M}$, ∃subm. A from N so that $M = A\oplus B$while$N \cap B$ is semi small subm.from B.Semi Locall s-lifitingmod. is a strong form from semi lifitingmod., where an $\mathcal{R}$-mod.$\mathcal{M}$is known semi locall s-lifitingmod. if $\mathcal{M}$has aunique semi max.subm. N whileallsubm. B from$\mathcal{M}$, ∃subm. A from N so that $\mathcal{M} = A\oplus B$while N ∩ B is semi small subm.from B.The current study deals with this class frommod.while give several fundamental properties related with this concept.

1. Introduction

Throughout the following convenient$\mathcal{R}$represents a commutative ring with identity, whileall$\mathcal{R}$-mod. are left until .[In this paper we denoted module by(mod.) and submodule by (subm.)and there exists by(∃) and finitely generated by(f.g)].A convenientsubm. A from an $\mathcal{R}$-mod.$\mathcal{M}$is known a small if A+B≠ $\mathcal{M}$for allconvenientsubm. B from$\mathcal{M}$[1].A nonzero mod.$\mathcal{M}$ known semi lifitingmod. if for every subm. N from$\mathcal{M}$,∃subm. A from N so that $\mathcal{M} = A\oplus B$while$N \cap B$semi small subm.from B[2] .A convenientsubm. N from an $\mathcal{R}$-mod.$\mathcal{M}$ known semi max.subm. in $\mathcal{M}$, if A is subm.from$\mathcal{M}$, with N is a convenientsubm.from A, so A = $\mathcal{M}$[3].An$\mathcal{R}$-mod.$\mathcal{M}$ known semi locall if $\mathcal{M}$has aunique semi max.subm. which contains allconvenientsubm.from$\mathcal{M}$[1].In this paper, we give a strong form from semi lifitingmod., we call it semi localls-lifitingmod. which is a mod. has aunique semi max.subm. N whileallsubm. B from$\mathcal{M}$,∃subm. A from N so that $\mathcal{M} = A\oplus B$while$N \cap B$ is a semi small subm.from B,this work contains three sections .In section one, we give the definition from semi localls-lifitingmod., we investigate the properties from this class frommod. In section two we investigate some conditions under which semi lifitingmod.while semi localls-lifitingmod. are equivalent[2]. The third section investigate the relation between the semi localls-lifitingmod.while other mod.all as amply supplemented[3], indecompsiblemod.while hollow mod.,[7].

.1- Semi locall S-lifitingmod

<u>Definition 1-1:</u>

An $\mathcal{R}$-mod.$\mathcal{M}$is said to be semi locall S-lifiting if $\mathcal{M}$hasaunique semi max.subm. N from$\mathcal{M}$,∃subm.s A from N while B from$\mathcal{M}$so that$\mathcal{M} = A\oplus B \ and \ N \cap B$ semi small subm.from B .

<u>Remark1-2:</u>

If $\mathcal{M}$is a mod. .Then $\mathcal{M}$is semi locall S-lifiting if while only if for every subm. N from$\mathcal{M}$,∃subm. A from N so that $\mathcal{M} = A\oplus B$to somesubm. B from$\mathcal{M}$while$N \cap B$ is a semi small subm.from$\mathcal{M}$.

<u>Examples 1-3:</u>

the Z_4 as Z-mod. is semi locall S-lifitingmod. because has one semi max.subm. $(\bar{2})$ while $(\bar{2})+(\bar{2})=Z_4$, $(\bar{2})\cap(\bar{2})$ is small subm.from $(\bar{2})$. But The Z-mod. Z_6 is not semi locall S-lifiting because Z_6 has two semi max. $(\bar{2})$ while $(\bar{3})$.

<u>Examples while Remarks 1-4:</u>

1- $Z_2\oplus Q$ is semi locall S-lifiting , but not semi locallmod. .Because$\{0\}\oplus Q$ is aunique semi max.subm.from"$Z_2\oplus Q$while$\{0\}\oplus\{0\}$ is a semi small subm.from$Z_2\oplus Q$while contained in$\{0\}\oplus Q$ .But $Z_2\oplus\{0\}$ a convenientsubm.from$Z_2\oplus Q$, but $Z_2\oplus\{0\}$ is not contain in $\{0\}\oplus Q$" .

2- Every semi locall S-lifitingmod. is semi lifitingmod..

3- Every semi locallmod. is semi locall S-lifitingmod.,but the convers is not true in general as we see in ex(1).

4-Every simple mod. is not semi locall S-lifitingmod..For example The Z-mod. Z_7 is simple mod. ,but not semi locall S-lifitingmod.. While every semi locall S-lifitingmod."is not simple mod..For example The Z-mod". Z_4 is semi locall S-lifitingmod. .But not simple mod..

<u>Proof:</u>

Suppose that $\mathcal{M}$is semi locall S-lifitingmod.Then,∃subm. A from N while B is subm.from$\mathcal{M}$so that$\mathcal{M}=A\oplus B$ while$N \cap B$ is semi small subm.from B in $\mathcal{M}$.Whilebecause N is subm.from$\mathcal{M}$.Then all semi small is contains in $\mathcal{M}$. By difinition from semi lifitingmod. then N is a semi small subm.from$\mathcal{M}$then that $\mathcal{M}$is semi lifitingmod,. .Wile the convarse(Remarek(1-4)(2) " is not true in general . For example $Z_p^{\infty}$is semi lifitingmod". but it is not semi locall S-lifitingmod..

Theorem1-5:

Let"$\mathcal{M}$be a mod. . The following statements are "equialent.

1)$\mathcal{M}$ semi locall S-lifiting

2) The semi max.subm. N from M can be written as $N = A \oplus S$, where A is a dirctsumandfrom$\mathcal{M}$while S is a semi small subm.from$\mathcal{M}$.

3) ∃aunique semi max.subm. N from$\mathcal{M}$, ∃ a dirctsumand K from$\mathcal{M}$so that K is subm.from N while N/K is subm.from$\mathcal{M}$/K .

(1)⇒(2)

Let $\mathcal{M}$be a semi locall S-lifitingmod.while N is aunique semi max.subm.from$\mathcal{M}$. ∃subm. K from N so that $\mathcal{M} = K \oplus B$while$N \cap B$ is semi small subm.from B . So , $N \cap B$ is a semi small subm.from$\mathcal{M}$by [10,prop(1,5)]

Now , N= $N \cap \mathcal{M}$

= $N \cap (K \oplus B)$

= $K \oplus N \cap B$, by the Modylar Law .

We take A=K while S= $N \cap B$. Therefore , $N = A \oplus S$ with A is a dirct sumandfrom$\mathcal{M}$while S is a semi small subm.from$\mathcal{M}$.

(2)⇒(3)

Let N be subm.from$\mathcal{M}$.By (2), $N = A \oplus S$ where A is a dirctsumandfrom$\mathcal{M}$while S is a semi small subm.from$\mathcal{M}$. We are done if we can show that N/A is a semi small subm.from$\mathcal{M}$/A . To see this suppose that N/A+L/A= $\mathcal{M}$/A .Then , $(A \oplus S)/A + L/A = \mathcal{M}/A$ Therefore , $\mathcal{M}$= A+S+L =S+L , because A is subm.from L

= L ,because S is subm.from$\mathcal{M}$.

Hence, N/A is subm.from M/A .

(3)⇒ (1)

Let N be aunique semi max.subm.from .∃subm. K from N so that $\mathcal{M} = K \oplus B$while N/K is a semi small subm.from$\mathcal{M}$/K . We want to show that $N \cap B + P = B$, where P is primary $\mathcal{R}$-subm.from B . Therefore , $\mathcal{M}$= K+B= K+ $N \cap B + P$ .Thus $\mathcal{M}$/K= (K+ $N \cap B + P)/K = (N \cap B + K)/K + (K + P)/K$. Because , $(N \cap B + K)/K$ is a sumod.from N/K , then $(N \cap B + K)/K$ is a small subm.from$\mathcal{M}$/K . Therefore , (K+P)/K= $\mathcal{M}$/K . Thus , $\mathcal{M}$=K+P . Because , B is primary $\mathcal{R}$-mod.from B . Hence $N \cap B$ is a semi small subm.from B .

In the following proposition we gives some from the basic prposition from semi locall S-lifitingmod..

Prposition1-6:

Epimorphic image from semi locall S-lifitingmod. is semi locall S-lifitingmod..

Proof:

Let $\mathcal{M}_1$ be semi locall S-lifitingwhilelet $f: \mathcal{M}_1 \to \mathcal{M}_2$ be an Epimorpihic with $\mathcal{M}_2$ an $\mathcal{R}$-mod. .Suppoose this N be aunique semi max.subm.from$\mathcal{M}_2$ with N+A= $\mathcal{M}_2$ where A is a convenientsubm.from$\mathcal{M}_2$. Now , f^{-1}(N) is aunique semi max.subm.from$\mathcal{M}_1$,because other wise f^{-1}(N)= $\mathcal{M}_1$, while hence f(f^{-1}(N))=f($\mathcal{M}_1$)= $\mathcal{M}_2$ implies that N= $\mathcal{M}_2$ which is A contradction . Thus N is aunique semi max.subm.from$\mathcal{M}_2$while

f^{-1}(N) is aunique semi max.subm.from$\mathcal{M}_1$. Because$\mathcal{M}_1$ is semi locall S-lifitingmod. , therefore ∃ A is subm.fromf^{-1}(N) while B is subm.from N so that

while$f^{-1}(N) \cap B$ is semi small subm.from B . Implies that $f(A) \oplus f(B) = f(\mathcal{M}_1) = \mathcal{M}_2$ $A \oplus B = \mathcal{M}_1$ (because f is an Epimorphic), so $f(f^{-1}(N)) \cap B$ is semi small subm.from f(B) . Therefore $\mathcal{M}$is semi locall S-lifitingmod. .

The following proposition describes more propertiesfrom semi locall S-lifitingmod.s.

Propostion1-7:

Let P be a semi small subm.frommod.$\mathcal{M}$, if M/P is semi locall S-lifitingmod. then $\mathcal{M}$is semi locall S-lifitingmod..

Proof:

Suppose that $\mathcal{M}$/P is semi locall S-lifitingmod. with P is a semi small subm.from$\mathcal{M}$then ∃ aunique semi max.subm. N/P from$\mathcal{M}$/P with A+B= $\mathcal{M}$where B is subm.from$\mathcal{M}$while A is a convenientsubm.from$\mathcal{M}$, then (A+B) /P = $\mathcal{M}$/P, implies that ((A+P) /P) +((B+P) /P) = $\mathcal{M}$/P . Because (A+P) /P is a convenientsubm.from N/P while$\mathcal{M}$/P is semi locall S-lifitingmod., then (A+P) /P is a semi small subm.from$\mathcal{M}$/P. Thus (B+P)/P = $\mathcal{M}$/P, so B+P= $\mathcal{M}$, Because P is a semi small subm.from$\mathcal{M}$, ∃ V is subm.from P while∃ aunique semi max.subm. N from$\mathcal{M}$while,∃ P is subm.mod. N while V is subm.from$\mathcal{M}$so that P⊕V= $\mathcal{M}$while N∩V is a semi small subm.from V then B= $\mathcal{M}$. Therefore $\mathcal{M}$is semi locall S-lifitingmod..

Corollary 1-8:

Let$\mathcal{M}$be an $\mathcal{R}$-mod., whether $\mathcal{M}$ semi locall S-lifitingmod. then $\mathcal{M}$/N is semi locall S-lifitingmod. where N has aunique semi max.subm.from$\mathcal{M}$.

Proof:

Let $\mathcal{M}$be a semi locall S-lifitingmod. then ∃aunique semi max. N,∃ A is subm.from N while B is subm.from$\mathcal{M}$so that A⨁B= $\mathcal{M}$, while N ∩B is semi small subm.from B. Let g:$\mathcal{M} \longrightarrow \mathcal{M}$/N be the natural epimorphism then $\mathcal{M}$/N is semi locall S-lifiting by prop (1-6).

The following proposition give more propertiesfrom semi locall S-lifitingmod.s.

Propostion1-9:

Let f: $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a projective cover from $\mathcal{M}_2$, if $\mathcal{M}_2$ is semi locall S-lifitingmod. then $\mathcal{M}_1$ is semi locall S-lifitingmod..

Proof :

Let $\mathcal{M}_2$ be a semi locall S-lifiting mod.whileBecause g: $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is an epimorphism then $\mathcal{M}_1$ /kerf isomorphism to $\mathcal{M}_2$while hence it is semi locall S-lifitingmod.while kerf is a semi small subm.from$\mathcal{M}_1$. Thus by prop(1-7)implies$\mathcal{M}_1$ is semi locall S-lifitingmod..

Proposition1-10:

Let$\mathcal{M}$be f.g$\mathcal{R}$-mod. .Then$\mathcal{M}$ semi locall S-lifitingmod. if while only if$\mathcal{M}$ cyclic while has aunique semi max.subm..

Proof:

Let $\mathcal{M}$be f.g semi locall S-lifitingmod. then $\mathcal{M}$= RX_1+RX_2 ...$+RX_n$ .If $\mathcal{M} \neq RX_1$ then RX_1 is a convenient sumod.from$\mathcal{M}$which implies that RX_1 is a semi max.subm.from$\mathcal{M}$. Hence $\mathcal{M}=RX_2+RX_3+ \dots +RX_n$. So , we cancelthe sumandone until we have$\mathcal{M}=RX_i$to some i . This $\mathcal{M}$ cyclicmod.whilebecause$\mathcal{M}$semi locall S-lifitingmod. ,so $\mathcal{M}$has auniquesemi max.subm. by def(1-1).

Conversely , let$\mathcal{M}$be cyclic mod. hasaunique semi max.subm. say N , $\mathcal{M}$is f.g . Let B a convenientsubm.from$\mathcal{M}$with A⨁B= $\mathcal{M}$to somesubm. A from$\mathcal{M}$.Now, if $N \cap A$ is not semi small subm.from A implies A≠ $\mathcal{M}$. Then A is a convenientsubm.from$\mathcal{M}$while A is subm.from N whilebecause$\mathcal{M}$is f.g , then A is contained in a semi max.subm. . But by assumption $\mathcal{M}$has aunique semi max.subm. N . Thus B is subm.from N (B is contaned in N). Therfore B+N=N= $\mathcal{M}$ which contradction. Hence A= $\mathcal{M}$,B subm.fromN while$N \cap A$ semi sma1l subm.from$\mathcal{M}$. Then $\mathcal{M}$ semi locall S-lifitingmod..

If$\mathcal{M}$f.g,then M/N f.g for allsubm. N from$\mathcal{M}$. "But the converse is not true in general".

"The following proposition shows if $\mathcal{M}$" semi locall S-lifitingmod.while$\mathcal{M}$/N is f.g then $\mathcal{M}$is also f.g where N is a semi max. sumod.from M .

Proposition1-11:

Let N be a semi max.subm.from an$\mathcal{R}$-mod .$\mathcal{M}$. If$\mathcal{M}$ semi locall S-lifitingmod.while$\mathcal{M}$/N f.g then $\mathcal{M}$f.g .

Proof:

Let N be convenientsubm.from semi locall S-lifitingmod.$\mathcal{M}$with $\mathcal{M}$/N is f.g . Then $\mathcal{M}/N=R(X_2+N)+R(X_2+N)+\dots+R(X_n+N)$ where $X_i \in m$ for all i=1,2,…,n we claim that $\mathcal{M}=RX_1+RX_2+\dots+RX_n$. Let $\mathcal{M} \in \mathcal{M}$then $\mathcal{M}+N \in \mathcal{M}/N$, impliies that $\mathcal{M}+N=r_1(x_1+N)+r_2(x_2+N)+\dots+r_n(x_n+N)=r_1x_1+r_2x_2+\dots+r_nx_n+N$. This impliies that

$= r_1x_1+r_2x_2+\dots+r_nx_n+N$ .To some$N \in \mathcal{M}$.Thus $\mathcal{M} = r_1x_1+r_2x_2+\dots+r_nx_n+N\mathcal{M}$

.whilebecause . $\exists RX_i \in \mathcal{M}$to some i while$RX_i \in N_N \cap RX_i, \mathcal{M}$is semi locall S-lifitingmod. , then $N \cap RX_i$ is a semi small subm.from RX_n is semi small subm.from$\mathcal{M}$which implies that $\mathcal{M} = r_1x_1+r_2x_2+\dots+r_nx_n+N$ .Thus $\mathcal{M}$is f.g.

.2-Locall S-lifitingmod.while semi locall S-lifitingmod

Recall that an $\mathcal{R}$-mod. is locall S-lifitingmod. if $\mathcal{M}$has auniquemax.subm. N, ∃subm. A from N while B from$\mathcal{M}$[2].

In the section one we said that every semi locall S-lifitingmod. is locallS-lifitingmod. .While we give an "example shows that the converse is not true .In this section we investigate conditions under which locall S-lifitingmod". can be semi locall S-lifitingmod..

Proposition 2-1:

Let $\mathcal{M}$be an $\mathcal{R}$-mod. , $\mathcal{M}$ semi locall S-lifitingmod.if while only if $\mathcal{M}$locall S-lifitingmod.while cycilc mod..

Proof:

Suppose that $\mathcal{M}$ semi locall S-lifitingmod.then it has aunique semi maxmial sumod. N ,∃ A issubm.from$\mathcal{M}$,while B subm.from N so that $A \oplus B = \mathcal{M}$so , $N \cap A$ is a semi small subm.from A . Let $X \in \mathcal{M}$with$X \notin N$ then RX subm.from$\mathcal{M}$.we claim that, RX= $\mathcal{M}$. If RX≠ $\mathcal{M}$, then $N \cap RX$ convenient semi small subm.from$\mathcal{M}$. While hence RX is subm.from N which implies that $X \in \mathcal{M}$which contradction. Thus

RX= $\mathcal{M}$ then $\mathcal{M}$ cycilc mod. .Now ,because $\mathcal{M}$ semi locall S-lifitingmod. then $\mathcal{M}$ is locall S-lifitingmod.by Remark (1-4)(2).

Convearsely, Supose that $\mathcal{M}$ locall S-lifitingmod.while cyclic mod., then it f.gmod.while hence $\mathcal{M}$ has aunique semi max.subm. N from M . So N is a semi max.subm.from $\mathcal{M}$ by[1,remark,(2-1-2)(2)] . Because $\mathcal{M}$ is locallS-lifitingmod. thus ∃ A is subm.from $\mathcal{M}$ while B is subm.from N so that A⊕B= $\mathcal{M}$. Let A be a convenient semi small subm.from $\mathcal{M}$., A+N= $\mathcal{M}$ then A⊕N= $\mathcal{M}$ thus N= $\mathcal{M}$ which "is a contradction. This implies that every convenient" small subm.from $\mathcal{M}$ contaned in N (i. e. N∩ A semi small subm.from A), thus $\mathcal{M}$ semi locall S-lifitingmod..

<u>Corollary 2-2:</u>

Let $\mathcal{M}$ be an $\mathcal{R}$-mod., $\mathcal{M}$ semi locall S-lifitingmod. if while only if $\mathcal{M}$ locall S-lifitingwhilef.gmod..

<u>Proof:</u>

Suppose that $\mathcal{M}$ is a semi locall S-lifitingmod..Then $\mathcal{M}$ locall S-lifitingmod.while cycilc by propo (2-1), thus $\mathcal{M}$ f.g.

Convearsely, let $\mathcal{M}$ be f.glocall S-lifitingmod. then $\mathcal{M}$=$RX_1+RX_2+\ldots+RX_n$. If M≠RX_i then RX_1 is a convenientsubm.from $\mathcal{M}$ which implies that RX_1 is a semi small subm.from $\mathcal{M}$.Hence $\mathcal{M}$=$RX_2+RX_3+\ldots+RX_n$. So ,wecancel.The sumand one by one unitil we have $\mathcal{M}$=RX_i to some i . Thus $\mathcal{M}$ cycilc mod. , while by prop(2-1)implies $\mathcal{M}$ semi locall S-lifitingmod..

<u>Proposition2-3:</u>

Let $\mathcal{M}$ be an $\mathcal{R}$-mod. ,$\mathcal{M}$ semi locall S-lifitingmod.if while only if $\mathcal{M}$ locall S-lifitingmod.while has aunique semi max.subm..

<u>Proof:</u>

Supose that $\mathcal{M}$ semi locall S-lifitingmod. , then $\mathcal{M}$ by Remark(1-4)(2),while by def(1-1) $\mathcal{M}$ has aunique semi max.subm. .

Convearsely , let $\mathcal{M}$ be localllifitingmod. which has aunique semi max.subm. , "say N , we only have to show that $\mathcal{M}$"f.g . Let $X_k \in \mathcal{M}$ while $X_k \notin N$, then R_k+N= $\mathcal{M}$ whilebecause $\mathcal{M}$ is a locall S-lifitingmod.while hence $\mathcal{M}$=RX_k .Therefore $\mathcal{M}$ is f.g , while by prop(2-1). Then $\mathcal{M}$ is semi locall S-lifitingmod. .

The next proposition gives a charcteriation for semi locall S-lifitingmod..

By [1,remark(2-1-2)(2)] Every max.subm. is semi max. , but the converse is not true unless $\mathcal{M}$ is f.g .Thus ,we can get the following corollary .

<u>Corollary 2-4:</u>

Let $\mathcal{M}$ be f.g$\mathcal{R}$-mod.. Then $\mathcal{M}$ semi locall S-lifitingmod. if while only if $\mathcal{M}$ locall S-lifitingmod.while has aunique semi max.subm..

<u>Proposition2-5:</u>

Let $\mathcal{M}$ be an $\mathcal{R}$-mod. , $\mathcal{M}$ semi locall S-lifitingmod. if while only if it cyclic mod.whileevery non zero faector mod.from $\mathcal{M}$ indecompsible .

<u>Proof:</u>

Let $\mathcal{M}$ be semi locall S-lifitingmod. , then by propo (2-1). $\mathcal{M}$ locall S-lifitingmod.while cyclic mod.while by [5,prop(1-2-11)]. "Then every non-zero factor mod.from $\mathcal{M}$"indecompsible.

Convearsely , let $\mathcal{M}$ be cycilc whileevery nonzero faector mod.from $\mathcal{M}$ indecompsible ,then by [5,prop(1-2-11)]. $\mathcal{M}$ locall S-lifitingmod.while by prop(2-1).Thus $\mathcal{M}$ semi locall S-lifitingmod. .

<u>Proposition 2-6:</u>

Let $\mathcal{M}$ be an $\mathcal{R}$-mod. , $\mathcal{M}$ semi locallS-lifitingmod. if while only if $\mathcal{M}$ locall S-lifitingmod.while Rad$\mathcal{M} \neq \mathcal{M}$.

<u>Proof:</u>

Let $\mathcal{M}$ be semi locall S-lifitingmod. , then $\mathcal{M}$ locall S-lifitingwhile cycilc mod. by prop(2-1). Whilebecause $\mathcal{M}$ cycilc mod. then $\mathcal{M}$ f.gwhile hence .Rad$\mathcal{M} \neq \mathcal{M}$.

Convearsely , let $\mathcal{M}$ locall S-lifitingmod.while Rad$\mathcal{M} \neq \mathcal{M}$, then Rad $\mathcal{M}$ semi small subm.from $\mathcal{M}$ .Also by [4, lemma (1-2-13)]. Rad $\mathcal{M}$ aunique semi max.subm.from $\mathcal{M}$ while thus $\mathcal{M}$/ Rad $\mathcal{M}$ simple mod.while hence cyclic.Implies that Rad $\mathcal{M}$=< m+ Rad $\mathcal{M}$>to some $m \in \mathcal{M}$. We claim that $\mathcal{M}$=Rm . Let $w \in \mathcal{M}$ then w+Rad $\mathcal{M} \in \mathcal{M}/Rad\mathcal{M}$,while therefore there is $r \in R$ so that w+Rad$\mathcal{M}$=r (m+Rad$\mathcal{M}$)= rm +Rad$\mathcal{M}$. Implies that w-r m∈ $Rad\ m$ which implies that w-r m =y to some $y \in Rad\mathcal{M}$.Thus w=r m +y∈Rm +Rad $\mathcal{M}$, hence $\mathcal{M}$= Rm +Rad$\mathcal{M}$ .But Rad $\mathcal{M}$ semi small subm.from $\mathcal{M}$ =Rm . Thus $\mathcal{M}$ cycilc mod.while by prop(2-1). Implies $\mathcal{M}$ semi locall S-lifitingmod. .

<u>Proposition 2-7:</u>

Let $\mathcal{M}$ be semi locall S-lifitingmod. if while only if Rad$\mathcal{M}$ semi small while semi max.subm.from $\mathcal{M}$.

<u>Proof:</u>

Suppose that Rad$\mathcal{M}$ be semi small while semi max.subm. , to prove that $\mathcal{M}$ semi locall S-lifitingmod. .First , we want to show that Rad $\mathcal{M}$ aunique semi max.subm. in $\mathcal{M}$,supose that B another semi max.subm. in $\mathcal{M}$, then $\mathcal{M}$=B +Rad$\mathcal{M}$, but Rad $\mathcal{M}$ is a semi small subm. which implies that B= $\mathcal{M}$, which is a contradction .Thus Rad $\mathcal{M}$ is aunique semi max.subm. in $\mathcal{M}$. We claim ∃ A is subm.from M . Let N be a semi small subm.from $\mathcal{M}$, if N is not contained" in Rad$\mathcal{M}$, then N+Rad$\mathcal{M}$= $\mathcal{M}$, but Rad $\mathcal{M}$" semi small subm.from$\mathcal{M}$which implies that N= $\mathcal{M}$then we have contradction. Therefore $\mathcal{M}$ semi locall S-lifitingmod..

Convearsely,suppose that $\mathcal{M}$ semi locall S-lifitingmod.,then by remark(1-4) (2) , implies$\mathcal{M}$locall S-lifitingmod.while by [4,lemma.(1-2-13)].Then Rad$\mathcal{M}$is a semi max.subm.whilebecause$\mathcal{M}$is a semi locall S-lifitingmod. . Thus Rad$\mathcal{M}$aunique semi max.subm.$\mathcal{M}$, hence Rad$\mathcal{M}$+N= $\mathcal{M}$to someconvenientsubm. N from$\mathcal{M}$. If Rad$\mathcal{M}$ is not semi small subm.from$\mathcal{M}$then N semi small subm.from$\mathcal{M}$, thus Rad$\mathcal{M}$= $\mathcal{M}$which is contradction by [5, prop(1-2-14)].Hence Rad$\mathcal{M}$ semi small subm.from$\mathcal{M}$.

3- Semi Locall S-lifitingMod.while Rlation Betwen Some Other Mod.s

We" study in this section the rlation betwen semi locall" S-lifitingmod.while other mod.s so as amply supplemented mod.s, indecompsiblemod.s while hollow mod.s.

<u>Definition3-1</u>:[3]:Let $\mathcal{M}$be a mod. , then $\mathcal{M}$is said to be ampily suplemented mod. , if for any two subm. U while V from$\mathcal{M}$with U+V= $\mathcal{M}$, V contains a suplement from U in $\mathcal{M}$.

<u>Proposition 3-2:</u>

Every semi locall S-lifitingmod. is ampily suplement mod..

<u>Proof:</u>

Let $\mathcal{M}$be semi locall S-lifitingmod. let N be aunique semi max.subm.from$\mathcal{M}$.Because$\mathcal{M}$is a semi locall S-lifitingmod. , then we have N+ $\mathcal{M}$= $\mathcal{M}$while$N \cap \mathcal{M} = N$ a semi small subm.from$\mathcal{M}$. Therefore $\mathcal{M}$is amply supplemented mod..

The coniverse from propo(3-2) is" not true in general , as we see in the following "example .

<u>Example 3-3:</u>

The Z-mod. Z_{10} is ampily suplemented mod. , but is not a semi locall S-lifitingmod. , because Z_{10} has two semi max.subm.s $(\bar{2})$while $(\bar{5})$. Thus it's not semi locall S-lifitingmod..

The following proposition show the relation between semi locall S-lifitingmod.while indecomposabe mod. .

<u>Definition3-4:</u>[8] An $\mathcal{R}$-mod.$\mathcal{M}$is indecomposible if M≠0 while the only a dirctsumands from$\mathcal{M}$are $\langle\bar{0}\rangle$while$\mathcal{M}$. Implies that has no a dirct sum from two non-zero subm..

<u>Proposition 3-5:</u>

Every semi locall S-lifitingmod. is an indecompsible.

<u>Proof:</u>

Let $\mathcal{M}$be a semi locall S-lifitingmod. , ∃aunique semi max.subm. N from$\mathcal{M}$. Suppose that M is not indecompsible , hence there are convenientsubm.s A while B from$\mathcal{M}$so that A is subm.from N while$A \oplus B = \mathcal{M}$.But $\mathcal{M}$is a semi locall S-lifitingmod. , hence $N \cap B$ is a semi small subm.from B implies that $\mathcal{M}$=A which is a contradction .Thus $\mathcal{M}$is an indecomposible mod..

The converse from the previous proposition is not true in general as we see in the following example.

<u>Example 3-6:</u>

Z_{10} is an indecompsible Z-mod. , because Z_{10} = $(\bar{2})\oplus(\bar{5})$ .But it's not semi locall S-lifitingmod. , because it has two semi max.subm.s $(\bar{2})$while $(\bar{5})$.

<u>Proposition 3-7:</u>

A cyclic whileindecompsiblemod. is a semi lifitingmod. .

<u>Proof:</u>

Let$\mathcal{M}$be an indecomposible mod.while let N be semi max.subm.from$\mathcal{M}$containes a non-zero subm. say L .Suppose that $\mathcal{M}$=L+K where K subm.from$\mathcal{M}$by[4,lemma(1-2-10)]implies$\mathcal{M}/(L \cap K) \simeq \mathcal{M}/L \oplus \mathcal{M}/K$ .But $\mathcal{M}/(L \cap K)$ an indecomposible mod. . Then by second isomorphism theorem implies either $\mathcal{M}$/L=0 or M/K =0 . Because L subm.from N while N semi max.subm.from$\mathcal{M}$, hence L convenientsubm.from$\mathcal{M}$.Then $\mathcal{M}$/L≠0, implies that $\mathcal{M}$/K=0 while hence $\mathcal{M}$=K . Therefore L semi small subm.from$\mathcal{M}$, so $\mathcal{M}$ semi locall S-lifitingmod. .

From the previuose propositions (3-5)while (3-7), implies the following resulet.

<u>Corollary 3-8:</u>

Let $\mathcal{M}$be cyclic mod. . Then $\mathcal{M}$ semi locall S-lifitingmod. if while only if$\mathcal{M}$ an indecompsible.

To show the rlation betwen a semi locall S-lifitingmod.while hollow mod. we have the folloing propsition .

<u>Proposition 3-9:</u>

Every semi locall S-lifitingmod. is hollow mod..

<u>Proof:</u>

Let $\mathcal{M}$be semi locall S-lifitingmod. , then ∃aunique semi max.subm. N from$\mathcal{M}$, ∃subm. A from N while B from$\mathcal{M}$so that $\mathcal{M} = A \oplus B$while$N \cap B$ is a semi small subm.from B. Then $\mathcal{M} = \mathcal{M} \oplus \{0\}$, where $\{0\}$ subm.from N , $N \cap \mathcal{M} = N$whilebecause$\mathcal{M}$ semi locall S-lifitingmod. . Then $N \cap \mathcal{M} = N$ semi small subm.from$\mathcal{M}$. Thus $\mathcal{M}$a hollow mod. .

The convearse from proposition (3-7)" is not true in general , as we see in the following" example.

<u>Example 3-10:</u>

The Z-mod. Z_6 is hollow mod. . But it's not semi locall S-lifitingmod. . "Howver have the following "result.

Proposition 3-11:
Let $\mathcal{M}$be cycilc whileindecompsiblemod. . If $\mathcal{M}$ hollow mod. . Then $\mathcal{M}$ semi locall S-lifitingmod..
Proof:
Let N be a convenientsubm.from$\mathcal{M}$, because$\mathcal{M}$ hollow mod. .Then $\mathcal{M}$=A+B , where A subm.while$N \cap A$ semi small subm.from A ,but $\mathcal{M}$ an indecompsiblemod. , thus B=0 while hence A= $\mathcal{M}$. whicih impilies that $N \cap \mathcal{M} = N$, hence N semi small subm.from$\mathcal{M}$. Hence $\mathcal{M}$lifitingmod.while$\mathcal{M}$cyclic mod. .Then $\mathcal{M}$ semi locall S-lifitingmod. by prop(2-1).
Another prooffrom proposition(3-11). Let $\mathcal{M}$ hollow mod.whilebecause$\mathcal{M}$indecompsible . Then $\mathcal{M}$lifitingwhilebecause$\mathcal{M}$ cyclic , then $\mathcal{M}$ semi locall S-lifiting .
Defintion3-12:[9] Let A while B are subm.s from a mod.$\mathcal{M}$.Then A supplement from B in $\mathcal{M}$if $\mathcal{M}$=A+B while$A \cap B$ is a small subm.from A.
Propostion 3-13:
Let A be semi max.subm.from an $\mathcal{R}$-mod.$\mathcal{M}$. If B supplement from A in $\mathcal{M}$, then B semi locall S-lifitingmod. .
Proof:
Let B be supplement from A while let B_1 be convenientsubm.from B with B_1+B_2=B to somesubm. B_2 from B . Now , A +B= $\mathcal{M}$then A+B_1+B_2= $\mathcal{M}$,while B_1 is subm.from A , because otherwise A . B_1 = $\mathcal{M}$whilebecause A is a semi max.subm.from$\mathcal{M}$implies B_1 =B , which is a contradction . Thus A+B_2= $\mathcal{M}$whilebecause A is semi max.subm.from$\mathcal{M}$implies B_2 = $\mathcal{M}$.Implies that B is a lifitingmod. . To show that B cyclic mod. , let $X \in \mathcal{M} and\ X \notin A$ then R_X+A= $\mathcal{M}$. While this implies that R_X =B by minmimality from B . While by prop(2-1), this B semi locall S-lifitingmod. .
Proposition 3-14:
If $\mathcal{M}$is locall S-lifitingmod. , then all non-zero coclosed subm.frommax.subm.from$\mathcal{M}$is semi locall S-lifitingmod. .
Proof:
Let $\mathcal{M}$be a locall S-lifitingmod. then $\mathcal{M}$has aunique semi max.subm. N from$\mathcal{M}$.Let A be a non-zero coclosed subm.from N . Then $\frac{N}{A}$ is a semi small subm.from$\frac{\mathcal{M}}{A}$ implies that N=A by[7,def (1,2,10], so N is semi max.subm.from$\mathcal{M}$by [1,corollary(2-3-6)] suppose L is a convenientsubm.from N . Because$\mathcal{M}$is locall S-lifitingmod. , then $A \oplus L = \mathcal{M}$while$N \cap L$ is semi small subm.from L so it's semi small subm.from A by[2, prop(1-1-4)(2)] . Hence A is semi locall S-lifitingmod..
Corolary 3-15:
If M is semi locall S-lifitingmod. , then all non-zero coclosed subm.from semi max.subm.from$\mathcal{M}$is semi locall S-lifitingmod. .
Propostion 3-16:
Let A be subm.from$\mathcal{R}$-mod.$\mathcal{M}$. If A is semi locall S-lifitingmod. , then either A semi small subm.from$\mathcal{M}$or coc.losed subm.from$\mathcal{M}$but not both .
Proof:
Supose that A is not coclosed subm.from$\mathcal{M}$. To prove that A semi small subm.from$\mathcal{M}$, ∃convenientsubm. B from M so that A/B is subm.from M/B . But A semi locall S-lifitingmod. . So implies A lifiting by Remark(1-4)(2) . Then by [2,prop (1-1-4)]implies B semi small subm.from A while hence A semi small subm.from M by [2,prop (1-1-6)]. Now ,we want to prove A is not co.colsed while A semi small subm.From$\mathcal{M}$we must show that A zero subm.from$\mathcal{M}$.Because A semi locall S-lifitingmod. then is not zero subm. which contradction .

Definition 3-17:[2]
A mod.$\mathcal{M}$is said to be semi simple , if every subm. is a dirctsumandfrom$\mathcal{M}$. It's clear that 0 is the only semi small subm. in a semi simple mod. .
Propostion 3-18:
Every semi simple mod. is a semi locall S-lifiting which has aunique semi max. .
Proof:
Let $\mathcal{M}$ be a semi simple mod.while N is aunique semi max.subm.from$\mathcal{M}$. Because , $\mathcal{M}$is semi simple , N is a dirctsumandfrom$\mathcal{M}$. i.e , $\mathcal{M} = N \oplus W$. $N \cap W = 0$ is semi locall S-lifiting .

Example: [7]
(1) J(Z)=0
(2) J(Z_4)=$\{\bar{0}, \bar{2}\}$.
(3) J(Z_p^n) $= Z_p^{n-1}$
(4) J(Q) =Q
(5) If $\mathcal{M}$ is semi simple , then J($\mathcal{M}$)=0

(6) If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, then $J(\mathcal{M}) = J(\mathcal{M}_1) \oplus J(\mathcal{M}_2)$

Propostion 3-19:

Any dirctsumandfrom a semi locall S-lifitingmod. is semi locall S-lifiting .

Proof:

Let $\mathcal{M}$ be a semi locall S-lifitingmod. . suppose that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. We want to show that $\mathcal{M}_1$ is semi locall S-lifiting . let N be aunique semi max.subm.from$\mathcal{M}_1$ so N is subm.from$\mathcal{M}$.By Theorem (1-5) ,$N = A \oplus S$, where A is a dirctsumandfrom$\mathcal{M}$while S is a a semi smallsubm.from$\mathcal{M}$. By [10, prop(1,5)], S is a semi small from$\mathcal{M}_1$ .Now , $\mathcal{M} = A \oplus T$, where T is subm.from$\mathcal{M}$, because , A is a dirctsumandfrom$\mathcal{M}$. We are done if we can show that A is a dirctsumandfrom$\mathcal{M}_1$ .Now $\mathcal{M}_1 = \mathcal{M}_1 \cap \mathcal{M}$

$=\mathcal{M}_1 \cap (A \oplus T)$

$= A \oplus (\mathcal{M}_1 \cap T)$, by the Mudular Law. Thus , A is a dirctsumandfrom$\mathcal{M}_1$ .By theorem (1-5), $\mathcal{M}_1$ is semi locall S-lifiting .

Propostion 3-20:

Let N while B be subm.from a mod.$\mathcal{M}$. The following are equivalent:

1) N is a supplement from B in M .

2) $\mathcal{M}$= N+ B while$N \cap B$ is a semi small subm.from N .

Proof:

:1) $\Longrightarrow$ 2)

Let N be a supplement from B in $\mathcal{M}$. Then $\mathcal{M}$=N+B while N is minimal with this convenientty. Now suppose that $N = N \cap B + A$, therefore , we have $\mathcal{M} = N \cap B + A + B = A + B$. By the minimality from N ,we have A=N . Hence $N \cap B$ is a semi small subm.from N .

:2) $\Longrightarrow$ 1)

Assume that $\mathcal{M}$=N+B while$N \cap B$ is a semi small subm.from N .we want to show that N is a supplement from B in $\mathcal{M}$. Suppose that ∃ A is subm.from N so that $\mathcal{M}$=A+B . Therefore , $N = N \cap \mathcal{M} = N \cap (A + B) = A + N \cap B$, but $N \cap B$ is a semi small subm.from N , so A =N . Hence ,N is a supplement from B in $\mathcal{M}$.

Lemma 3-21:

Let $\mathcal{M}$be a semi locall S-lifitingmod. . Let X be subm.from N while Y be subm.from$\mathcal{M}$so that $\mathcal{M}$=X+Y , then ∃ a dirctsumand A from$\mathcal{M}$so that $\mathcal{M}$=X+A while A is subm.from Y .

Proof:

Because , $\mathcal{M}$is semi locall S-lifitingmod. ,$Y = A \oplus S$ *where* A is a dirctsumandfrom$\mathcal{M}$while S is a semi small subm.from$\mathcal{M}$. Now $\mathcal{M}$=X+Y=X+A+S=X+A .

The following proposition gives another characterization from semi locall S-lifitingmod. .

Proposition 3-22: [7]

Let $\mathcal{M}$be a mod. . The following statements are equivalent:

1) $\mathcal{M}$is semi locall S-lifiting .

2) $\mathcal{M}$is amply supplement while every supplement subm.from$\mathcal{M}$is a dirctsumandfrom$\mathcal{M}$.

Proof:

1) $\Longrightarrow$ 2):

Let $\mathcal{M}$=X+Y .We have to show that Y contains a supplement from X.By Lemma (3-21) , we may assume that Y is a dirctsumandfrom$\mathcal{M}$. $X \cap Y = A_1 \oplus S_1$ where A_1 is a dirctsumandfrom$\mathcal{M}$while S_1 is a semi small subm.from$\mathcal{M}$.because Y is a dirctsumandfrom$\mathcal{M}$then S_1 is a semi small subm.from Y . A_1 is a dirctsumandfrom$\mathcal{M}$, so ∃ T subm.from$\mathcal{M}$so that $\mathcal{M} = A_1 \oplus T$ .Now , $Y = Y \cap \mathcal{M} = A_1 \oplus Y \cap T$, because , A_1 is subm.from$X \cap Y$while . $X \cap Y$ is subm.from Y . Therefore , A_1 is a dirctsumandfrom Y , $Y = A_1 \oplus A_2$ .Consider the projection $\pi: A_1 \oplus A_2 \rightarrow A_2$ .Then , $X \cap Y = (X \cap Y) \cap Y = (X \cap Y) \cap (A_1 \oplus A_2) =$, because A_1 is a dirctsumandfrom$X \cap Y$. Now , $X \cap A_2 = (X \cap \mathrm{Y}) \cap A_2 = \pi(\mathrm{X} \cap \mathrm{Y}) = A_1 \oplus \big((X \cap Y) \cap A_2\big)$ $\pi(A_1 + S_1) = \pi(S_1)$. Because , S_1 is a semi small subm.from Y ,we have $X \cap A_2$ is a semi small subm.from A_2 .Now , $\mathcal{M}$=X+Y = X+A_1 +A_2 =X +A_2 while$X \cap A_2$ is a semi small subm.from A_2 . By proposition(3-20), A_2 is a supplement from X in $\mathcal{M}$. Hence , $\mathcal{M}$is amply supplement . Now, let P be a supplement subm.from$\mathcal{M}$. ∃subm. K from$\mathcal{M}$so that P is minimal with the convenientty $\mathcal{M}$=K+P .By Theorem (1-5), $P = L \oplus T$, where L is a dirctsumandfrom$\mathcal{M}$while T is a semi small subm.from$\mathcal{M}$.Now $\mathcal{M}$=K+P=K+L+T=K+L ,because T is a semi small subm.from$\mathcal{M}$.By the minimality from P , we have P=L .Hence , P is a dirctsumandfrom$\mathcal{M}$.

2) $\Longrightarrow$ 1):

Let L be aunique semi max.subm.from M . Because M is amply supplemented ,therefore, $\mathcal{M}$ is supplemented. Then ,L has a supplement N in $\mathcal{M}$, i.e. , $\mathcal{M}$ =N+L while$N \cap L$ is a semi small subm.from N . Because , $\mathcal{M}$ is amply supplemented while$\mathcal{M}$=L+N, then L contains a supplement $\mathcal{M}_1$ from N .By assumption $\mathcal{M}_1$ is a dirctsumandfrom $\mathcal{M}$ i.e., $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ .Now ,$L = L \cap \mathcal{M} = L \cap (\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{M}_1 \oplus L \cap \mathcal{M}_2$ .Also , $\mathcal{M}$= $\mathcal{M}_1$ +N ,because$\mathcal{M}_1$ is a supplement from N in $\mathcal{M}$ .Therefore , we obtain $L = L \cap \mathcal{M} = L \cap (\mathcal{M}_1 + N) = \mathcal{M}_1 + L \cap N$. Consider the projection $\pi: \mathcal{M}_1 \oplus \mathcal{M}_2 \rightarrow \mathcal{M}_2$. Then ,$L \cap \mathcal{M}_2 = \pi(L) =$

$\pi(\mathcal{M}_1 + L \cap N) = \pi(L \cap N)$. Because , $L \cap N$ is a semi small subm.from $\mathcal{M}$, $L \cap \mathcal{M}_2$ is a semi small subm.from$\mathcal{M}_2$ .Hence $\mathcal{M}$is semi locall S-lifiting.

Definition 3-23: [7] let $\mathcal{M}$be a mod. , $\mathcal{M}$is said to be D_3 if K_1while K_2 are dirctsumands from$\mathcal{M}$with $\mathcal{M}$=K_1 +K_2 then $K_1 \cap K_2$ is also a dirctsumandfrom$\mathcal{M}$.

Corollary 3-24:

Every semi locall S-lifiting is amply supplemented .

If , moreover M is (D_3)-mod. ,we have the following proposition [16].

Proposition 3-25:

Let $\mathcal{M}$be a (D_3) – mod. . The following statements are equivalent:

1) $\mathcal{M}$is a semi locall S-lifiting.

2) $\mathcal{M}$is amply supplemented while$\mathcal{M} = X \oplus Y$to some mutual supplements X while Y in $\mathcal{M}$.

Proof:

1) $\Longrightarrow$ 2):

Let X e subm.from N if $\mathcal{M}$has aunique semi max.subm. N while Y be subm.from$\mathcal{M}$which are mutual supplements in $\mathcal{M}$.Because$\mathcal{M}$is semi locall S-lifiting ,by proposition(3-22), $\mathcal{M}$is amply supplemented while both X while Y are dirctsumands from$\mathcal{M}$.By (D_3) , $X \cap Y$ is a dirctsumandfrom$\mathcal{M}$.Therefore , $X \cap Y$ is a semi small subm.from$\mathcal{M}$.[$X \cap Y$ is a semi small subm.from Y,because Y is a supplement from X . Therefore , $X \cap Y$is a semi small subm.from$\mathcal{M}$] .Hence , $X \cap Y = 0$. [$\mathcal{M} = X \cap Y \oplus T \Longrightarrow T \cap (X \cap Y) = 0$. Because , $X \cap Y$ is a semi small subm.from$\mathcal{M} \Longrightarrow$ T= $\mathcal{M} \Longrightarrow \mathcal{M} \cap (X \cap Y) = 0 \Longrightarrow X \cap Y = 0$] . Thus , $\mathcal{M} = X \oplus Y$.

2) $\Longrightarrow$ 1):

is semi locall S-lifiting by prop(3-22).$\mathcal{M}$

Theorem 3-26:

Let $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ be an amply supplemented mod. .Then the following ststements are equivalent:

1) $\mathcal{M}$is semi locall S-lifitingmod. .

2) Every co-closed subm. A from$\mathcal{M}$so that either $\mathcal{M}$=A+ $\mathcal{M}_1$ or $\mathcal{M}$= A+ $\mathcal{M}_2$ is a dirctsumandfrom$\mathcal{M}$.

Proof:

1) $\Longrightarrow$ 2):

Let $\mathcal{M}$ be a semi locall S-lifitingmod. if $\mathcal{M}$ has aunique semi max.subm. N . Let A be a co-closed subm.from$\mathcal{M}$so that either $\mathcal{M}$=A+ $\mathcal{M}_1$ or $\mathcal{M}$= A+ $\mathcal{M}_2$. By [7,prop(1-2-11)], A is a supplement subm.from$\mathcal{M}$ .By prop(3-22) , A is a dirctsumandfrom$\mathcal{M}$.

2) $\Longrightarrow$ 1):

Suppose that A is a co-closed subm.from$\mathcal{M}$so that either $\mathcal{M}$=A+ $\mathcal{M}_1$ or $\mathcal{M}$=A+ $\mathcal{M}_2$ .Then ,A is a dirctsumandfrom$\mathcal{M}$ .By [7,prop(1-2-12)], every co-closed subm.from$\mathcal{M}$ is a dirctsumand. Now let N be supplement unique semi max.subm.from$\mathcal{M}$ .By [7,prop(1-2-11)], N is co-closed in $\mathcal{M}$, so N is a dirctsumandfrom$\mathcal{M}$. By proposition (3-22), $\mathcal{M}$ is semi locall S-lifiting .

Proposition 3-27:

Every semi locall S-lifitingmod. is completely $\oplus$-supplemented.

Proof:

Let $\mathcal{M}$ be a semi locall S-lifitingmod. .Let N be aunique semi max.subm.from$\mathcal{M}$. By prop(3-19),N is semi locall S-lifiting . Let B be subm.from N . By prop(3-22), B has a supplement which is a dirctsumandfrom$\mathcal{M}$. Hence N is $\oplus$-supplemented .

Note that the converse from this remark is not true in general.

Corollary 3-28:

Every semi locall S-lifitingmod. is $\oplus$-supplemented.

Propostion 3-29:

Every semi locall S-lifitingmod. is H- supplemented.

Proof:

Let $\mathcal{M}$ be a semi locall S-lifitingmod.while A is aunique semi max.subm.from$\mathcal{M}$. Suppose that $\mathcal{M}$=A+X .By remark(1-2), $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_1$ is subm.from A while$A \cap \mathcal{M}_2$ is a semi small subm.from$\mathcal{M}$. Now , $A = A \cap \mathcal{M} = A \cap (M_1 + M_2) = M_1 + A \cap M_2$ .Therefore , we have M=A+X=M_1 +$A \cap M_2 + X = M_1 + X$, because$A \cap M_2$ is a semi small subm.from M .If $\mathcal{M}$= $\mathcal{M}_1$ +X , then $\mathcal{M}$= A+X , because , $\mathcal{M}_1$ is subm.from A .Hence $\mathcal{M}$ is H- supplemented.

Conclusion

The main results are as follows .Every semi localllifitingmod. semi lifitingmod., while the convers is not true in general (seeRemark while Example) (1.4) (2) , while the convers is true under certain conditions (cyclic, unique max.subm. ,RadM $\neq$ M) , every semi locallmod.s issemi localllifitingmod.s, but the convers is not true in general(see Remark while Example) (1.4) (3), while the convers is true under certain conditions ,every semi localllifitingmod. is amply supplemented ,while the convers is not true in general (see proposition) (3.2) , while the convers is true under certain condition , every semi localllifitingmod. is indecompsiblemod., while the convers is not true in general (see proposition) (3.5) , while the convers is true under certain conditions (cyclic mod. see proposition)(3.7), whileimplies every semi localllifitingmod. is hollow mod., while the convers is not true in general(see proposition 3.8) , while the convers is true under certain condition (cyclic indecompsible see proposition 3.11)

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