Bδg- Separation Axioms

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Abstract: In this paper we introduce Bδg separation axioms and study their basic properties.
We also studied the relations between these axioms and Bδg closed sets and Bδg open sets We give necessary and sufficient condition for a singleton sub set of closed set to be Bδg-T_0 space. Also we have given the equivalent condition for a space X to be a Bδg-T_1 space. A space X is Bδg-T_2 space if and only if for each pair of distinct points x,y∈X, there exists a Bδg-clopen set containing one of them but not the other.

Keywords: BδgT_0- space,BδgT_1-space,Bδg T_2-space

1. Introduction

The relationships among the several separation axioms such as $B\delta g$-T_0, $B\delta g$ T_1, Bδg -T_2, - are explored in the framework of topological spaces. Separation axioms [1, 2, 3, 4, 5] in topological spaces are primarily formulated so as to identify non-homeomorphic topological spaces. If X and Y are two topological spaces such that X satisfies a separation axiom while Y does not satisfy it, then X and Y are not homeomorphic. These are fundamental constructs and permeates everywhere in the study of topological spaces and their applications. The structure and properties of some of these are explored in recent years in generalized topological spaces [1, 2, 3, 4, 5].

1.Preliminaries

Definition 1.1: A space X is said to be:

(i) $\delta - T_0$ space, if for each pair of distinct points $X,$ there exists a $\delta -$ open set containing one of them but not the other.
(ii) $\delta - T_1$ space, if for each pair of distinct points x and y in $X,$ there exists two $\delta -$ open sets U and V containing x and y respectively, such that $x \notin V$ and $y \notin U$.
(iii) $\delta - T_2$ space, if for each pair of distinct points x and y in $X,$ there exists two disjoint $\delta -$ open sets U and V such that $x \in U$ and $y \in V$.

Definition 1.2: A space X is said to be:

(iv) $B\delta g - T_0$ space, if for each pair of distinct points $X,$ there exists a $B\delta g -$ open set containing one of them but not the other.
(v) $B\delta g - T_1$ space, if for each pair of distinct points x and y in $X,$ there exists two $B\delta g -$ open sets U and V containing x and y respectively, such that $x \notin V$ and $y \notin U$.
(vi) $B\delta g - T_2$ space, if for each pair of distinct points x and y in $X,$ there exists two disjoint $B\delta g -$ open sets U and V such that $x \in U$ and $y \in V$.

Remark 1.3: From the above definitions it is clear that every $\delta - T_i$ space is $B\delta g - T_i$ space, for $i = 0,1,2,$ since every $\delta -$open set is $B\delta g -$ open.

Definition 1.4: Let X be a space and let $x \in X$, then a subset A of X is said to be $B\delta g -$ neighborhood (briefly, $B\delta g$nbd) of x if there exists $B\delta g -$ open set U in X such that $x \in U \subseteq A$.

Definition 1.5: The union of all $B\delta g -$ open sets which are contained in A is called the $B\delta g -$ interior of A and it is denoted by $B\delta g\text{int}(A)$.

Definition 1.6: $B\delta g -$ closure of a subset A of a space X is the intersection of all $B\delta g -$ closed sets containing A and it is denoted by $B\delta g - \text{cl}(A)$

Theorem 1.7: Let X be a space and $A, B \subseteq X$. If $A \subseteq B$, then $B\delta g - \text{cl}(A) \subseteq B\delta g - \text{cl}(B)$

Lemma 1.8: The set A is $B\delta g$ −open in the space X if and only if for each $x \in A$, there exists a $B\delta g$ −open set B such that $x \in B \subseteq A$.

Lemma 1.9: For any subset A of a space X, $B\delta g - \text{cl}(A) = A \cup B\delta g - \text{D}(A)$, where $B\delta g - \text{D}(A)$ stands for the set of all $B\delta g -$ limit points of A in X.

Theorem 1.10: Let Y be a regular closed subset of X. If A is a $B\delta g$ −open subset of Y, then A is $B\delta g$ −open in X.

Theorem 1.11: Let $f: X \to Y$ be a homeomorphism. If A is a $B\delta g$ −open set in X, then $f(A)$ is a $B\delta g$ −open set in Y.

Theorem 1.12: A function $f: X \to Y$ is $B\delta g$ −continuous if and only if for every open subset O of Y, $f^{-1}(O)$ is $B\delta g$ −open in X.

Theorem 1.13: Let $f: X \to Y$ be continuous and open function, then $f^{-1}(B)$ is $B\delta g$ −open in X for any B is $B\delta g$ −open in Y.

2. Bδg − Separation Axioms

Theorem 2.1:

A space X is $B\delta g - T_0$ space if and only if $B\delta g$ −closure of distinct points are distinct.

Proof:

Let X be a $B\delta g - T_0$ space and $x, y \in X$ such that $x \neq y$.

Since $x \neq y$ and X is a $B\delta g - T_0$ space, there exists a $B\delta g$ −open set G contains one of them, say x, and not the other. Then $X \backslash G$ is $B\delta g$ −closed set in X contains y but not x, but $B\delta g\text{cl}(\{y\}) \subseteq X \backslash G$ and since $x \notin X \backslash G$ implies that $x \notin B\delta g\text{cl}(\{y\})$, so $B\delta g\text{cl}(\{x\}) \neq B\delta g\text{cl}(\{y\})$.

Conversely, let $x, y \in X$ such that $x \neq y$.

By hypothesis, $B\delta g\text{cl}(\{x\}) \neq B\delta g\text{cl}(\{y\})$, then there exists at least one point p of X which belongs to one of them, say $B\delta g\text{cl}(\{x\})$ and does not belongs to $B\delta g\text{cl}(\{y\})$.

If $x \in B\delta g\text{cl}(\{x\})$, then $\{x\} \subseteq B\delta g\text{cl}(\{y\})$. This implies that by Theorem 1.7, $B\delta g\text{cl}(\{x\}) \subseteq B\delta g\text{cl}(\{y\})$ which is a contradiction to the fact that $p \notin B\delta g\text{cl}(\{y\})$ but $p \in B\delta g\text{cl}(\{x\})$, so $x \notin B\delta g\text{cl}(\{x\})$.

Hence, $x \in X \backslash B\delta g\text{cl}(\{y\})$ and $X \backslash B\delta g\text{cl}(\{y\})$ is $B\delta g$ −open set containing x but not y. Thus, X is a $B\delta g - T_0$ space.

Lemma 2.2: A subset of a space X is $B\delta g$ −open set if and only if it is a $B\delta g$nbd of each of its points.

Proof:

This follows immediately from the definition of $B\delta g$nbd and the property of a topology that the union of a collection of open sets is again open.

Theorem 2.3:

A space X is $B\delta g - T_1$ space if and only if every singleton subset of X is $B\delta g$ −closed.

Proof:

Let X be a $B\delta g - T_1$ space and $x \in X$. Let $y \in X \backslash \{x\}$ implies that $y \neq x$.

Since X is $B\delta g - T_1$ space, there exists two $B\delta g$ −open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V, y \in V$. This implies that $y \in V \subseteq X \backslash \{x\}$ so that by Lemma 1.8, $X \backslash \{x\}$ is a $B\delta g$ −open set.

Hence, $\{x\}$ is $B\delta g$ −closed.

Conversely, let $x, y \in X$ such that $x \neq y$, implies that $\{x\}, \{y\}$ are two $B\delta g$ −closed sets in X. Then $X \backslash \{x\}$ and $X \backslash \{y\}$ are two $B\delta g$ −open sets and $X \backslash \{x\}$ contains y but not x, also $X \backslash \{y\}$ contains x but not y. Therefore, X is $B\delta g - T_1$ space.

Theorem 2.4:

For any space X, the following statements are equivalent:

(i) X is a $B\delta g-T_1$ space
(ii) Each subset of X is the intersection of all $B\delta g$ −open sets containing it.
(iii) The intersection of all $B\delta g$ −open sets containing the point $x \in X$ is the set $\{x\}$.

Proof:

(i) $\Longrightarrow$ (ii): Let X be a $B\delta g-T_1$ space and $A \subseteq X$.

Then for each $y \notin A$, there exists a set $X\backslash\{y\}$ such that $A \subseteq X\backslash\{y\}$ and by Theorem 2.3, the set $X\backslash\{y\}$ is $B\delta g$ −open for every y. This implies that $A = \cap\{X\backslash\{y\}: y \in X\backslash A\}$, so the intersection of all $B\delta g$ −open sets containing A is A itself.

(ii) $\Longrightarrow$ (iii): Let $x \in X$, then $\{x\} \subseteq X$ so by (ii), the intersection of all $B\delta g$ −open sets containing $\{x\}$ is $\{x\}$ itself. Hence, the intersection of all $B\delta g$ −open sets containing x is $\{x\}$.

(iii) $\Longrightarrow$ (i): Let $x, y \in X$ such that $x \neq y$ implies that by (iii), the intersection of all $B\delta g$ − open sets containing x and y are $\{x\}$ and $\{y\}$ respectively, then for each $x \in X$ there exists a $B\delta g$ −open set G_x such that $x \in G_x$ and $y \notin G_x$. Similarly for $y \in X$ there exists a such that $y \in G_y$ and $x \notin G_y$ this implies that X is a $B\delta g-T_1$ space.

Theorem 2.5:

A space X is a $B\delta g-T_1$ space if and only if $B\delta g\mathrm{D}(\{x\}) = \emptyset$ for each $x \in X$.

Proof:

Let X be a $B\delta g-T_1$ space and $x \in X$.

If possible suppose that $B\delta g\mathrm{D}(\{x\}) \neq \emptyset$ implies that there exists $y \in B\delta g\mathrm{D}(\{x\})$ and $y \neq x$ and since X is $B\delta g-T_1$ space, so there exists a $B\delta g$ −open set U in X such that $y \in U$ and $x \notin U$ implies that $\{x\} \cap U = \emptyset$, then $y \in B\delta g\mathrm{D}(\{x\})$ which is a contradiction. Thus $B\delta g\mathrm{D}(\{x\}) = \emptyset$ for each $x \in X$.

Conversely, let $B\delta g\mathrm{D}(\{x\}) = \emptyset$ for each $x \in X$. Then by Lemma 1.9, $B\delta g\mathrm{cl}(\{x\}) = \{x\}$ which is $B\delta g$ −closed set in X. This implies that each singleton set in X is $B\delta g$ −closed. Thus by Theorem 2.3, X is a $B\delta g-T_1$ space.

Lemma 2.6:

If every finite subset of a space X is $B\delta g$ −closed, then X is $B\delta g-T_1$ space.

Proof:

Let $x, y \in X$ such that $x \neq y$. Then by hypothesis, $\{x\}$ and $\{y\}$ are $B\delta g$ −closed sets which implies that $X\backslash\{x\}$ and $X\backslash\{y\}$ are $B\delta g$ −open sets such that $x \in X\backslash\{y\}$ and $y \in X\backslash\{x\}$. Hence X is $B\delta g-T_1$ space.

Theorem 2.7:

If X is $B\delta g-T_0$ space, then $B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{x\})\big) \cap B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big) = \emptyset$ for each pair of distinct points x and y in X.

Proof:

Let X be a $B\delta g-T_0$ space and $x, y \in X$ such that $x \neq y$. Then there exists a $B\delta g$ −open set G containing one of the point, say x, and not the other implies that $x \in G$ and $y \notin G$, then $y \in X\backslash G$ and $X\backslash G$ is $B\delta g$ −closed.

Now $B\delta g\mathrm{int}(\{y\}) \subseteq B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big) \subseteq X\backslash G$ this implies that $G \cap B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big) = \emptyset$, then $G \subseteq X\backslash B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big)$.

But $x \in G \subseteq X\backslash B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big)$, then $B\delta g\mathrm{cl}(\{x\}) \subseteq X\backslash B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big)$ this implies that $B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{x\})\big) \subseteq B\delta g\mathrm{cl}(\{x\}) \subseteq X\backslash B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big)$. Therefore, $B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{x\})\big) \cap B\delta g\mathrm{int}\big(B\delta g\mathrm{cl}(\{y\})\big) = \emptyset$.

Theorem 2.8:

If for each $x \in X$, there exists a regular closed set U containing x such that U is $B\delta g-T_0$ subspace of X, then the space X is $B\delta g-T_0$.

Proof:

Let $x, y \in X$ and $x \neq y$. Then by hypothesis, there exists regular closed sets U and V such that $x \in U$, $y \in V$ and U, V are $B\delta g - T_0$ subspaces. Now, if $y \notin U$ then the proof is complete. But if $y \in U$ and since U is $B\delta g - T_0$ subspace, so there exists a $B\delta g$ −open set W in U such that $y \in W$ and $x \notin W$ and since U is regular closed set so by Theorem 1.10, W is a $B\delta g -$ open set in X containing y but not x. Thus, X is $B\delta g - T_0$.

Theorem 2.9:

If for each $x \in X$, there exists a regular closed set U containing x such that U is $B\delta g - T_1$ subspace of X, then the space X is $B\delta g - T_1$.

Proof:

Let $x, y \in X$ and $x \neq y$. Then by hypothesis, there exists regular closed sets U and V such that $x \in U$, $y \in V$ and U, V are $B\delta g - T_1$ subspaces.

Now, if $y \notin U$, $x \notin V$ then the proof is complete.

But if $y \in U$ and since U is $B\delta g - T_1$ subspace, so there exists a $B\delta g$ −open set W in U such that $y \in W_y$ and $x \notin W_y$ and since U is regular closed set so by theorem 1.10, W_y is a $B\delta g -$ open set in X containing y but not x.

Similarly, if $x \in V$ and since V is $B\delta g - T_1$ subspace, so there exists a $B\delta g$ −open set W_x in U such that $x \in W_x$ and $y \notin W_x$ and since U is regular closed set so by theorem 1.10, W is a $B\delta g -$ open set in X containing x but not y.

Thus, X is $B\delta g - T_1$.

Theorem 2.10:

For a space X the following statements are equivalent:

(i) X is a $B\delta g - T_2$ space
(ii) If $x \in X$, then for each $y \neq x$ there exists a $B\delta g$nbd N of x such that $y \notin B\delta g\mathrm{cl}(N)$.
(iii) For each $x \in X$, $\cap \{B\delta g\mathrm{cl}(N): N$ is a $B\delta g$nbd of $x\} = \{x\}$.

Proof:

(i) $\Longrightarrow$ (ii): Let X be a $B\delta g - T_2$ space and let $x, y \in X$, then for each $y \neq x$ there exists two disjoint $B\delta g -$ open sets U and V such that $x \in U$ and $y \in V$. This implies that $x \in U \subseteq X\backslash V$, so by Definition 1.4, $X\backslash V$ is a $B\delta g$nbd N of x which is $B\delta g$ −closed set in X and $x \in X\backslash V$ implies that $y \notin B\delta g\mathrm{cl}(X\backslash V)$.

(ii) $\Longrightarrow$ (i): let $x, y \in X$ such that $x \neq y$, then by hypothesis, there exists a $B\delta g$nbd N of x such that $y \notin B\delta g\mathrm{cl}(N)$ implies that $y \in X\backslash B\delta g\mathrm{cl}(N)$ and $x \notin X\backslash B\delta g\mathrm{cl}(N)$. But $X\backslash B\delta g\mathrm{cl}(N)$ is $B\delta g$ −open set also since N is $B\delta g$nbd of x, then there exist $B\delta g -$ open set G of X such that $x \in G \subseteq N$ this implies that $G \cap \left(X\backslash B\delta g\mathrm{cl}(N)\right) = \emptyset$. Hence X is $B\delta g - T_2$ space.

(ii) $\Longrightarrow$ (iii): Let $x \in X$. If $\cap \{B\delta g\mathrm{cl}(N): N$ is $B\delta g$nbd of $x\} \neq \{x\}$, then there exists $y \in \cap \{B\delta g\mathrm{cl}(N): N$ is $B\delta g$nbd of $x\}$ such that $y \neq x$ so by (ii), there exists a $B\delta g$nbd M of x such that $y \notin B\delta g\mathrm{cl}(M)$ which is contradiction to the fact that $y \in \cap \{B\delta g\mathrm{cl}(N): N$ is $B\delta g$nbd of $x\}$. Thus, $\cap \{B\delta g\mathrm{cl}(N): N$ is a $B\delta g$nbd of $x\} = \{x\}$.

(iii) $\Longrightarrow$ (ii): Let $x \in X$, so by hypothesis, we have $\cap \{B\delta g\mathrm{cl}(N): N$ is a $B\delta g$nbd of $x\} = \{x\}$.

Now if $x \in X$ and $y \neq x$, then $y \notin \cap \{B\delta g\mathrm{cl}(N): N$ is a $B\delta g$nbd of $x\} = \{x\}$ and hence there exists a $B\delta g$nbd M of x such that $y \notin B\delta g\mathrm{cl}(M)$.

Lemma 2.11:

Let Y be a regular closed subset of the space X, then any $B\delta g$nbd of the point x in Y is a $B\delta g$nbd of x in X.

Proof:

Let N be any $B\delta g$nbd of $x \in X$ this implies that by Definition 1.4, there exists a $B\delta g$ −open set G in Y such that $x \in G \subseteq N$. Since Y is regular closed set in X, so by Theorem 1.10, G is a $B\delta g$ −open set in X which implies that N is a $B\delta g$nbd of x in X.

Lemma 2.12:

Let Y be a regular closed subset of the space X and $A \subseteq Y$, then $B\delta g\mathrm{cl}(A) \subseteq B\delta g\mathrm{cl_y}(A)$.

Proof:

Let $x \notin B\delta g\mathrm{cl_y}(A)$ implies that there exists a $B\delta g$ −open set U in Y containing x such that $U \cap A = \emptyset$. Since Y is regular closed set in X, by Theorem 1.10, U is $B\delta g$ − open set in X implies that $x \notin B\delta g\mathrm{cl}(A)$, so $B\delta g\mathrm{cl}(A) \subseteq B\delta g\mathrm{cl_y}(A)$.

Theorem 2.13:

If for each point x of a space X there exists a regular closed subset A containing x and A is $B\delta g - T_2$ subspace of X, then X is $B\delta g - T_2$ space.

Proof:

Let $x \in X$, then by hypothesis, there exists a regular closed set A containing x and A is $B\delta g - T_2$ subspace. Hence, by theorem 2.10, we have $\cap\, \{B\delta g\mathrm{cl}_A(N): N$ is $B\delta g$nbd of x in $A\} = \{x\}$ and since A is regular closed set in X, so by Lemma 2.12, $B\delta g\mathrm{cl}(N) \subseteq B\delta g\mathrm{cl}_A(N)$ and by Lemma 2.11, N is $B\delta g$nbd of x in $X\} = \{x\}$. Therefore, by theorem 2.10, X is $B\delta g - T_2$ space.

Theorem 2.14:

A space X is $B\delta g - T_2$ space if and only if for each pair of distinct points $x, y \in X$, there exists a $B\delta g$ −clopen set containing one of them but not the other.

Proof:

Let X be a $B\delta g - T_2$ space and $x, y \in X$ such that $x \neq y$ implies that there exists two disjoint $B\delta g$ −open sets U and V such that $x \in U$ and $y \in V$. Now since $U \cap V = \emptyset$ and V is $B\delta g$ −open set implies that $x \in U \subseteq X\backslash V$ and $X\backslash V$ is $B\delta g$ −closed set, since X is $B\delta g - T_2$ space so for each $x \in X\backslash V$ there exists a $B\delta g$ −open set U_x such that $x \in U_x \subseteq X\backslash V$, then by Lemma 1.8, $X\backslash V$ is $B\delta g$ −open set. Thus $X\backslash V$ is $B\delta g$ −clopen set.

Conversely, let for each pair of distinct points $x, y \in X$, there exists a $B\delta g$ −clopen set containing x but not y implies that $X\backslash U$ is also $B\delta g$ −open set and $y \in X\backslash U$, since $U \cap (X\backslash U) = \emptyset$ so X is $B\delta g - T_2$ space.

Theorem 2.15:

A space X is $B\delta g - T_2$ space if for any pair of distinct points $x, y \in X$, there exists a $B\delta g$ − continuous function f of X into a T_2 −space Y such that $f(x) \neq f(y)$.

Proof:

Let x and y be any two distinct points in X. Then by hypothesis, there exists a $B\delta g$ −continuous function f of X into a T_2 −space Y such that $f(x) \neq f(y)$. But $f(x), f(y) \in Y$ and since Y is a T_2 −space so there exists two disjoint open sets U_x and V_y such that $f(x) \in U_x$ and $f(y) \in V_y$ implies that $x \in f^{-1}(U_x)$ and $y \in f^{-1}(V_y)$ and since f is $B\delta g$ − continuous function, so by Theorem 1.12, $f^{-1}(U_x)$ and $f^{-1}(V_y)$ are $B\delta g$ − open sets and $f^{-1}(U_x) \cap f^{-1}(V_y) = \emptyset$. This implies that X is $B\delta g - T_2$ space.

Theorem 2.16:

For a space X the following statements are equivalent:

(i) X is $B\delta g - T_2$ space
(ii) The intersection of all $B\delta g$ −clopen sets of each point in X is singleton.
(iii) For a finite number of distinct points x_i $(1 \leq i \leq n)$, there exists a $B\delta g$ −open set G_i such that G_i $(1 \leq i \leq n)$ are pairwise disjoint.

Proof:

(i) ⟹ (ii): Let X be a $B\delta g - T_2$ space and $x \in X$.

To show: $\cap\, \{G : G$ is $B\delta g$ −clopen set and $x \in G\} = \{x\}$.

If $\cap\, \{G : G$ is $B\delta g$ −clopen set and $x \in G\} = \{x, y\}$ where $x \neq y$. Then since X is $B\delta g - T_2$ space so there exists two disjoint $B\delta g$ −open sets U and V such that $x \in U$ and $y \in V$, implies that $x \in U \subseteq X\backslash V$ so by Lemma 1.8, $X\backslash V$ is $B\delta g$ −open set and also it is $B\delta g$ −closed set this implies that $X\backslash V$ is $B\delta g$ −clopen containing x but not y which is a contradiction. Thus the intersection of all $B\delta g$ −clopen sets containing x is $\{x\}$.

(ii) $\Longrightarrow$ (iii): Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a finite number of distinct points of X, then by (ii), $\{x_i\} = \cap \{F : F$ is $B\delta g -$clopen set and $x_i \in F\}$ for $i = 1,2, \ldots, n$. Since $x_i \notin \{x_j\}$ for $i, j = 1,2, \ldots, n$ and $i \neq j$, so there exists a $B\delta g -$clopen set F_0 such that $x_i \in F_0$ and $x_j \notin F_0$ for $i \neq j$, $(1 \leq i, j \leq n)$ implies that $x_j \in X \backslash F_0$, where $X \backslash F_0$ is also $B\delta g -$clopen set and $F_0 \cap (X \backslash F_0) = \emptyset$. Therefore, $X \backslash F_0$ is $B\delta g -$open set containing x_j, that is for each i there exist pairwise disjoint $B\delta g -$open sets N_i for x_i $(1 \leq i \leq n)$.

(iii) $\Longrightarrow$ (i): Obvious

Theorem 2.17:

If f and g are strongly $\delta -$continuous functions on a space X into a $B\delta g - T_2$ space Y, then the set of all point x in X such that $f(x) = g(x)$ is closed set in X.

Proof:

Let $A = \{x \in X : f(x) = g(x)\}$.

It is enough to show that $X \backslash A$ is an open set in X. So let $a \in X \backslash A$, then $f(a) \neq g(a)$ and $f(a), g(a) \in Y$, but Y is $B\delta g - T_2$ space, hence, there exist two disjoint $B\delta g -$open sets U and V in Y such that $f(a) \in U$ and $g(a) \in V$. Since f and g are strongly $\delta -$continuous functions and U and V are $\delta g -$ open sets, so by Definition 1.3, we obtain that $f^{-1}(U)$ and $g^{-1}(V)$ are open sets containing a. This implies that $a \in f^{-1}(U) \cap g^{-1}(V)$ and $f^{-1}(U) \cap g^{-1}(V)$ is an open set.

Now let $G = f^{-1}(U) \cap g^{-1}(V)$.

To show that $G \subseteq X \backslash A$. If possible, suppose that there exists one point $b \in G$ but $b \notin X \backslash A$, then $b \in A$. Therefore, $f(a) = f(b)$ and since $b \in E$, then $b \in f^{-1}(U)$ and $b \in g^{-1}(V)$. This implies that $f(b) \in U$ and $g(b) \in V$, but $f(b) = g(b)$ so $U \cap V \neq \emptyset$ which is contradiction. Thus, $a \in G \subseteq X \backslash A$ implies that $X \backslash A$ is a neighborhood of each of its points, so $X \backslash A$ is open set. Thus, A is closed set in X.

Corollary 2.18:

If f and g are strongly $\delta -$continuous functions on a space X into a $B\delta g - T_2$ space Y and the set of all point x in X such that $f(x) = g(x)$ is dense in X, then $f = g$.

Proof:

By Theorem 2.17, the set $A = \{x \in X : f(x) = g(x)\}$ is closed in X, that is $A = \mathrm{cl}(A)$ and from the hypothesis A is dense implies that $A = \mathrm{cl}(A) = \mathrm{X}$. Therefore, $fx) = g(x)$ for all $x \in X$. Hence, $f = g$.

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