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Research Article

On some parameters of Parity Signed Graphs

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Abstract: The parameters of parity signed graph mentioned in this paper are- rna and adhika number. The rna number of a parity signed graph S∗ is the minimum number of negative edges among all possible parity labelling of it's underlying graph G, whereas adhika number is the maximum number of positive edges among all possible parity labelling of it's underlying graph G.This paper mainly focuses on rna number and adhika number for certain classes of parity signed graphs..

Keywords: Signed Graph, Parity Signed Graph, rna number, adhika number .

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1. Introduction

Graph Theory deals with the study of graphs. A graph G= (V_G, E_G, R_G) is an ordered triplet where,

V_G: vertex set of G,

E_G: edge set of G and

R_G: function which is defined from set E_G to an unordered pair of (same/distinct) vertices of V_G which are called its endpoints.

We consider graphs with atmost one edge between every pair of vertices. For terminologies related to graph we refer to [1,2]. Also $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n and $\lceil n \rceil$ denotes the lowest integer greater than or equal to n.

A Signed graph is a special kind of graph, where each edge receives either a positive sign or negative sign. The idea of Signed graphs has been first introduced by Frank Harary in [3] and this concept was used by Frank Harary and Dorwin Cartwright to handle a problem in social psychology. We represent signed graph as $S^* = (G, \alpha)$ where G = (V_G, E_G) is called underlying graph of S^* and α: $E_G \rightarrow \{+, -\}$ is called the signature of S^* . The set $E^+(S^*)$ indicates set of positive edges (i.e, edges receiving positive sign). The set $E^-(S^*)$ indicates set of negative edges(i.e, edges receiving negative sign). An all-positive signed graph is a signed graph in which there are no negative edges, whereas all-negative signed graph is a signed graph in which there are no positive edges. If a Signed graph S^* is either all- positive or all-negative then it is homogeneous, otherwise it is heterogeneous. For a signed graph S^*, if we take product of edge signs around every cycle and if we get positive, then we say that signed graph S^* is balanced, otherwise it is unbalanced.

Signed graph has applications in industrial as well as theoretical fields. It has most of it's applications in the field of social networking i.e., they were mostly used for demonstrating social scenario among group of people where vertices represent people, positive edges represent friendship and negative edges represents enmities between them. We refer [7] for signed graphs. To know more detailed information in signed graphs, refer to [8].

M. Acharya et al. in [4] has introduced the concept of parity signed graph. In this article, they initiated the study on parity labelling in signed graphs and also found rna number for some classes of parity signed graphs. M. Acharya et al. in [5] has given fundamental descriptions for graphs like- parity signed stars, bistars, cycles, paths and complete bipartite graphs and also they found rna number for few parity signed graphs. Athira et al. in [6] has initiated the research on sum signed graphs and they also established rna number on some graph classifications and presented some of the features of sum signed graphs. T. Zaslavsky in [7] initiated the conception of matroids of signed graphs, which generalise both polygon matroids and even- circle matroids of ordinary graphs. Debanjan et al. in [9] has considered different problems like representation of fair friendship, representation of unfair friendship, educational field problems like mark distribution of some students in certain subjects, industrial problems like which machine is performing badly so that it can be replaced by better ones, medical problems like study of spreading of cancer and all these were studied with the help of signed graphs.

Consider a graph G and a function δ: $V_G \rightarrow \{1,2, \dots \dots, n\}$. Let α : $E_G \rightarrow \{+, -\}$ be a function in order that for any edge xy in G, $\alpha(xy) = +$, if $\delta(x)$ and $\delta(y)$ are of same parity (i.e, both end vertices should be either odd or even) and $\alpha(xy) = -$, if $\delta(x)$ and $\delta(y)$ are of opposite parity. $S^* = (G, \alpha)$ is parity signed graph if δ is bijective. Now we define two parameters of parity signed graph - rna number and adhika number.

The Sanskrit word for '-' is 'rna' which implies debt. The term 'rna' is pronounced as **rina** where **ri** is same as **ri** in **ri**bbon and **na** is same as **na** in coro**na**. The rna number of a parity signed graph S^* is the minimum number of negative edges among all possible parity labelling of it's underlying graph G, which is indicated by the symbol $\alpha^-(S^*)$. The main application of rna number is mainly found in sociology. 'rna' number gives us the slightest amount of discomfort among a group of people i.e, rna number is directly proportional to amount of discomfort. In other words, when the value of rna number is small, the discomfort among people is low. Whereas, adhika number is the maximum number of positive edges among all possible parity labelling of it's underlying graph G, which is indicated by the symbol $\alpha^-(S^*)$. We can also define adhika number as shown below,

$\alpha^+(S^*)=$ Number of edges - $\alpha^-(S^*)$

The following inequalities are true for a parity signed graph S^* [4].

$$\alpha^-(S^*)\leq| E^-(S^*) | \quad \text{and} \quad | E^+(S^*)|\leq \alpha^+(S^*).$$

We recall few results on rna number of parity signed graphs.

Proposition 1. *[4]*

If $G\cong P_m, \alpha^-(P_m) = 1$, where P_m is the path on m vertices.

Proposition 2. [4]

If $G\cong C_n, \alpha^-(C_n) = 2$, where C_n is the cycle on n vertices.

Proposition 3. [4]

If $G \cong K_{1,m}$, $\alpha^-(K_{1,m}) = \left\lceil \frac{m}{2} \right\rceil$, where $K_{1,m}$ is the star graph on (m+1) vertices.

Proposition 4. [4]

If $G \cong K_m$, $\alpha^-(K_m) = \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil$, where K_m is the complete graph on m vertices and $m \geq 2$.

Proposition 5. [4]

Let Z be a parity signed tree. Then $\alpha^-(Z) = |E^-(Z)|$ iff

$Z \cong K_{1,n}$, where $n \in N$ is odd.

Proposition 6. [5]

If $G \cong W_m$, $\alpha^-(W_m) = \left\lceil \frac{m+4}{2} \right\rceil$, where W_m is the wheel graph on m vertices.

Proposition 7. [5] Let $G \cong P_n$(or C_n). The rna number of P_n, where $n \geq 2$ is 1 . The rna number of C_n is 2 . Also , $\alpha^-(G) = \alpha^+(G)$ iff $G \cong P_3$(or C_4).

Proposition 8. [5] There exists a parity signed graph S^* such that $\alpha^-(S^*)$= k, where $k \in \mathbb{N}$.

Proposition 9. [5] If S^* is a parity signed graph such that

$\alpha^-(S^*) = 1$, then S^* has a cut-edge.

Proposition 10. [5] If S^* is a parity signed graph having a cut-edge which joins two graphs having orders which differ by at most one, then $\alpha^-(S^*) = 1$. .

In this paper, we found rna number and adhika number for some parity signed graph classes whose underlying graphs are Complete Bipartite graph, Centipede graph, Barbell graph, Fan graph, Helm graph, Sunlet graph and Friendship graph.

rna number and adhika number

2.Complete Bipartite Graph [10]

A complete bipartite graph $K_{p,q}$ where p,q > 0 is a graph whose vertex set can be partitioned into two disjoint sets in such a way that every vertices of the first set are adjacent to every vertices of the second set. This graph has total (p + q) vertices and pq edges.

Theorem 1. If $G \cong K_{p,q}$ where p,q > 0,

$$\alpha^-(K_{p,q}) = \left\lceil \frac{pq}{2} \right\rceil$$

Proof. Let $K_{p,q}$ be a complete bipartite graph. Let the first vertex set be X = $\{v_i : 1 \le i \le p\}$ and the other vertex set be

Y = $\{u_i : 1 \le i \le q\}$.

Let $\gamma : V(K_{p,q}) \rightarrow \{1,2,3, \ldots\ldots (p+q)\}$ be the vertex labelling function. Let us label the vertices $v_1, v_2 \ldots, v_p$ of the first set X with integers consecutively and then vertices $u_1, u_2 \ldots, u_q$ of the second set with remaining integers consecutively. Then we get 2 cases:

Case 1 : pq is even

In this case, $\left(\frac{pq}{2}\right)$ pairs of end vertices are of different parity. Hence there will be $\left(\frac{pq}{2}\right)$ negative edges. Therefore,

$$\alpha^-(K_{p,q}) = \left(\frac{pq}{2}\right).$$

Case 2 : pq is odd

In this case, $\left(\frac{pq+1}{2}\right)$ pairs of end vertices are of different parity. Hence,there will be $\left(\frac{pq+1}{2}\right)$ negative edges. Therefore,

$$\alpha^-(K_{p,q}) = \left(\frac{pq+1}{2}\right) = \left\lceil \frac{pq}{2} \right\rceil$$

Thus we can conclude,

$$\alpha^-(K_{p,q}) = \left\lceil \frac{pq}{2} \right\rceil . \square$$

Corollary 1.

For any complete bipartite graph $K_{p,q}$ where p,q > 0

$$\alpha^+(K_{p,q}) = pq - \left\lceil \frac{pq}{2} \right\rceil$$

3.. Centipede Graph [10]

m-Centipede graph C_m where $m \ge 1$ is a graph which is obtained by connecting one pendant edge to every vertex of the path P_m. This graph has total 2m vertices and 2m-1 edges.

Theorem 2. *If* $G \cong C_m$ *with* $m > 1$,

$$\alpha^-(C_m) = \begin{cases} 1, & if\ m\ is\ even \\ 2, & if\ m\ is\ odd \end{cases}$$

Proof.

Let C_m be m-centipede graph. Let the pendant vertices be

$\{v_i : 1 \le i \le m\}$ and let the vertices of the path be $\{u_i : 1 \le i \le m\}$. Let f :V $(C_m) \rightarrow \{1,2...2m\}$ be the vertex labelling function. Then we have m vertices with odd labels and m vertices with even labels.

Case 1 : m is even

For $1 \le i \le \frac{m}{2}$ label v_i, u_i's with odd integers and for $\left(\frac{m}{2}\right) + 1 \le i \le m$, label v_i,, u_i's with even integers. From this labelling one edge in the path having end vertices

of opposite parity receives negative sign. Therefore, $\alpha^-(C_m) = 1$..

Case 2 : m is odd.

For $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$, label vertices v_i, u_i's with odd integers and for $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m$, label vertices v_i, u_i's with even integers. Then there will be one pendant edge in middle, label that $v_i u_i$ with remaining integers of different parity. From this labelling one edge in path and one pendant edge have end vertices of different parity. Hence they both will receive negative sign. Therefore,

$$\alpha^{-}(C_m) = 2$$

Thus we conclude that

$$\alpha^{-}(C_m) = \begin{cases} 1, & \text{if } m \text{ is even} \\ 2, & \text{if } m \text{ is odd} \end{cases}$$

□

Corollary 2.

For any m-centipede graph C_m where $m > 1$,

$$\boldsymbol{\alpha}^{+}(\mathbf{C_m}) = \begin{cases} (2m-1)-1 = 2m-2, & \text{if m is even} \\ (2m-1)-2 = 2m-3, & \text{if m is odd} \end{cases}$$

4.Barbell Graph [11]

t-Barbell graph B_t where $t \geq 2$ is the simple graph which we get by joining two replicas of K_t with a bridge, where K_t denotes complete graph on t vertices. This graph has total 2t vertices and $2.\binom{t}{2}+1$ edges.

Theorem 3.

If $G \cong B_t$ with $t \geq 2$,then $\alpha^{-}(B_t) = 1$.

Proof. Let B_t be t-Barbell graph. Let $X = \{u_i : 1 \leq i \leq t\}$ be the vertices of first replica of K_t and let $Y = \{v_i : 1 \leq i \leq t\}$ be the vertices of second replica of K_t. Let u_1v_1 be the bridge connecting two replica's of K_t.

Let $f : V(B_t) \rightarrow \{1,2,...,2t\}$ be the vertex labelling function. Label the vertices of two sets with function $f(u_i) = 2i-1$ and $f(v_i) = 2i$, where $i=1,2,...,t$.

Then we get, the bridge u_1v_1 which connects the two replica's of K_t have end vertices of different parity and all other edges have end vertices of same parity. Thus it is clear that only one edge receives negative sign, which is bridge. Therefore,

$$\alpha^{-}(B_t) = 1. \qquad \square$$

Corollary 3. For any t-Barbell graph where $t \geq 2$,

$$\alpha^{+}(B_t) = (2.\binom{t}{2}+1)-1 = 2.\binom{t}{2} = t^2 - t.$$

Fan Graph [10]

A Fan graph f_q where $q \geq 2$ is obtained by joining all vertices of P_q (path graph on q vertices) to a common vertex called center. A Fan graph has total (q+1) vertices and (2q-1) edges.

Theorem 4. If $G \cong f_q$, $q \geq 2$ then,

$$\alpha^{-}(f_q) = \begin{cases} \frac{q+2}{2}, & \text{if q is even} \\ \frac{q+3}{2}, & \text{if q is odd} \end{cases}$$

Proof. Let f_q be a fan graph. Let the vertices of path be

$\{v_i : 1 \leq i \leq q\}$. Let the central vertex be c.

Let $\delta : V(f_q) \rightarrow \{1,2...,q+1\}$ be the vertex labelling function such that $\delta(c) = 1$.

Case 1 : q is even.

In path P_q , label first $\frac{q}{2}$ vertices with odd integers and the remaining $\frac{q}{2}$ vertices with even integers.

Then we get, $\frac{q}{2}$ edges connecting central vertex receive negative sign. In addition to that, one middle edge of P_q also receives negative sign. Hence, $\left(\frac{q}{2}\right)+1 = \left(\frac{q+2}{2}\right)$ edges receives negative sign.

Therefore,

$$\alpha^{-}(f_q) = \left(\frac{q+2}{2}\right).$$

Case 2 : q is odd.

In path P_q , label first $\left\lfloor \frac{q}{2} \right\rfloor$ vertices with odd integers and the remaining $\left\lceil \frac{q}{2} \right\rceil$ vertices with even integers Then we get $\left(\frac{q+1}{2}\right)$ edges connecting central vertex and one edge in P_q have end vertices of opposite parity. Hence $\left(\frac{q+1}{2}\right) + 1 = \left(\frac{q+3}{2}\right)$ edges receives negative sign.

Therefore,

$$\alpha^-(f_q) = \left(\frac{q+3}{2}\right).$$

Thus, we can conclude that

$$\alpha^-(f_q) = \begin{cases} \frac{q+2}{2}, & \text{if q is even} \\ \frac{q+3}{2}, & \text{if q is odd} \end{cases}$$

□

Corollary 4. For any fan graph f_q , where $q \geq 2$,

$$\alpha^+(f_q) = \begin{cases} (2q-1) - \frac{q+2}{2} = \frac{3q-4}{2}, & \text{if q is even} \\ (2q-1) - \frac{q+3}{2} = \frac{3q-5}{2}, & \text{if q is odd} \end{cases}$$

5.Helm Graph [10]

Helm Graph H_m where $m \geq 3$ is a graph which is obtained from W_m, by attaching a single pendant edge to every vertex of the cycle in W_m, where W_m denotes wheel on m vertices. This graph has total 2m+1 vertices and 3m edges.

Theorem 5. If $G \cong H_m$ where $m \geq 3$,

$$\alpha^-(H_m) = m.$$

Proof. Let H_m be a Helm graph. Let the hub (central vertex) be c. Let the pendant vertices be $\{v_i : 1 \leq i \leq m\}$ and let the vertices of the cycle be $\{u_i : 1 \leq i \leq m\}$.

Let $\gamma: V(H_m) \to \{1,2, \ldots, (2m+1)\}$ be the vertex labelling function such that $\gamma(c) = 1$.

First we label the vertices of the cycle u_i with odd integers and then label the pendant vertices v_i with even integers. From this labelling, all the m pendant edges receives negative sign. Therefore,

$$\alpha^-(H_m) = m.$$

Corollary 5. For any Helm graph H_m where $m \geq 3$,

$$\alpha^+(H_m) = (3m) - m = 2m.$$

6.Sunlet Graph [10]

p-Sunlet graph S_p where $p \geq 3$ is a graph which we get by adding pendant edges to every vertices of cycle graph C_p. This graph has total 2p vertices and 2p edges.

Theorem 6. If $G \cong S_P$ where $p \leq 3$,

$$\alpha^-(S_p) = \begin{cases} 3, & \text{if p is odd} \\ 2, & \text{if p is even} \end{cases}$$

Proof. Let S_P be p-Sunlet graph. Let the pendant vertices be $\{v_i : 1 \leq i \leq p\}$

and let vertices of cycle be $\{u_i : 1 \leq i \leq p\}$.

Let $\beta: V(S_p) \to \{1,2, \ldots \ldots, 2p\}$ be the vertex labelling function. Then we have p vertices with odd labels and p vertices with even labels.

Case 1: p is odd

We label the $(p-1)$ pair of v_iu_i with integers of same parity. Then we get one pair of v_iu_i and two edges of cycle have end vertices of different parity. Hence, these three edges receives negative sign. Therefore, $\alpha^-(S_p) = 3$.

Case 2: p is even

We label the p pair of v_iu_i with integers of same parity. Then we get two edges of cycle having end vertices of different parity and hence these two edges receives negative sign. Therefore, $\alpha^-(S_p) = 2$.

Hence, we can conclude that

$$\alpha^-(S_p) = \begin{cases} 3, & \text{if p is odd} \\ 2, & \text{if p is even} \end{cases}$$

Corollary 6. For any p-Sunlet graph S_p where $p \geq 3$

$$\alpha^+(S_p) = \begin{cases} (2p) - 3, & \text{if p is odd} \\ (2p) - 2, & \text{if p is even} \end{cases}$$

7.Friendship Graph [10]

Friendship graph $F_m$where $m \geq 2$ is a planar undirected graph which is obtained by connecting m identicals of cycle graph C_3 to a common vertex. It has total (2m+1) vertices and 3m edges.

Theorem 7. If $G \cong F_m$ where $m \geq 2$, then

$$\alpha^-(F_m) = \begin{cases} m, & \text{if m is even} \\ m+1, & \text{if m is odd} \end{cases}$$

Proof. Let central vertex of F_m be c. Let v_iu_i where $i = 1,2, \dots \dots \dots, m$denote the edges for each cycle of F_m. Let $\rho: V(F_m) \to \{1,2, \dots \dots, 2m+1\}$ be the vertex labelling function such that $\rho(c) = 1$. Then m vertices recieves even labels and (m+1) vertices receives odd labels.

Case 1: m is even

Label the vertices v_i, u_i of each cycle with integers of same parity. Then m edges connecting central vertex c with even label vertices receives negative sign. Therefore, $\alpha^-(F_m) = m$

Case 2: m is odd

Label the vertices v_i, u_i of each cycle in clockwise direction as follows.

For $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$, label vertices v_iu_i's with odd integers and

for $\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m$

,label vertices v_iu_i's with even integers. Then there will be one edge v_iu_i of a cycle, label that v_iu_i with remaining integers. From this labelling, m edges connecting central vertex c with even label vertices and one edge v_iu_i of a cycle have end vertices of different parity and hence, receives negative sign.

Therefore,

$$\alpha^-(F_m) = m + 1$$

Thus we conclude that,

$$\alpha^-(F_m) = \begin{cases} m, & \text{if m is even} \\ m+1, & \text{if m is odd} \end{cases}$$

Corollary 7. For any Friendship graph F_m where $m \geq 2$,

$$\alpha^+(F_m) = \begin{cases} (3m) - m = 2m, & \text{if m is even} \\ (3m) - (m+1) = 2m - 1, & \text{if m is odd} \end{cases}$$

8..Conclusion

In this paper, we found rna number and adhika number for certain classes of parity signed graphs whose underlying graphs are Complete Bipartite Graph, Centipede Graph, Barbell Graph, Fan Graph, Helm Graph, Sunlet Graph and Friendship Graph. We can study the parameters of derived signed graphs of these graph classes.It can be further extended to graph operations of signed graphs

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