Turkish Journal of Computer and Mathematics Education Vol.12 No.13 (2021), 1980-1986

Research Article

Global Domination Number of Squares of Certain Graphs

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 4 June 2021

Abstract: The square of a graph G= (V, E) is the graph G2 with the same vertex set as G and every two vertices u, v ϵ V are adjacent in G2 if and only if they are adjacent in G by a path of length one or two. Throughout this paper an attempt has been done to analyse the global domination of squares of certain graphs. We considered some connected graphs like paths, cycles, wheel graphs, complete tripartite, windmill graphs and some tree graphs. We characterised the global domination number of squares of a graph. We obtained a relationship between the global domination number of the square of some graphs and the domination number of the given graph. We also obtained the global domination number of squares of certain graphs.

AMS Subject Classification: 05C69

Keywords: Global domination, Square of graphs, Windmill graphs, Double star, Caterpillar graphs, Complete Tripartite Graphs

1. Introduction

All the graphs that we considered here are connected, undirected and non-trivial with neither loops nor multiple edges. Let $G = (V, E)$ be the graph with the set of all vertices V and the set of all edges E. The idea of domination is based on the sets of vertices that are near to all the vertices of a graph.

A set S is said to be ***dominating set*** in graph theory, if $S \subseteq V$ in G such that every vertex in V - S is adjacent to at least one element of S [9]. A dominating set is a minimal dominating set if it is not a proper subset of any other dominating set and its cardinality is minimum. The cardinality of smallest dominating set in G is called the ***domination number*** $\gamma(G)$ of G. The corresponding dominating set is called γ- set of G. The ***complement*** $\bar{G}$ of a graph G is an operation of a graph with the vertex set V such that two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

The concept of Global domination was introduced in the paper 'The global domination number of a Graph' by Sampath Kumar [7]. The Thesis on "Domination in Graphs" by Jennifer M. Tarr [9] gives the best basics on the concept domination which is a salient part of the subject matter. The research article 'Some New Perspective on Global Domination in Graphs' by S K Vaidya and R M Pandit [10] gives a good detailed explanation about the possibility to state the domination and global domination number of graphs obtained by various graph operations, whereas the research paper 'Trees with the same global domination number as their square' by D.A Mojdeh, M. Alishahi and M.Chellali [5] discussed the characterization of all trees whose global domination number equals to the global domination number of their squares. The research paper 'On certain graph domination and their applications' by V.Yegnanarayanan, Valentina E Balas & G. Chitra [13] discussed graph domination and its applications. For the basic terms and notions about graphs mentioned here, we referred the book "Introduction to Graph Theory, by D B West and for the terminologies and notions related with the domination of graphs we stick to "Fundamentals of Domination in Graphs" by Haynes, Teresa W, Stephen Hedetniemi and Peter Slater [3].

Global domination has many applications in various sectors. It is used to characterize the asymptotic behaviour of a number of dynamic processes related to the web [14]. Global domination is used in communication networks modelled by a graph G with subnetworks defined by matching M_i of cardinality k [5].

A dominating set is called a ***Global dominating set***, if it is a dominating set for a graph G and its complement $\bar{G}$.

For any graph $G = (V, E)$, the global domination number $\gamma_g(G) = \max\{\gamma(G), \gamma\overline{(G)}\}$ where $\gamma(G)$ is the domination of the graph and $\gamma\overline{(G)}$ be the domination of the complement of the graph G [1].

In this paper, we analyzed the global domination number of squares of some connected graphs like path graphs (P_n), the cycle graphs (C_n), the wheel graphs (W_n),), complete graphs (K_n), fan graph ($F_{m,n}$), windmill graphs ($W_d\ (n, m$), Complete tripartite graphs ($K_{m,n,r}$) Caterpillar trees and Double stars. We obtained certain relationship between global domination number of squares of the given graphs and its domination number. We also characterized the global domination number of squares of some graphs.

2. Preliminary Definitions:

2.1 A dominating set is called a **Global dominating set**, if it is a dominating set for a graph *G* and its complement G. For any graph *G* = (*V*, *E*), γ_g(G) = max {γ (G), γ ($\bar{G}$)} where γ (G) is the domination of the graph and γ ($\bar{G}$) be the domination of the complement of the graph G. [1]

2.2 The **square** of a graph, G^2 is the graph with the same vertex set *V* and two vertices are adjacent in G^2 if they are joined in G by a path of length at most two.[5]

An attempt has been made to characterize the global domination square of some connected graphs and to obtain a sharp bound.

Theorem 2.1: Let G be a graph with n vertices, then the following assertions hold

(a) If P_n is the path graph with n vertices, $\gamma_g(P_n{}^2) = \begin{cases} 2 & if \ \ n = 2 \\ 3 & if \ \ n \geq 4 \end{cases}$

(b) If C_n is the cycle graph with n vertices $\gamma_g(C_n{}^2) = \begin{cases} n & if n \leq 7 \\ 3 & if \ \ n \geq 8 \end{cases}$

(c) If K_n is the complete graph with n vertices,

$$\gamma_g(K_n{}^2) = n \text{ , for all n.}$$

(d) $K_{m,n}$ is the complete bipartite graph

$$\gamma_g(K_{m,n}{}^2) \begin{cases} = m + n & if \ \ m \leq 3, n \leq 4 \\ \leq 6 \ if \ 4 \leq m \leq 8; \ 5 \leq n \leq 9 \end{cases}$$

(e) S_n is a star graph with n vertices.

$$\gamma_g(S_n{}^2) = n \ \ if \ \ n \geq 4 .$$

3. Complete tripartite Graphs:

A complete tripartite graph G, designated K m, n, r has the following properties.

- The vertices can be partitioned into 3 subsets, m, n and r.
- Each vertex in m is connected to all vertices in n and r. Similarly for the vertices in n and r.
- No vertex in m is connected to any other vertices in m. Similarly for the vertices in n and r

Theorem 3.1: The global domination of square of complete tripartite graph $K_{m,n,r}$ is

γ_g ($K_{m, n, r}$) = m + n + r, if m = 1,2: n = 1, 2 and r = 1,2,3 where $|v_1|$ = m, $|v_2|$ = n and $|v_3|$ = r are the partition sizes of three independent sets V_1, V_2, V_3 of vertices in $K_{m,n,r}$.

Proof: Consider a complete tripartite graph $K_{m,n,r}$ with partition size $|V_1|$ = m, $|V_2|$= n and $|V_3|$ = r where

$1 \leq |V_1| \leq 2$; $1 \leq |V_2| \leq 2$; $1 \leq |V_3| \leq 3$.

By definition V_1, V_2, V_3 are independent sets. Since the vertices in the three independent sets are adjacent in the complement of $K_{m, n, r}$ the domination number of the complement of the complete tripartite graph will be equal to 3.i.e., γ ($\overline{K_{m,n,r}}$) = 3

Case 1: m = 1; n ≤ 2 or r ≤ 3 respectively.

In either case the domination number γ ($K_{m, n, r}$) = 1 as every vertex in the set V_1 is adjacent to all vertices in $V_2$and V_3 .This result holds good if n = 1, m ≤ 2 or r ≤ 3 or r =1; m ≤ 2 or n ≤ 2.

The global domination number γ_g ($K_{m, n, r}$) = max {γ ($K_{m, n, r}$), γ ($\overline{K_{m,n,r}}$) } = max {1,3} = 3

Case 2: m = 2, n, r ≤ 3.

We have two non-adjacent vertices in V_1 and hence γ ($K_{2, n, r}$) = 2

The global domination number γ_g ($K_{m, n, r}$) = max {γ ($K_{m, n, r}$, γ $\overline{(K_{m,n,r}}$) } = {2,3} = 3.

Now, consider the square of $K_{m,n,r}$. Since every vertex in $(K_{m,n,r})^2$ is adjacent to every other vertex, the domination number of the square of $K_{m,n,r}$ will be equal to 1; where m =1,2, n =1,2 and r=1,2,3.

The complement of the square of $K_{m,n,r}$ will be the vertices in the independent sets $V_1\ V_2 and\ V_3$. Hence domination of its complement will be equal to m + n+ r, the cardinality of V_1, V_2, V_3 .

Therefore,

$\gamma_g (K_{m,n,r})^2 = \max\ \{\gamma (K_{m,n,r})^2, \gamma\ \overline{(K_{m,n,r})^2}$

Hence the global domination number of squares of $K_{m,n,r}$ is equal to the sum of the number of vertices in V_1, V_2 , and V_3 .

i.e., $\gamma_g (K_{m,n,r})^2 = m + n + r$

Theorem 4: The global domination number of a wheel W_n graph $\gamma_g (W_n^2) = n$ for all n.

Proof: Let W_n be a wheel graph connecting a single vertex to all other vertices in the graph. While observing the global domination of the square of wheel graphs, any two vertices with a distance of at most two must have an edge inserted in between them. All vertices in the wheel graph are centrally connected to a single vertex and all vertices are at a distance of at most two. We come across the following cases

Case 1: When $n = 2k$, where k = {2, 3, 4…, n

Taking the square of W_n where $n = 2k$, we obtain a complete graph K_{2k-1} whose domination number is 1, whereas the complement of a complete graph leaves isolated vertices whose domination number is n.

$\gamma_g (W_n^2) = Max\ \{n, 1\} = n$

Case 2: When $n = 2k + 1$, where $k = \{2, 3, 4\ldots, n\}$

Taking the square of W_n where $n = 2k + 1$, we obtain a complete graph K_{2k} whose domination number is 1, whereas the complement of a complete graph leaves isolated vertices whose domination number is n.

$\gamma_g (W_n^2) = Max\ \{n, 1\} = n$

Thus, $\gamma_g (W_n^2) = n$ for all values of $n \in N$.

5. Fan graph:

A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$ where $\overline{K_m}$ is the empty graph on m vertices and P_n is the path graph on n vertices.

Theorem 5.1: The global domination number square of a fan graph $F_{m,n}$

$\gamma_g (F_{m,n})^2 = m + n$, for all $m, n \geq 3$.

Proof:

Let $F_{m,n}$ be a fan graph, for n ≥ 3, all the vertices of $F_{m,n}$ are joined to a central vertex.

We consider the following cases.

Case 1: When $m = 1$ and $n \geq 3$

Since the vertex connectivity of the $F_{1,n}$ graph is 1, the domination number of $F_{1,n}$ is equal to 1.

The complement of $F_{1,n}$graph leaves an isolated common vertex in $F_{1,n}$ and other vertices in the path graph are connected to all other vertices alternatively.

Therefore $\gamma (F_{1,n}) = 2 + 1$, Hence, $\gamma_g (F_{1,n}) = \max\ \{\gamma (F_{1,n}), \gamma\ (\overline{F_{1,n}})\} = \max\ \{1, 3\} = 3$

The square of $F_{1,n}$will have a vertex connectivity 1. So $\gamma (F_{1,n})^2 = 1$ and the complement of $F_{1,n}{}^2$ will give an isolated common vertex and an edge between every pair of vertices of distance two in the corresponding path graph. But all the vertices in the path graph are connected to the common vertex and the distance between any vertex in the path and other vertices is at most two if the path is chosen through the central vertex.

Therefore $\gamma\ (\overline{F_{1,n}{}^2}) = n + 1$

Hence, $\gamma_g (F_{1,n})^2 = \max\ \{\gamma (F_{1,n})^2, \gamma\ \overline{(F_{1,n})^2}\ \} = \max\ \{1, n + 1\} = 1 + n$.

Case 2: If $m \geq 2, n \geq 3, m < n$

$\gamma_g (F_{m,n})^2 = m + n$

Let $F_{m,n}$ be a fan graph with domination 1 and the complement of $F_{m,n}$ will be m isolated vertices and a null graph of order n. So $\gamma_g (\overline{F_{m,n})}) = m + n$

Hence $\gamma_g (F_{m,n}) = \max \{\gamma (F_{m,n}, \gamma (\overline{F_{m,n}})\} = max \{m, m + n\} = m + n$.

Taking the square of fan graph will give a vertex connectivity 1 and thereby $\gamma (F_{m,n})^2 = 1$

And the complement of $F_{m,n}{}^2$ will have m isolated vertices and the dominating vertices of the path graph will be the n isolated vertices.

Therefore $\gamma (F_{m,n})^2 = m + n$

And hence $\gamma_g (F_{m,n})^2 = \max \{\gamma (F_{m,n})^2, \gamma \overline{(F_{m,n})^2}\} = \max \{1, m + n\} = m + n$

Hence proved.

6. Windmill graph: The windmill graph $Wd (n, m)$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

Theorem 6.1: For windmill graph $Wd (n, m)$ of n ≥ 4, $\gamma_g (Wd (n, m)^2) = N$, where N is the total number of its vertices.

Proof: Let $Wd (n, m)$ be a windmill graph obtained by joining m copies of a complete graph K_n with a common vertex. Hence the number of vertices of $Wd (n, m)$ is given by $N = m (n- 1) + 1$.

By definition, the windmill graph has vertex connectivity 1 so that its domination number is $\boldsymbol{\gamma} (Wd (n, m)) = 1$.

Now, the square of the windmill graph has vertex connectivity 1 and is a complete graph.

Hence $\gamma ((Wd (n, m)^2) =1$. Clearly its complement will be equal to the total number of vertices N.

Hence $\gamma_g (Wd (n, m)^2) = \max \{\gamma ((Wd (n, m)^2), \gamma \overline{(Wd (n,m))^2}\} = max \{1, N\} = N$

Thus, for a windmill graph $\gamma_g (Wd (n, m)^2) = N$

Hence proved.

Lemma 7: If G is a connected graph that has at least one vertex of degree n - 1, then the global domination number of its square is n.

Proof: Suppose G be a connected graph with n vertices so that G has at least one vertex of degree n - 1, to prove that $\gamma_g (G^2) = n$

Since G has at least one vertex of degree $n-1$, clearly $\gamma (G) =1$. This means that all the other vertices in G are connected with other vertices with a minimum distance of one or two. According to the definition, the domination number of squares of the graph G, $\gamma (G^2)$ is also 1. While finding the square of a graph, the vertices having distance at most two will be connected by an edge and hence all the vertices in G^2 are connected by a path of length at least one Thus the domination number of $\gamma (\overline{G}^2)$ is n. Hence $\gamma_g (G^2) = max \{1, n\} = n$.

Example: Consider graphs K_n, S_n, $Wd (n, m)$ and $F_{m,n}$. Here, all these four graphs have a vertex connectivity one and hence $\gamma(G) =1$ and $\gamma_g (G^2) = \text{Max} \{1, n\} = n$

Theorem 7.1: $\gamma_g (G^2) = n$ if and only if $\gamma (G) = 1$, where G is a connected graph of order n.

Proof:

Case 1: Let $\gamma_g (G^2) = n$. By definition $\gamma_g (G^2) = max \{\gamma (G^2), \gamma (\overline{G^2})\} = n$.

i.e.; either $\gamma (G^2) = n$ or $\gamma (\overline{G^2}) = n$.

If $\gamma (G^2) = n$ then the n vertices in G^2 are isolated vertices, which is not possible since G is connected.

If $\gamma (\overline{G^2}) = n$ then $\overline{G^2}$ has n isolated vertices. Clearly G is a connected graph with $\gamma (G) = 1$.

Case 2: The converse of the theorem can be proved by using Lemma 7.

8. Double star: A tree containing exactly two non-pendant vertices is called a double-star.

8.1 Caterpillar tree: A caterpillar or caterpillar tree is a tree in which all the vertices are within distance one of a central path and the removal of its endpoints leaves a path graph.

8.2 Observations:

1: For caterpillar trees $\gamma_g(G^2) \le \gamma_g(G) + 2$.

2: For a path graph is P_n, $\gamma_g(G^2) \le \gamma(G) + 2$.

3: For fan graph F_n and star graphs, $K_{1,n}$, where $n \ge 7$ is $\gamma_g(G^2) = \gamma(G) + 2$.

Theorem 8.3: For cycle graphs, fan graphs, star graphs, Double star graph, Caterpillar trees and Windmill graphs, $\gamma_g(G^2) = \gamma(\overline{G^2})$.

Proof:

Case 1: Graphs with $\gamma(G) = 1$

Consider Fan graph $F_{m,n}$ star graph Sn and Windmill graph *Wd* (*n, m*). All the graphs that we considered here have domination 1. So, it is clear that domination of their square will also be equal to one i.e.; $\gamma(G^2) = 1$ for all these graphs. Since the square of all the graphs we considered are complete graphs, the complement of their squares leaves isolated vertices which are equal to the total number of vertices in the graph, say n. Hence

$\gamma(\overline{G^2}) = n$

Also, by the lemma if a connected graph has at least one vertex of degree n-1, the global domination number of its square is equal to n. This is true in the case of all the given graphs $\gamma_g(G^2) == n$,

Hence $\gamma_g(G^2) = \gamma(\overline{G^2})$.

Case 2: Graphs with $\gamma(G) \ne 1$

Consider cycle graph, double star and caterpillar trees.

We know that the domination of the square graph is less than the domination of its complement.

ie; $\gamma(G^2) \le \gamma(\overline{G^2})$.

But $\gamma_g(G^2) = \text{Max}\ \{\gamma(G^2), \gamma(\overline{G^2})\}$.

Hence $\gamma_g(G^2) = \gamma(\overline{G^2})$.

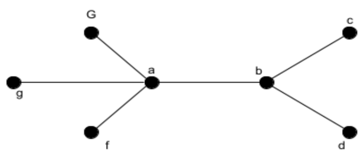

Figure 1: Graph G

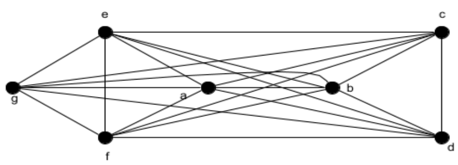

Figure 2:

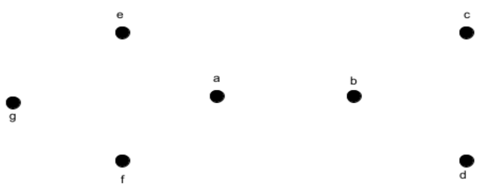

Figure 3:

Double Star Graph

In the Figure; $\gamma(G^2) = 1$; $\gamma(\overline{G^2}) = 7$

Global domination of square of a double star graph is,

$\gamma_g(G^2)$ = *max* {γ (G^2), γ ($\overline{G^2}$) } = {1, 7} = 7= γ ($\overline{G^2}$)

Hence $\gamma_g(G^2)$ = γ ($\overline{G^2}$)

Theorem 8.4: For any graph G, *Min* {γ (G), γ ($\bar{G}$)} ≤ $\gamma_g(G^2)$ ≤ γ (G) + γ ($\overline{G^2}$)

Where γ (G) be the domination number of G, γ ($\bar{G}$) be the domination number of the complement of G and γ ($\overline{G^2}$) be the domination of complement of the square of graph G.

Proof:

First let us consider the graphs with γ (*G*) = 1 and we choose γ (G) < γ ($\bar{G}$).

Also, γ (*G*) = 1 ⇒ γ (G^2) = 1 and $\gamma_g(G^2) = n$ by Theorem 6.1.

Thus, $\gamma_g(G^2)$ = *Max* {γ (G^2), γ ($\overline{G^2}$) } = *n*.

Hence γ ($\overline{G^2}$) = n

Also, $n < n + 1$= γ ($\overline{G^2}$) + γ (G)

Hence, min {γ (*G*), γ ($\bar{G}$)} is γ (*G*) and γ (*G*) < $\gamma_g(G^2) = n <$ γ ($\overline{G^2}$) + γ (G)

Now, consider the graphs with γ (*G*) ≠ 1.

Here consider the two cases i) γ (*G*) ≤ γ ($\bar{G}$) or ii) γ ($\bar{G}$) ≤ γ (G) in both cases γ_g (*G*) > Min {γ (*G*), γ ($\bar{G}$)}.

But γ_g (*G*) ≤ $\gamma_g(G^2)$

Hence Min {γ (*G*), γ ($\bar{G}$)} ≤ γ g (G^2)

To prove γ_g (G^2) ≤ γ (*G*) + γ ($\overline{G^2}$). For some graphs in theorem8.3, it is true that γ_g (G^2) = γ ($\overline{G^2}$).

Hence it is obvious that $\gamma_g(G^2)$ ≤ γ (G) + γ ($\overline{G^2}$).

γ_g (G^2) = Max {γ (G^2), γ ($\overline{G^2}$)}, if γ (G^2) < γ ($\overline{G^2}$), γ_g (G^2) = γ ($\overline{G^2}$) and γ (G) < γ_g (G^2).

Thus $\gamma_g(G^2)$ ≤ γ (G) + γ ($\overline{G^2}$).

9. Scope of further research

While analyzing the relationship between global domination number of squares of certain graphs, we came across many observations which could not be analyzed completely due to the scarcity of time. The results can be expanded further in more graphs so that global domination of the squares of connected graphs can be characterized in trees as well.

10. Conclusion:

The global domination number of the squares of certain graphs like C_n, P_n along with some specific graphs whose domination number is one namely, K_n, S_n, W_d (n, m), $F_{m,n}$, Wheel graph (W_n) and a few trees like double star and Caterpillar trees were analyzed. And we tried to characterize the square of global domination number of a general graph. An attempt has been made to obtain a sharp bound for squares of global domination number of some graphs

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