Solving Some Evolution Equations with Mixed Partial Derivatives by Using Laplace Substitution - Variation Iteration Method

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Abstract: The aim of this paper is to investigate the application of integral transform combined with variation iteration method to solve evolution partial differential equations. The combined form of the Laplace substitution and variation iteration method is implemented efficiently in finding the analytical and numerical solutions of nonlinear evolution partial differential equations with mixed partial derivatives. The obtained solutions are compared to the exact solutions and other existing methods. Illustrative examples show the efficiency and the powerful of the used method.

Keywords: Laplace substitution, variational iteration method; nonlinear partial differential equation**.**

1. Introduction

Various phenomena arising in natural, nonlinear physical sciences [1, 2], and engineering [3, 4] are modeled by a class of integrable nonlinear evolution equations which can be expressed in terms of nonlinear partial differential equations (NLPDEs). Those problems have important effects in applied mathematics. Many authors had paid great attention in developing different methods for finding exact and/or approximate solutions of such models [5-13] and the references therein.

Nonlinear evolution partial differential equations involving mixed partial derivatives appear in several fields of science, physics and engineering. The important applications of these equations have obtained so much interesting from many author scientists. Until now getting the exact or approximation solutions for the most models of these equations have big problem. Solving these equations need some different methods. In the recent period, many researchers mainly had paid attention to studying the solution of these equations by using various methods [14-23]. The paper is devoted to solve some nonlinear evolution partial differential equations involving mixed partial derivatives by using a hybrid method combined the Laplace substitution method (LS) and the variational iteration method (VIM).

The concept of LS [21] was proposed by Sujit Handibag and B. D. Karande in 2012. The method is based on the application of the well-known Laplace transform. On the other hand, the VIM was developed by He [24-27] for solving linear and nonlinear PDEs. The goal of this work is to extend the application of LS with combination of VIM (LS-VIM) for solving nonlinear evolution PDEs involving mixed derivatives.

The rest of this paper is organized as follows. In Section 2, the brief description of the LS-VIM is given. In Section 3, we apply the proposed method for solving NLPDEs involving mixed partial derivatives. Finally, the conclusions are given in Section 4.

2. Description of Method

Consider the following general form of NLPDE involving mixed derivatives with initial conditions

$$\mathcal{L}u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = \hbar(x,t) \qquad (1)$$

$$u(x,0) = f(x), \quad u_t(0,t) = g(t) \qquad (2)$$

where $L=\frac{\partial^2}{\partial x\partial t}$ is a linear operator, $\mathcal{R}u(x,t)$ is the remaining of linear operator, $\mathcal{N}u(x,t)$ represents the nonlinear operator, and $\hbar(x,t)$ is the source term.

Then equation (1) can be written as:

$$\frac{\partial^2}{\partial x\partial t}u(x,t)+\mathcal{R}u(x,t)+\mathcal{N}u(x,t)=\hbar(x,t) \tag{3}$$

Let $\frac{\partial}{\partial t}=U$ then replace it in equation (3) we get

$$\frac{\partial U}{\partial x}u(x,t)+\mathcal{R}u(x,t)+\mathcal{N}u(x,t)=\hbar(x,t) \tag{4}$$

Taking Laplace transform with respect to *x* of both sides of equation (4) and apply the differentiation property of Laplace transform, we get

$$L_x\left[\frac{\partial U}{\partial x}u(x,t)\right]+L_x[\mathcal{R}u(x,t)]+L_x[\mathcal{N}u(x,t)]=L_x[\hbar(x,t)] \tag{5}$$

$$[sU(s,t)-U(0,t)]+L_x[\mathcal{R}u(x,t)]+L_x[\mathcal{N}u(x,t)]=L_x[\hbar(x,t)] \tag{6}$$

Multiplying the both sides of equation (6) by $\frac{1}{s}$ and substitution the initial conditions given in equation (2), we get

$$\left[U(s,t)-\frac{1}{s}g(t)\right]+\frac{1}{s}L_x[\mathcal{R}u(x,t)]+\frac{1}{s}L_x[\mathcal{N}u(x,t)]=\frac{1}{s}L_x[\hbar(x,t)] \tag{7}$$

Now, applying the inverse Laplace transform with respect to *x* of both sides of equation (7), yields

$$U(x,t)=[g(t)]-L_x^{-1}\left\{\frac{1}{s}L_x[\mathcal{R}u(x,t)]-\frac{1}{s}L_x[\mathcal{N}u(x,t)]+\frac{1}{s}L_x[\hbar(x,t)]\right\} \tag{8}$$

Re-substitute $\frac{\partial}{\partial t}=U$ in equation (8), we have

$$\frac{\partial U(x,t)}{\partial t}=[g(t)]-L_x^{-1}\left\{\frac{1}{s}L_x[\mathcal{R}u(x,t)]-\frac{1}{s}L_x[\mathcal{N}u(x,t)]+\frac{1}{s}L_x[\hbar(x,t)]\right\} \tag{9}$$

Now, by taking the Laplace transform of equation (9) with respect to *t* and Multiplying by $\frac{1}{s}$, then using the initial condition given in equation (2) and applying the inverse Laplace transform with respect to *t* , we get

$$u(x,t)=f(x)+L_t^{-1}\left(\frac{1}{s}L_t\left[[g(t)]-L_x^{-1}\left\{\frac{1}{s}L_x[\mathcal{R}u(x,t)]-\frac{1}{s}L_x[\mathcal{N}u(x,t)]+\frac{1}{s}L_x[\hbar(x,t)]\right\}\right]\right) \tag{10}$$

$$u_{xt}(x,t)=\frac{\partial^2}{\partial x\partial t}f(x)+\frac{\partial^2}{\partial x\partial t}L_t^{-1}\left(\frac{1}{s}L_t\left[[g(t)]-L_x^{-1}\left\{\frac{1}{s}L_x[\mathcal{R}u(x,t)]-\frac{1}{s}L_x[\mathcal{N}u(x,t)]+\frac{1}{s}L_x[\hbar(x,t)]\right\}\right]\right) \tag{11}$$

Using correction function of the variation iteration method with Lagrange multiplier $\lambda=-1$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x\int_0^t(u_n)_{xt}(x,t)-\frac{\partial^2}{\partial x\partial t}f(x)-\frac{\partial^2}{\partial x\partial t}L_t^{-1}\left(\frac{1}{s}L_t\left[[g(t)]-L_x^{-1}\left\{\frac{1}{s}L_x[\mathcal{R}u(x,t)]-\frac{1}{s}L_x[\mathcal{N}u(x,t)]+\frac{1}{s}L_x[\hbar(x,t)]\right\}\right]\right)dxdt \tag{12}$$

The solution of the given NLPDEs in equation (1) represented by equation (12) with correction function and the solution $u(x,t)$ is given by

$$u(x,t)=\lim_{n\to\infty}u_n(x,t) \tag{13}$$

3. Application of the Proposed Method

Example (3.1): [15] consider the following nonlinear partial differential equation which involving mixed derivatives,

$$\frac{\partial^2 u}{\partial x \partial t} - \left(\frac{\partial u}{\partial x}\right)^2 + u^2 = e^x \tag{11}$$

subject to the initial condition

$$u(0,t) = t \qquad , \qquad u_x(x,0) = 0 \tag{12}$$

with exact solution

$$u(x,t) = e^x t \tag{13}$$

Let $\frac{\partial u}{\partial x} = U$

$$\frac{\partial U}{\partial t} - (U)^2 + u^2 = e^x \tag{14}$$

Taking Laplace transform with respect to *t* of both sides of equation (17) and apply the differentiation property of Laplace transform, we get

$$L_t\left(\frac{\partial U}{\partial t}\right) - L_t(U)^2 + L_t(u^2) = L_t e^x \tag{15}$$

$$(sU(x,s) - u(x,0)) - L_t(U)^2 + L_t(u^2) = L_t e^x \tag{16}$$

Dividing equation (19) by *s* and substitution the initial conditions given in equation (15), we have

$$\left(U(x,s) - \frac{1}{s}0\right) - L_t^{-1}\left[\frac{1}{s}L_t(U)^2\right] + \frac{1}{s}L_t(u^2) = \frac{1}{s^2}e^x \tag{17}$$

Now, applying the inverse Laplace transform with respect to *t* of both sides of equation (20), yields

$$U(x,t) - L_t^{-1}\left[\frac{1}{s}L_t(U)^2\right] + L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right] = e^x t \tag{18}$$

Go back to $\frac{\partial u}{\partial x} = U$

$$\frac{\partial u}{\partial x} - L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right] + L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right] = e^x t \tag{19}$$

Taking Laplace transform with respect to *x* of both sides of equation (22) and apply the differentiation property of Laplace transform, we get

$$su(s,t) - u(0,t) - L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\} + L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right) = L_x(e^x t) \tag{20}$$

Dividing both sides of equation (23) by *s* and substitution the initial conditions given in equation (15), yields

$$u(s,t) - \frac{1}{s}t - \frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\} + \frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right) = \frac{1}{s}L_x(e^x t) \tag{21}$$

Applying the inverse Laplace transform with respect to *x* of both sides of equation (24), we get

$$u(x,t) - t - L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right) + L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right)\right) = e^x t - t \tag{22}$$

$$u(x,t) = L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right) - L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right)\right) + e^x t \tag{23}$$

Now,

$$u_{xt}(x,t) = \frac{\partial^2 u}{\partial x \partial t}\left[L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right)\right] - \frac{\partial^2 u}{\partial x \partial t}\left\{L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right)\right)\right\} + e^x.$$

$$u_{xt}(x,t) - \frac{\partial^2 u}{\partial x \partial t}\left[L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right)\right] + \frac{\partial^2 u}{\partial x \partial t}\left\{L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(u^2)\right]\right)\right)\right\} - e^x = 0.$$

Using correction function and $\lambda = -1$

$$\boldsymbol{u_{n+1}(x,t) = u_n(x,t) - \int_0^x \int_0^t \left((u_n)_{xt}(x,t) - \frac{\partial^2 u}{\partial x \partial t}\left[L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u_n}{\partial x}\right)^2\right]\right\}\right)\right] + \frac{\partial^2 u}{\partial x \partial t}\left\{L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t({u_n}^2)\right]\right)\right)\right\} - e^x\, dxdt\right)}$$

(24)

When $n = 0$ and $u_0(x,t) = e^x t$, we have

$$\boldsymbol{u_1(x,t) = u_0(x,t) - \int_0^x \int_0^t \left((u_0)_{xt}(x,t) - \frac{\partial^2 u}{\partial x \partial t}\left[L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right\}\right)\right] + \frac{\partial^2 u}{\partial x \partial t}\left\{L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t({u_0}^2)\right]\right)\right)\right\} - e^x\, dxdt\right)}$$

(25)

$$\boldsymbol{u_1(x,t) = e^x t - \int_0^x \int_0^t \left(e^x + \frac{\partial^2 u}{\partial x \partial t}\left[L_x^{-1}\left(\frac{1}{s}L_x\left\{L_t^{-1}\left[\frac{1}{s}L_t(e^{2x}t^2)\right]\right\}\right)\right] + \frac{\partial^2 u}{\partial x \partial t}\left\{L_x^{-1}\left(\frac{1}{s}L_x\left(L_t^{-1}\left[\frac{1}{s}L_t(e^{2x}t^2)\right]\right)\right)\right\} - e^x\, dxdt\right)}$$

(26)

$$\boldsymbol{u_1(x,t) = e^x t} \quad (27)$$

Equation (30) represents the exact solution.

Note that this example cannot be solved by LS because the equation (14) is not linear, i.e. $\mathcal{R}u(x,t) \neq \mathbf{0}$, the term $u(x,t)$ appears in the both sides of the equation and cannot be combined in a single side. The proposed LS-VIM overcomes this limitation efficiently. The exact solution of the given equation is obtained in one iteration, while it was obtained after substituting all values of $u_n(x,t)$ by using the method of Laplace substitution combined with the Adomian decomposition method [15].

Example (3.2): [22] consider the following nonlinear partial differential equation which involving mixed derivatives,

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial x} + u = 6x^2 t} \quad (28)$$

subject to the initial conditions $\boldsymbol{u(x,0) = 1}$, $\boldsymbol{u(0,t) = t}$, $\boldsymbol{u_t(0,t) = 0}$. (29)

with exact solution $\boldsymbol{u\,(x,t) = 1 - tx + t^2 x^3}$. (30)

Let $\frac{\partial u}{\partial t} = U$

$$\boldsymbol{\frac{\partial U}{\partial x} + \frac{\partial u}{\partial x} + u = 6x^2 t} \quad (31)$$

Taking Laplace transform with respect to *x* of both sides of equation (35) and apply the differentiation property of Laplace transform, we get

$$\boldsymbol{L_x\left(\frac{\partial U}{\partial x}\right) + L_x\left(\frac{\partial u}{\partial x}\right) + L_x(u) = L_x(6x^2 t)} \quad (32)$$

$$\boldsymbol{sU(s,t) - U(0,t) + su(s,t) - u(0,t) + L_x(u) = \frac{12t}{s^3}} \quad (33)$$

Dividing equation (37) by *s* and substitution the initial conditions given in equation (33), we have

$$\boldsymbol{U(s,t) - 0 + u(s,t) - \frac{1}{s}t + \frac{1}{s}L_x(u) = \frac{12t}{s^4}} \tag{34}$$

Now, applying the inverse Laplace transform with respect to *x* of both sides of equation (38), yields

$$\boldsymbol{\frac{\partial u}{\partial t} + u(x,t) - t + L_x^{-1}\left(\frac{1}{s}L_x(u)\right) = L_x^{-1}\left(\frac{12t}{s^4}\right)} \tag{35}$$

$$\boldsymbol{\frac{\partial u}{\partial t} + u(x,t) - t + L_x^{-1}\left(\frac{1}{s}L_x(u)\right) = 2tx^3} \tag{36}$$

Taking Laplace transform with respect to *t* of both sides of equation (40) and apply the differentiation property of Laplace transform, we get

$$\boldsymbol{u(x,s) - \frac{1}{s} + \frac{1}{s}L_t\big(u(x,t)\big) - \frac{1}{s}\left(\frac{1}{s^2}\right) + \frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u)\right)\right) = \frac{1}{s}L_t(2tx^3)} \tag{37}$$

Applying the inverse Laplace transform with respect to *t* of both sides of equation (41), we get

$$\boldsymbol{u(x,t) - 1 + L_t^{-1}\left(\frac{1}{s}L_t\big(u(x,t)\big)\right) - \frac{t^2}{2} + L_t^{-1}\left(\frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u)\right)\right)\right) = t^2x^3} \tag{38}$$

$$\boldsymbol{u(x,t) = 1 - L_t^{-1}\left(\frac{1}{s}L_t\big(u(x,t)\big)\right) + \frac{t^2}{2} - L_t^{-1}\left(\frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u)\right)\right)\right) + t^2x^3} \tag{39}$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) = -\frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left(\frac{1}{s}L_t\big(u(x,t)\big)\right) - \frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left(\frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u)\right)\right)\right) + 6x^2t}$$

(40)

Now using correction function and $\lambda = -1$

$$\boldsymbol{u_{n+1}(x,t) = u_n(x,t) - \int_0^x \int_0^t \left((u_n)_{xt}(x,t) + \frac{\partial^2 u}{\partial x \partial t}\left[L_t^{-1}\left(\frac{1}{s}L_t\big(u_n(x,t)\big)\right)\right] + \frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left(\frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u_n)\right)\right)\right) - 6tx^2\, dxdt\right)}$$

(41)

Taking $n = 0$ and $u_0 = 1$.

$$\boldsymbol{u_1(x,t) = u_0(x,t) - \int_0^x \int_0^t \left((u_0)_{xt}(x,t) + \frac{\partial^2 u}{\partial x \partial t}\left[L_t^{-1}\left(\frac{1}{s}L_t\big(u_0(x,t)\big)\right)\right] + \frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left(\frac{1}{s}L_t\left(L_x^{-1}\left(\frac{1}{s}L_x(u_0)\right)\right)\right) - 6tx^2\, dxdt\right)}$$

(42)

$$\boldsymbol{u_1(x,t) = 1 - \int_0^x \int_0^t (0 + 0 + 1 - 6tx^2\, dxdt)} \tag{43}$$

$$\boldsymbol{u_1(x,t) = 1 - \int_0^x \int_0^t \left(\frac{\partial^2 u}{\partial x \partial t}[t] + \frac{\partial^2 u}{\partial x \partial t}(xt) - 6tx^2\, dxdt\right)} \tag{44}$$

$$\boldsymbol{u_1(x,t) = 1 - \int_0^x \int_0^t (1 - 6tx^2\, dxdt)} \tag{45}$$

$\boldsymbol{u_1(x,t) = 1 - tx + t^2x^3}$ which is the exact solution $\boldsymbol{u(x,t)}$. (46)

The exact solution was obtained after two iterations by using other existing methods such as Laplace substation method, the Adomian decomposition method, and the homotopy perturbation method [22].

Example (3.3): [23] consider the nonlinear partial differential equation which involving mixed derivatives

$$\boldsymbol{u_{xt} + u = 0} \tag{47}$$

with exact solution

$$\boldsymbol{u(x,t) = \cosh(x-t)} \tag{48}$$

subject to the initial conditions

$$\boldsymbol{u(x,0) = \cosh(x), \quad u_t(0,t) = \sinh(t).} \tag{49}$$

Let $\frac{\partial u}{\partial t} = U$

$$\boldsymbol{\frac{\partial U}{\partial x} + u = 0} \tag{50}$$

Taking Laplace transform with respect to *x* of both sides of equation (54) and apply the differentiation property of Laplace transform, we get

$$\boldsymbol{sU(s,t) - U(0,t) + L_x(u) = 0} \tag{51}$$

Dividing equation (55) by *s* and substitution the initial conditions given in equation (53), we have

$$\boldsymbol{U(s,t) - \frac{1}{s}\sinh(t) + \frac{1}{s}L_x(u) = 0} \tag{52}$$

Now, applying the inverse Laplace transform with respect to *x* of both sides of equation (56), yields

$$\boldsymbol{U(x,t) - \sinh(t) + L_x^{-1}\left[\frac{1}{s}L_x(u)\right] = 0} \tag{53}$$

$$\boldsymbol{\frac{\partial u}{\partial t}(x,t) - \sinh(t) + L_x^{-1}\left[\frac{1}{s}L_x(u)\right] = 0} \tag{54}$$

Applying the inverse Laplace transform with respect to *t* of both sides of equation (58), we get

$$\boldsymbol{su(x,s) - u(x,0) - \sinh(t) + L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right] = 0} \tag{55}$$

Dividing equation (59) by *s* and substitution the initial conditions given in equation (53), we have

$$\boldsymbol{u(x,s) - \frac{1}{s}\cosh(x) \quad - \frac{1}{s}L_t[\sinh(t)] + \frac{1}{s}\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right] = 0} \tag{56}$$

Now, applying the inverse Laplace transform with respect to *t* of both sides of equation (60), we get

$$\boldsymbol{u(x,t) - \cosh(x) \quad - \cosh(t) - 1 + L_t^{-1}\left[\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right]\right] = 0} \tag{57}$$

$$\boldsymbol{u(x,t) = [\cosh(x) \quad + \cosh(t) + 1 - L_t^{-1}\left[\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right]\right]} \tag{58}$$

$$\boldsymbol{\frac{\partial u}{\partial t} = \sinh(t) - \frac{\partial u}{\partial t}L_t^{-1}\left[\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right]\right]} \tag{59}$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial t} = -\frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left[\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right]\right]} \tag{60}$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t}L_t^{-1}\left[\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s}L_x(u)\right]\right]\right] = 0} \tag{61}$$

Using correction function and $\lambda = -1$,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \int_0^t \left((u_n)_{xt}(x,t) + \frac{\partial^2 u}{\partial x \partial t} \left[L_t^{-1} \left[\frac{1}{s} L_t \left[L_x^{-1} \left[\frac{1}{s} L_x(u_n) \right] \right] \right] dxdt \right) \tag{62}$$

Now taking $n = 0$ and $u_0 = \cosh(x) + \cosh(t) - 1$

$$u_1(x,t) = u_0(x,t) - \int_0^x \int_0^t \left((u_0)_{xt}(x,t) + \frac{\partial^2 u}{\partial x \partial t} \left[L_t^{-1} \left[\frac{1}{s} L_t \left[L_x^{-1} \left[\frac{1}{s} L_x(u_0) \right] \right] \right] dxdt \right) \tag{63}$$

$$u_1(x,t) = \cosh(x) + \cosh(t) - 1 - \frac{1}{2}(-e^{-x}t + e^x t - e^{-t}x + e^t x - 2tx) \tag{64}$$

$$u_1(x,t) = \cosh(x-t) \tag{65}$$

$$u_1(x,t) = u(x,t) \tag{66}$$

Note that the exact solution of equation (51) was obtained in u_7 by using multi-Laplace transform method [23] comparing to the proposed LS-VIM.

Example (3.4): [20] Let us consider the nonlinear equation which reads

$$u_t - u_{xxt} + \left(\frac{u^2}{2}\right)_x = 0 \tag{67}$$

with the initial conditions $u(x,0) = x$, $u_t(0,t) = 0$ (68)

and $u_{xt}(0,t) = \frac{-1}{(1+t)^2}$ (69)

The exact solution $u(x,t) = \frac{x}{1+t}$ $\quad -\infty \le x \le \infty \quad t \ge 0$ (70)

Let $\frac{\partial u}{\partial t} = U$, then

$$U - \frac{\partial^2 U}{\partial x^2} + \left(\frac{u^2}{2}\right)_x = 0 \tag{71}$$

Taking Laplace transform with respect to *x* of the both sides of equation (75) and apply the differentiation property of Laplace transform, we get

$$L_x(U) - L_x\left(\frac{\partial^2 U}{\partial x^2}\right) + L_x\left(\frac{u^2}{2}\right)_x = 0 \tag{72}$$

$$L_x(U) - \left[s^2 U(s,t) + \frac{1}{(1+t)^2}\right] + L_x\left(\frac{u^2}{2}\right)_x = 0 \tag{73}$$

$$\frac{1}{s^2} L_x(U) - u(s,t) + \frac{1}{s^2}\frac{1}{(1+t)^2} + \frac{1}{s^2} L_x\left(\frac{u^2}{2}\right)_x = 0 \tag{74}$$

Now, applying the inverse Laplace transform with respect to *x* of both sides of equation (78), yields

$$L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{\partial u}{\partial t}\right)\right] - \frac{\partial u}{\partial t} + \left[\frac{x}{(1+t)^2}\right] + L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{u^2}{2}\right)_x\right] = 0 \tag{75}$$

Applying the inverse Laplace transform with respect to *t* of both sides of equation (79), we get

$$L_t\left[L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{\partial u}{\partial t}\right)\right]\right] - (su(x,s) - u(x,0)) + L_t\left[\frac{x}{(1+t)^2}\right] + L_t\left[L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{u^2}{2}\right)_x\right]\right] = 0 \tag{76}$$

$$\frac{1}{s} L_t\left[L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{\partial u}{\partial t}\right)\right]\right] - \left(u(x,s) - \frac{1}{s}x\right) + \frac{1}{s} L_t\left[\frac{x}{(1+t)^2}\right] + \frac{1}{s} L_t\left[L_x^{-1}\left[\frac{1}{s^2} L_x\left(\frac{u^2}{2}\right)_x\right]\right] = 0 \tag{77}$$

Now, applying the inverse Laplace transform with respect to *t* of both sides of equation (81), we get

$$L_t^{-1}\left\{\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{\partial u}{\partial t}\right)\right]\right]\right\}-(u(x,t)-x)+\frac{tx}{1+t}+L_t^{-1}\left(\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{u^2}{2}\right)_x\right]\right]\right)=0 \quad (78)$$

$$u(x,t)=L_t^{-1}\left\{\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{\partial u}{\partial t}\right)\right]\right]\right\}-x+\frac{tx}{1+t}+L_t^{-1}\left(\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{u^2}{2}\right)_x\right]\right]\right) \quad (79)$$

$$u_{xxt}(x,t)=\frac{\partial^3 u}{\partial x^2\partial t}L_t^{-1}\left\{\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{\partial u}{\partial t}\right)\right]\right]\right\}+\frac{\partial^3 u}{\partial x^2\partial t}L_t^{-1}\left(\frac{1}{s}L_t\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left(\frac{u^2}{2}\right)_x\right]\right]\right) \quad (80)$$

By substituting in the correction function and using Lagrange multiplier $\lambda=-1$ also continuing by selecting $u_0=x-xt$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x\int_0^t t+xt^2\ dxdt \quad (81)$$

Taking $u_0(x,t)=x-xt$ and $n=0$

$$u_1(x,t)=x-xt+\frac{xt^2}{2}+\frac{x^2t^3}{6} \quad (82)$$

Table (1) shows the results of $u_1(x,t)$ obtained by using LS-VIM comparing to the Exact solution and the other existing methods such as VIM, and HPM, while table (2) showing the absolute error for LS-VIM, VIM, and HPM compared to the exact solution. Figure (1) depicts the results of u_1 obtained by the proposed LS-VIM, the exact solution, the VIM, and the HPM when $t=0.1$.

Table (1) The results of u_1 obtained by LS-VIM comparing to the exact solution, the VIM, and the HPM when $t=0.1$.

x	Exact	LS-VIM U1	VIM U1	HPM U1
0.1	0.091	0.091	0.09	-0.021
0.2	0.182	0.181	0.18	-0.04
0.3	0.273	0.272	0.27	-0.06
0.4	0.364	0.362	0.36	-0.08
0.5	0.455	0.453	0.45	-.105

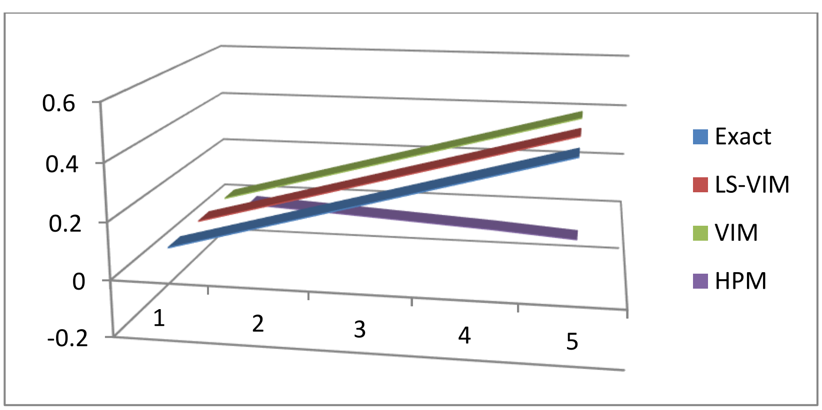

Figure (1) The result of u_1 obtained by LS-VIM, the exact solution , the VIM, and the HPM when $t=0.1$

Table (2) The absolute error of solution $u_1(x,t)$ by using the LS-VIM, the VIM, and the HPM at the values of *x* and *t* used in table (1).

Error of LS-VIM	Error of VIM	Error of HPM
0	0.01	0.112
0.001	0.002	0.222
0.001	0.003	0.333
0.002	0.004	0.444
0.002	0.005	0.56

4. Conclusions

The combined form of the Laplace substitution method together with the variational iteration method presented in this paper has been successfully implemented to solve nonlinear evolution partial differential equations including mixed derivatives. Illustrative examples show the efficiency of the proposed method throughout getting the exact and/or the numerical solutions from the first iteration. The method gives accurate results comparing with some of the existing techniques such as the variational iteration method and the homotopy perturbation method, as shown in table (1), and it is capable to solve several different types of nonlinear partial differential equations including mixed derivatives.

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