A note on extension of semi-commutative module

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Abstract: Let $\boldsymbol{\alpha}$ be endomorphism of an associative ring $\boldsymbol{R}$ and $\boldsymbol{M}$ be right $\boldsymbol{R}$ module of ring $\boldsymbol{R}$. In the present study, we investigated a relation between power series extensions of an $\boldsymbol{\alpha}$ Armendariz module $\boldsymbol{M}$ and extended the results of $\boldsymbol{\alpha}$-semicommutative rings to $\boldsymbol{\alpha}$-semicommutative modules. Some of the well-established results which were related to $\boldsymbol{\alpha}$-rigid rings, $\boldsymbol{\alpha}$-semicommutative rings and $\boldsymbol{\alpha}$-rigid modules are obtained as the corollaries of the outcomes given here.

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1. Introduction

In the present study $\boldsymbol{R}$ represents one associative ring with an identity while $\boldsymbol{M}$ represents a **right** $\boldsymbol{R}$-module. Recall that the ring $\boldsymbol{R}$ is $\boldsymbol{\alpha}$-semicommutative if $\boldsymbol{ab = 0 \Rightarrow aRb = 0}$ and $\boldsymbol{a\alpha(b) = 0 \Leftrightarrow ab = 0}$, where $\boldsymbol{\alpha}$ being an endomorphism of $\boldsymbol{R}$. $\boldsymbol{M}$ module is $\boldsymbol{\alpha}$-semicommutative for the case, (i) $\boldsymbol{ma = 0 \Rightarrow mRa = 0}$, (ii) $\boldsymbol{ma = 0 \Leftrightarrow m\alpha(a) = 0}$, where $\boldsymbol{\alpha}$ be an endomorphism of $\boldsymbol{R}$ [6]. **Kaplansky** presented the Baer rings for abstracting different von-Neumann algebras properties & AW *-Algebras. The Bear ring $\boldsymbol{R}$ is right annihilator for its non-empty subsets as generated by idempotent [5].

Von-Neumann Algebras are a type of Baer ring. As a generalization of Baer ring, **Clark** [3] defined quasi-Baer ring & used it for characterizing the finite dimension Algebra with unity over a field that is algebraically closed and is isomorphic to a twisted-matrix units semigroup algebra. A different Baer ring generalization is a p.p. ring. If the right (respectively left) annihilator of an element of R was formed by an idempotent, then R is a right (respectively left) p.p. ring.

Zhou and Lee [6] developed the p.p. modules, quasi-Baer and Baer, as below conditions,

(i) M is p.p. module if, for any element $m \in M$, $ann_R(m) = eR$, where $e^2 = e \in R$.

(ii) M is quasi-Baer if, for any sub-module $N \subseteq M$, $ann_R(N) = eR$, where $e^2 = e \in R$.

(iii) M is Baer if, for any subset N of M, $ann_R(N) = eR$, where $e^2 = e \in R$.

We write $R[x]$ is a polynomial ring, $R[[x]]$ a power series ring, $R[x, x^{-1}]$ Laurent polynomial ring and $R[[x, x^{-1}]]$ is a Laurent power series ring over a ring R. **Zhou and Lee** [6] did introduce a notation for M module as,

$$M[x;\alpha] = \left\{ \sum_{i=0}^{p} m_i x^i \mid p \geq 0, m_i \in M \right\}$$

$$M[[x;\alpha]] = \left\{ \sum_{i=0}^{\infty} m_i x^i \mid m_i \in M \right\}$$

Each of the above is abelian group underneath the addition condition. Furthermore, $M[x;\alpha]$ is a module for $R[x;\alpha]$ under the product operation as:

$$m(x) \qquad = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha],$$

$$f(x) = \sum_{j=0}^{q} f_q x^q \in R[x;\alpha]$$
$$m(x)f(x) = \sum_{k=0}^{p+q} (\sum_{i+j=k} m_i \alpha^i(f_j))x^k$$

In the same way, $\boldsymbol{M}[[\boldsymbol{x};\boldsymbol{\alpha}]]$ transforms into a module on $\boldsymbol{R}[[\boldsymbol{x};\boldsymbol{\alpha}]]$. The $\boldsymbol{M}[[\boldsymbol{x};\boldsymbol{\alpha}]]$ is skew power series extension and $\boldsymbol{M}[\boldsymbol{x};\boldsymbol{\alpha}]$ is skew polynomial extension module for $\boldsymbol{M}$**.**

Again, from [6], module M is known as α-Armendariz of power-series kind as per the below conditions:

(i) For $m \in M$ and $a \in R$, $ma = 0$ for the case if $m\alpha(a) = 0$

(ii) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$

$$m(x)f(x) = 0 \quad \text{imply} \quad m_i\alpha^i(a_j) = 0 \text{ for all i and j.}$$

In [2], **Baser et al**. proved some results for α-semicommutative, quasi-Baer rings, p.p. rings. Motivated by above results, we generalize the results for α-semicommutative module which is an α-rigid module generalization and the extension of semi commutative module.

2. Main Results

Under the following section, the main results are proved. To prove these results, we have used some lemmas which have been taken from [1].

Lemma 2.1 *Assume* $\boldsymbol{R}$ *is a ring for* $ab = 0$ *which imply that,* $aRb = 0$, *for* $a, b \in R$, *then* $\alpha(e) = e$ *for idempotent* $e \in R$.

Proof Refer to [1, Lemma 2.3]

Example 2.2 ([4, Example 2]) *If the module* $\boldsymbol{M}$ *is semicommutative, it is not compulsory* $M[[x]]$ $(M[[x;\alpha]])$ *to be semicommutative (*α*-semicommutaitve).*

Theorem 2.3 *Assume that* M *is* α*-Armendariz power series module. Then* $M[[x;\alpha]]$ *is semicommutative in case,* M *is* α*-semicommutative.*

Proof: Let, $M[[x;\alpha]]$ is semicommutative and M be an α-Armendariz module.

Let $ma = 0$ for any $m \in M$ and $a \in R$.

Since $M[[x;\alpha]]$ is semicommutative so $mR[[x;\alpha]]a = 0$. Thus,

$$mf(x)a = 0, \quad \forall\ f(x) = \sum_{i=0}^{\infty} f_i x^i \in R[[x;\alpha]]$$

$$\Rightarrow m(f_0 + f_1 x + \cdots)a = 0$$

$$\Rightarrow mf_0 a + mf_1 xa + mf_2 x^2 a + \cdots = 0$$

$$\Rightarrow mf_0 a + mf_1 \alpha(a)x + mf_2 \alpha^2(a)x^2 + \cdots = 0$$

$$\Rightarrow mf_0 a = 0, mf_1 \alpha(a) = 0$$

$$\Rightarrow mR\alpha(a) = 0.$$

So, M is α-semicommutative.

Further, suppose M is α-semicommutative and α-Armendariz power series module.

Assume $a \in R$ and $m \in M$ with $ma = 0 \Rightarrow mR\alpha(a) = 0$. Let $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $g(x) = \sum_{j=0}^{\infty} g_j x^j \in R[[x;\alpha]]$ such that $m(x)g(x) = 0 \Rightarrow m_i \alpha^i(g_j) = 0$, $\forall\ i, j$. Where M is α-Armendariz module of power series type so $m_i g_j = 0$, $\forall\ i$ and j. Also M is α-semicommutative so $m_i R\alpha(g_j) = 0$, $\forall\ k = 1,2,3,4, \ldots.$

Now take any

$$h(x) \quad = \sum_{k=0}^{\infty} h_k x^k \in R[[x;\alpha]].$$

$$m(x)R[[x;\alpha]]g(x) = m(x)h(x)g(x), \forall h(x) \in R[[x;\alpha]]$$

$$= (m_0 + m_1 x + m_2 x^2 + \cdots)(h_0 + h_1 x + h_2 x^2 + \cdots)$$

$$(f_0 + f_1 x + f_2 x^2 + \cdots)$$

$$= (m_0 h_0 f_0) + (m_0 h_0 f_1 + m_0 h_1 \alpha(f_0) + m_1 \alpha(h_0) f_0)x + \cdots$$

$$= 0 + 0 + 0 + 0$$

Since M is α-Armendariz module and α-semicommutative.

So $m(x)h(x)g(x) = 0$, $\forall\ h(x) \in R[[x;\alpha]]$.

Hence $M[[x;\alpha]]$ is semicommutative.

Baser et al. proved that in case, R is α-SPS Armendariz ring, in that time R is the Baer (quasi-Baer) ring only in case $R[[x;\alpha]]$ is also a Baer ring (quasi-Baer) [1]. Here we extend these results for power series module $M[[x;\alpha]]$.

Theorem 2.4 *Let M is α-Armendariz power series module. Then M is a quasi-Baer module only in case $M[[x;\alpha]]$ is quasi-Baer module.*

Proof: Suppose M is quasi-Baer, now for proving, $M[[x;\alpha]]$ is quasi-Baer module, i.e., $r_{R[[x;\alpha]]}(N) = eR[[x;\alpha]]$ for a subset $N \subseteq M[[x;\alpha]]$ and $e^2 = e \in R$. Let N' be right sub-module of M that is generated by the coefficients of N. As M is the quasi-Baer module so $r_R(N') = eR$, for $e \in R$. Here, M is α-Armendariz power series module and $e \in R[[x;\alpha]]$ so by [5, Lemma 3.5], $eR[[x;\alpha]] \subseteq r_{R[[x;\alpha]]}(N)$. Now suppose any

$$f(x) = \sum_{j=0}^{\infty} f_j x^j \in r_{R[[x;\alpha]]}(N)$$

Where $m_i \in N'$.

Again M is a power series type α-Armendariz module, so $m_i \alpha^i(f_j) = 0 \Rightarrow m_i f_j = 0$, $\forall\ i, j$. Thus $f_j \in r_R(N')$. Also M is quasi-Baer module so $f_j \in r_R(N') = eR$ where $e^2 = e \in R$. Hence there exists, $r_0, r_1, r_2, \ldots \in R$ such that $f_0 = er_0$, $f_1 = er_1$, $f_2 = er_2$ .... Thus, $f_j = er_j \Rightarrow f(x) = er(x) \in eR[[x;\alpha]]$. So $r_{R[[x;\alpha]]}(N) \subseteq eR[[x;\alpha]]$. Therefore $M[[x;\alpha]]$ is quasi-Baer module.

Contrarily, suppose, $M[[x;\alpha]]$ be quasi-Baer module where, K is a non-empty sub-module of M. Here, $r_{R[[x;\alpha]]}(K) = eR[[x;\alpha]]$ for $e^2 = e \in R$ idempotent .

Hence,

$$r_R(K) = r_{R[[x;\alpha]]}(K) \cap R$$

$= eR[[x;\alpha]] \cap R$

$= eR$

and Hence, the quasi-Baer is given by M

Corollary 2.5 *Assume M is a α-rigid module. Hence, M becomes Baer module only when $M[[x;\alpha]]$ is the Baer module.*

Corollary 2.6 ([2, Theorem 3.6 (1)]). *Assume R stands for α-SPS Armendariz ring. Hence, R is Baer (resp. quasi-Baer) ring only in case $R[[x;\alpha]]$ is Baer ring.*

Theorem 2.7 *Suppose M is a α-Armendariz module of type power series. If $M[[x;\alpha]]$ is a p.p. module, in that case M too is a p.p. module.*

Proof Suppose M be an α-Armendariz module of power series type and $M[[x;\alpha]]$ is a p.p. module. Now suppose any element $m \in M$ so $m \in M[[x;\alpha]]$ and there is an idempotent being $e(x) = e_0 + e_1x + e_2x^2 + \cdots \in R[[x;\alpha]]$ such that $r_{R[[x;\alpha]]}(m) = e(x)R[[x;\alpha]]$. So $me(x) = 0 \Rightarrow me_0 = 0$. Thus $e_0R \subseteq r_R(m)$. Let any element $a \in r_R(m) \Rightarrow a \in r_{R[[x;\alpha]]}(m)$. Since M is an α-Armendariz power series module. Now $M[[x;\alpha]]$ is p.p. module so $a = e(x)a \Rightarrow a = e_0a \Rightarrow a \in e_0R \Rightarrow r_R(m) \subseteq e_0R$, hence M is p.p. module.

Corollary 2.8 ([2, Theorem 3.6 (2)]) *Let R be a α-SPS-Armendariz ring and if $R[[x;\alpha]]$ is right p.p. ring, then R too will be right p.p. ring.*

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