Research Article

Estimation of Parameters Model Based on a Modification of Newton-Raphson Method

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Abstract. The study concern with two main ideas. The first one focuses on generating an odd distribution that involves a combination of Lomax distribution with extreme value distribution. The second one is devoted to estimate the parameter model that will be obtained from the generated odd distribution. A simulation part of the modification technique of Newton-Raphson method has been proposed to evaluate the parameters values by using an initial value to each parameter. We summarize this work with some statements that highlight the proposed ideas.

1. Introduction

This study concerned with a generating an odd distribution by combing the extreme value distribution with the Lomax distribution, denoted by the ODEL. Afterwards, we estimate the new parameter model obtained from this combination by using the numerical methods like the Newton-Raphson method. The usage of such method is crucial due to the number of nonlinear equations that each one is contained. Also, there is an extension has been made by replacing the location parameter of the ODEL-distribution with a linear model $\bar{\beta}'\bar{z}, (\bar{\beta} = (\beta_o, \beta_1, \dots, \beta_n), \bar{z} = (z_o, z_1, \dots, z_n))$. The extended model of the ODEL-distribution is denoted by EODEL-distribution. This approach aims to enhance the analysis of data set that have been used to compare the ODEL-distribution to its extended formula. We have also used the measure-of-goodness to find out the best distribution that could fits data among others.

Extreme value distributions (EVDs) are necessary to demonstrate and quantify events that occur with low probability and have been used extensively as part of risk management, finance, insurance, financial aspects, mathematics, hydrogen, materials science, broadcast communications, and many data sets of extreme events. The EVDs category includes three types of EVD in the literature: (Type I) Gumbel, (Type Two) Fréchet, (Type Three) Weibull These diseases are generalized by combining position, scale, and power parameters leading to maximum value distributions (GEVDs): generalized Gumbel, generalized Frèchet and generalized Weibull.

Maximum probability is a relatively simple way to construct the estimator of unknown parameter 0. It was introduced by R.A Fisher, a great English mathematical statistician, in 1912. The Maximum Likelihood Estimation (MLE) can be applied to most problems, has an intuitive gravitational value, and often results in a reasonable estimate of 0. Moreover, if the sample is large, Then the method will produce an excellent estimate of 0. For these reasons, the maximum likelihood method is perhaps the most widely used estimation method in statistics.

In the modern era, there has been an increasing interest in introducing new generators of univariate distributions through at least one additional shape parameter events for the baseline distribution. This additional parameter value was demonstrated in the tail properties in addition to increasing the proposed family suitability. Some common generators are beta-G [1]. gamma-G (type 1) [2],Kumaraswamy-G [3], gamma-G (type 2) [4], McDonald-G [5], gamma-G (type 3) [6], exponentiated generalized-G [7], Transformed-Transformer (T-X) [8], Weibull-G [9], Garhy-G [10], exponentiated Weibull-G [11], Kumaraswamy Weibull-G [12], type II half logistic-G [13] and exponentiated extended-G family [14].

This paper consist of four parts. In the next part we present the most basic concepts related to the odd distribution and its properties. Part three is devoted to explain our methodology, derivations, properties, analysis of data set, and comparison. The last part is devoted to present summaries and conclusion related to the results that we have gain within this study.

2. Basic Notions

We review some basic notions related to extreme value distribution, and Lomax distribution. The pdf, the cdf, survival function $S(x)$ and hazard function $h(x)$ of the extreme value distribution are, respectively

$$f(x) = \frac{1}{\theta} e^{-(\frac{x-\lambda}{\theta})} e^{-e^{-\left(\frac{x-\lambda}{\theta}\right)}}, -\infty < x < \infty \tag{2.1}$$

$$F(x) = e^{-e^{-\left(\frac{x-\lambda}{\theta}\right)}},\ -\infty < \lambda < \infty, \theta > 0 \tag{2.2}$$

$$S(x) = 1 - F(x),\ -\infty < \lambda < \infty, \theta > 0 \tag{2.3}$$

$$h(x) = \frac{f(x)}{S(x)} \tag{2.4}$$

Also, the pdf, cdf, survival function and hazard function of Lomax distribution are, respectively.

$$g(x) = \frac{\alpha}{\sigma}(1 + \frac{x}{\sigma})^{-(\alpha+1)} \qquad 0 < x < \infty\ , \alpha, \sigma > 0 \tag{2.5}$$

Where, $\alpha\ and\ \sigma$ are the shape parameter and scale parameter.

$$G(x) = 1 - \left(1 + \frac{x}{\sigma}\right)^{-\alpha}, \bar{G}(x) = \left(1 + \frac{x}{\sigma}\right)^{-\alpha}, \qquad 0 < x < \infty\ , \alpha, \sigma > 0 \tag{2.6}$$

In particular, we combine extreme value distribution with Lomax distribution in order to obtain an odd distribution. We have named it by ODEL-distribution. It is firstly necessary to recall cdf, and survival functions of Lomax distributions in (2.6). Therefore, the odd ratio that represents the random variable of the generating odd distribution has the following formula

$$\mathrm{X} = \frac{G(x)}{\bar{G}(x)} = \left(1 + \frac{x}{\sigma}\right)^{\alpha} - 1, \quad 0 < x < \infty\ , \alpha, \sigma > 0 \tag{2.7}$$

In fact, the random variable in equation (2.7) will be the new random variable. By substituting such random variable into the distribution function in (2.2), it yields that

$$F_1(x) = \mathrm{e}^{\left[-e^{-\frac{1}{\theta}\left[\left(1+\frac{x}{\sigma}\right)^{\alpha} - (1+\lambda)\right]}\right]}, \quad -\infty < x < \infty, -\infty < \lambda < \infty\ , \alpha, \sigma > 0 \tag{2.8}$$

By differentiate $F_1(x)$ with respect to x, we obtain the pdf of ODEL-distribution, that is

$$f_1(x) = \frac{\alpha}{\theta\sigma}\left(1 + \frac{x}{\sigma}\right)^{\alpha-1} \mathrm{e}^{-\frac{1}{\theta}\left[\left(1+\frac{x}{\sigma}\right)^{\alpha} - (1+\lambda)\right]}\ e^{-\mathrm{e}^{-\frac{1}{\theta}\left[(1+\frac{x}{\sigma})^{\alpha} - (1+\lambda)\right]}}, -\infty < x < \infty \tag{2.9}$$

The pdf in (2.9) can be modified by replacing its location parameter λ by a linear model $\bar{\beta}'\bar{z}, (\bar{\beta} = (\beta_o, \beta_1, \dots, \beta_n), \bar{z} = (z_o, z_1, \dots, z_n))$. This modification is an extension to the ODEL-distribution, denoted by EODEL-distribution. Then, the pdf of EODEL-distribution is

$$f_2(x) = \frac{\alpha}{\theta\sigma}\left(1 + \frac{x}{\sigma}\right)^{\alpha-1} \mathrm{e}^{-\frac{1}{\theta}\left[\left(1+\frac{x}{\sigma}\right)^{\alpha} - (1+\bar{\beta}\prime\bar{z})\right]} e^{-\mathrm{e}^{-\frac{1}{\theta}\left[(1+\frac{x}{\sigma})^{\alpha} - (1+\bar{\beta}\prime\bar{z})\right]}}, \tag{2.10}$$

And its distribution function is

$$F_2(x) = e^{\left[-\mathrm{e}^{-\frac{1}{\theta}\left[(1+\frac{x}{\sigma})^{\alpha} - (1+\bar{\beta}\prime\bar{z})\right]}\right]}, \qquad -\infty < x < \infty, \sigma, \alpha > 0, \tag{2.11}$$

The last notion that we need to present in this part related to the measures of quality that we will use to compare the generated odd distribution to its extended one. These measures of goodness-of-fit are called Akanke information criterion (AIC), consistent Akanke information criterion (CAIC), Haman-Quinn information criterion (HQIC), and Bayesian information criterion (BIC), respectively. They are widely used to find out which distribution is the best to fit data among others. Also, they are very well-known statistical tools in many statistical inferences, and they have the following forms, respectively.

$$AIC = -2\,\hat{\ell} + 2h \tag{2.12}$$

$$BIC = -2\,\hat{\ell} + h\ln(n) \tag{2.13}$$

$$CAIC = -2\,\hat{\ell} + \frac{2hn}{n-h-1} \tag{2.14}$$

$$HQIC = -2\,\hat{\ell} + 2h\ln(\ln(n)) \tag{2.15}$$

Where, $\hat{\ell}$ is the maximum log-likelihood function, h is the number of the parameters of each estimated distribution within the maximum likelihood method, n is the sample size of the desired data.

$$\ell(\theta) \;=\; log\;\ell(\theta) \;=\; \log\prod_{i=1}^{n} f(X_i;\theta) \;=\; \sum_{i=1}^{1}\log f(X_i;\theta) \tag{2.16}$$

Maximizing $\ell(\theta)$ with respect to θ will give us the MLE estimation.

It is well-known fact that maximum likelihood method sometimes leads to nonlinear equations that need numerical methods to find their solutions. One of the best common method associated with the maximum likelihood method is well-known by the Newton-Raphson method that has the following general formula:

$$x_{\boldsymbol{i+1}} = x_i + \frac{f(x)}{\acute{f}(x)}, \qquad\qquad i = 1,..,n \tag{2.17}$$

3. Modification, and Simulation

3.1. Modification, and Derivations

In [15], it has shown in detail the log-likelihood of the ODEL-distribution, and the EODEL-distribution, respectively. Their general forms are

$$\ell_1 = \ln(l_1) = \mathrm{n}\ln\alpha - n\ln\theta - \mathrm{n}\ln\sigma + (\alpha-1)\sum_{i=1}^{n}\ln[1+\frac{x_i}{\sigma}] - \sum\nolimits_{i=1}^{n} W_i - \sum_{i=1}^{n} e^{-W_i}\,, \tag{3.1}$$

$$\ell_2 = \ln(l_2) = \mathrm{n}\ln\alpha - n\ln\theta - \mathrm{n}\ln\sigma + (\alpha-1)\sum_{i=1}^{n}\ln[1+\frac{x_i}{\sigma}] - \sum\nolimits_{i=1}^{n} M_i - \sum_{i=1}^{n} e^{-M_i}\,, \tag{3.2}$$

where $W_i = \frac{1}{\theta}\left[\left(1+\frac{x_i}{\sigma}\right)^{\alpha} - (1+\lambda)\right]$ and $M_i = \frac{1}{\theta}\left[\left(1+\frac{x_i}{\sigma}\right)^{\alpha} - \left(1+\bar{\beta}'\bar{z}\right)\right]$

We estimate the parameters of each distribution by implementing the maximum likelihood method the log-likelihood functions in (3.1), (3.2), so this yields the following functions

$$f_1^{\,*}(\theta,\alpha,\sigma,\lambda,\check{x}) = \frac{\partial\ell_1}{\partial\theta} = \frac{-n}{\theta} + \frac{1}{\theta}\sum_{i=1}^{n} W_i\,[1-e^{-W_i}] = 0, \tag{3.3}$$

$$f_2^{\,*}(\theta,\alpha,\sigma,\lambda,\tilde{x}) = \frac{\partial\ell_1}{\partial\alpha} = \frac{\mathrm{n}}{\alpha} + \sum_{\mathrm{i}=1}^{\mathrm{n}}\ln\left[1+\frac{x_i}{\sigma}\right] - \frac{1}{\theta}\sum_{\mathrm{i}=1}^{\mathrm{n}}\ln\left[1+\frac{x_i}{\sigma}\right]\left(1+\frac{x_i}{\sigma}\right)^{\alpha}[1-\mathrm{e}^{-W_i}] = 0, \tag{3.4}$$

$$f_3^{\,*}(\theta,\alpha,\sigma,\lambda,\tilde{x}) = \frac{\partial\ell_1}{\partial\sigma} = \frac{-\mathrm{n}}{\sigma} - \frac{(\alpha-1)}{\sigma^2}\sum_{\mathrm{i}=1}^{\mathrm{n}}\frac{x_i}{\left(1+\frac{x_i}{\sigma}\right)} + \frac{\alpha}{\theta\sigma^2}\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{x_i}\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}[1-\mathrm{e}^{-\mathrm{W_i}}] = 0, \tag{3.5}$$

$$f_4^{\,*}(\theta,\alpha,\sigma,\lambda,\tilde{x}) = \frac{\partial\ell_1}{\partial\lambda} = \frac{n}{\theta} - \frac{1}{\theta}\sum_{i=1}^{n} e^{-W_i} = 0, \tag{3.6}$$

We also need to show the elements of variance-covariance matrix, that are

$$\frac{\partial f_1{}^*}{\partial \theta}=\frac{n}{\theta^2}-\frac{2}{\theta^2}\sum_{i=1}^{n}W_i+\frac{1}{\theta^2}\sum_{i=1}^{n}W_ie^{-W_i}(2-\frac{1}{\theta}W_i) \tag{3.7}$$

$$\frac{\partial f_1{}^*}{\partial \alpha}=\frac{1}{\theta^2}\sum_{i=1}^{n}\mathrm{Log}[1+\frac{x_i}{\sigma}](1+\frac{x_i}{\sigma})^{\alpha}-\frac{1}{\theta^2}\sum_{i=1}^{n}e^{-W_i}\mathrm{Log}\left[1+\frac{x_i}{\sigma}\right]\left(1+\frac{x_i}{\sigma}\right)^{\alpha}[1-W_i] \tag{3.8}$$

$$\frac{\partial f_1{}^*}{\partial \sigma}=-\frac{\alpha}{\theta^2\sigma^2}\sum_{i=1}^{n}x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}+\frac{\alpha}{\theta^2\sigma^2}\sum_{i=1}^{n}x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}e^{-W_i}[1-W_i] \tag{3.9}$$

$$\frac{\partial f_1{}^*}{\partial \lambda}=-\frac{n}{\theta^2}+\frac{1}{\theta^2}\sum_{i=1}^{n}e^{-W_i}[1-W_i] \tag{3.10}$$

$$\frac{\partial f_2{}^*}{\partial \alpha}=-\frac{n}{\alpha^2}-\frac{1}{\theta}\sum_{i=1}^{n}\mathrm{Log}[1+\frac{x_i}{\sigma}]^2(1+\frac{x_i}{\sigma})^{\alpha}+\frac{1}{\theta}\sum_{i=1}^{n}(\mathrm{Log}[1+\frac{x_i}{\sigma}])^2(1+\frac{x_i}{\sigma})^{\alpha}e^{-W_i}\left[1-\frac{1}{\theta}(1+\frac{x_i}{\sigma})^{\alpha}\right], \tag{3.11}$$

$$\frac{\partial f_2{}^*}{\partial \sigma}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{x_i}{\left(1+\frac{x_i}{\sigma}\right)}+\frac{1}{\theta\sigma^2}\sum_{i=1}^{n}\left(x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}+\alpha\mathrm{Log}\left[1+\frac{x_i}{\sigma}\right]x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}\right)[1-e^{-W_i}]$$
$$+\frac{\alpha}{\theta^2\sigma^2}\sum_{i=1}^{n}x_i\left(1+\frac{x_i}{\sigma}\right)^{2\alpha-1}(\mathrm{Log}\left[1+\frac{x_i}{\sigma}\right])e^{-W_i}, \tag{3.12}$$

$$\frac{\partial f_2{}^*}{\partial \lambda}=\frac{1}{\theta^2}\sum_{i=1}^{n}\mathrm{Log}[1+\frac{x_i}{\sigma}](1+\frac{x_i}{\sigma})^{\alpha}e^{-W_i} \tag{3.13}$$

$$\frac{\partial f_3{}^*}{\partial \sigma}=\frac{n}{\sigma^2}+\frac{(\alpha-1)}{\sigma^3}\sum_{i=1}^{n}(\frac{2x_i}{(1+\frac{x_i}{\sigma})}-\frac{x_i{}^2}{\sigma(1+\frac{x_i}{\sigma})^2})-\frac{\alpha^2}{\theta^2\sigma^4}\sum_{i=1}^{n}x_i{}^2\left(1+\frac{x_i}{\sigma}\right)^{2(\alpha-1)}e^{-W_i}$$
$$+\frac{\alpha}{\theta\sigma^4}\sum_{i=1}^{n}\left((\alpha-1)x_i{}^2\left(1+\frac{x_i}{\sigma}\right)^{\alpha-2}+2\sigma x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}\right)[e^{-W_i}-1], \tag{3.14}$$

$$\frac{\partial f_3{}^*}{\partial \lambda}=-\frac{\alpha}{\theta^2\sigma^2}\sum_{i=1}^{n}x_i\left(1+\frac{x_i}{\sigma}\right)^{\alpha-1}e^{-W_i}, \tag{3.15}$$

$$\frac{\partial f_4{}^*}{\partial \lambda}=-\frac{1}{\theta^2}\sum_{i=1}^{n}e^{-W_i}, \tag{3.16}$$

Note

$$\frac{\partial f_1{}^*}{\partial \alpha}=\frac{\partial f_2{}^*}{\partial \theta},\qquad \frac{\partial f_1{}^*}{\partial \sigma}=\frac{\partial f_3{}^*}{\partial \theta},\qquad \frac{\partial f_1{}^*}{\partial \lambda}=\frac{\partial f_4{}^*}{\partial \theta},$$

$$\frac{\partial f_2{}^*}{\partial \sigma}=\frac{\partial f_3{}^*}{\partial \alpha},\qquad \frac{\partial f_2{}^*}{\partial \lambda}=\frac{\partial f_4{}^*}{\partial \alpha},\qquad \frac{\partial f_3{}^*}{\partial \lambda}=\frac{\partial f_4{}^*}{\partial \sigma}$$

In particular, when $\lambda = \bar{\beta}'\bar{z}$, and when $k = 2$, $\lambda = \beta_0 + \beta_1 Z_1$ we can find out that the function $f^*{}_1, f^*{}_2, f^*{}_3, f^*{}_4$ with respect to the linear model, they have the same forms in (3.3), (3.4), (3.5) and (3.6). Therefore, we only need to replace the function symbols like h to refer to the new obtained functions with that extension. In fact, due to the linear model $\lambda = \beta_0 + \beta_1 Z_1$, we obtain one more function of the parameter β_1, that is

$$f_5{}^*(\theta, \alpha, \sigma, \lambda, \tilde{x}) = \frac{\partial \ell_2}{\partial \beta_1} = \frac{1}{\theta}\sum_{i=1}^{n} Z_{i1} - \frac{1}{\theta}\sum_{i=1}^{n} Z_{i1}\, e^{-M_i} = 0, \tag{3.17}$$

$$\frac{\partial f_5{}^*}{\partial \theta} = \frac{-1}{\theta^2}\sum_{i=1}^{n} Z_{i1} + \frac{1}{\theta^2}\sum_{i=1}^{n} Z_{i1}\, e^{-M_i}(1 - M_i) \tag{3.18}$$

$$\frac{\partial f_5{}^*}{\partial \alpha} = \frac{1}{\theta^2}\sum_{i=1}^{n} Z_{i1}\, \mathrm{g}[1 + \frac{x_i}{\sigma}](1 + \frac{x_i}{\sigma})^{\alpha} e^{-M_i} \tag{3.19}$$

$$\frac{\partial f_5{}^*}{\partial \sigma} = -\frac{\alpha}{\theta^2\sigma^2}\sum_{i=1}^{n} Z_{i1}\, x_i \left(1 + \frac{x_i}{\sigma}\right)^{\alpha-1} e^{-M_i} \tag{3.20}$$

$$\frac{\partial f_5{}^*}{\partial \beta_0} = -\frac{1}{\theta^2}\sum_{i=1}^{n} Z_{i1}\, e^{-M_i} \tag{3.21}$$

$$\frac{\partial f_5{}^*}{\partial \beta_1} = -\frac{1}{\theta^2}\sum_{i=1}^{n} Z_{i1}{}^2\, e^{-M_i} \tag{3.22}$$

We can see that the forms below are equivalent.

$$\frac{\partial f_5{}^*}{\partial \theta} = \frac{\partial f_1{}^*}{\partial \beta_1}, \quad \frac{\partial f_5{}^*}{\partial \alpha} = \frac{\partial f_2{}^*}{\partial \beta_1}, \quad \frac{\partial f_5{}^*}{\partial \sigma} = \frac{\partial f_3{}^*}{\partial \beta_1}, \quad \frac{\partial f_5{}^*}{\partial \beta_0} = \frac{\partial f_4{}^*}{\partial \beta_1}$$

In order to provide the remaining terminologies of the Newton-Raphson method in (2.17), we need to find the partial derivatives of the function in (3.3), (3.4), (3.5) and (3.6), and also the function in (3.17).

The proposed modification considers that we have to set an initial value to each parameter in (3.1), (3.2).

In this case, we set $\theta = \sigma = \alpha = 1$ and $\lambda = 10$, while the extra parameters β_0, β_1 have the initial values $\beta_0 = 10,\ \beta_1 = 0$. By implementing the Newton-Raphson general form, we can write the estimation formula of each parameter, by the following way

$$\hat{\theta}^{\delta+1} = \hat{\theta}^{\delta} + \gamma_1 \tag{3.23}$$

$$\hat{\alpha}^{\delta+1} = \hat{\alpha}^{\delta} + \gamma_2 \tag{3.24}$$

$$\hat{\sigma}^{\delta+1} = \hat{\sigma}^{\delta} + \gamma_3 \tag{3.25}$$

$$\hat{\lambda}^{\delta+1} = \hat{\lambda}^{\delta} + \gamma_4 \tag{3.26}$$

By replacing $\hat{\lambda}$ with a linear model $\hat{\beta}_o + \hat{\beta}_1 z_1$, the formula in (3.26), can be written as.

$$\hat{\beta}_o{}^{\delta+1} = \hat{\beta}_o{}^{\delta} + \gamma_4 \tag{3.27}$$

$$\hat{\beta}_1{}^{\delta+1} = \hat{\beta}_1{}^{\delta} + \gamma_5 \tag{3.28}$$

$$\gamma_j = -A^{-1}\bar{f}^*{}_j(x), \qquad j = 1, \ldots, 5 \tag{3.29}$$

$$\bar{f}^{*}(x)=\begin{bmatrix} f_1^{*} \\ f_2^{*} \\ f_3^{*} \\ f_4^{*} \\ f_5^{*} \end{bmatrix} \quad and \quad B=\begin{bmatrix} \frac{\partial f_1^{*}}{\partial\theta} & \frac{\partial f_1^{*}}{\partial\alpha} & \frac{\partial f_1^{*}}{\partial\sigma} & \frac{\partial f_1^{*}}{\partial\beta_0} & \frac{\partial f_1^{*}}{\partial\beta_1} \\ \frac{\partial f_2^{*}}{\partial\theta} & \frac{\partial f_2^{*}}{\partial\alpha} & \frac{\partial f_2^{*}}{\partial\sigma} & \frac{\partial f_2^{*}}{\partial\beta_0} & \frac{\partial f_2^{*}}{\partial\beta_1} \\ \frac{\partial f_3^{*}}{\partial\theta} & \frac{\partial f_3^{*}}{\partial\alpha} & \frac{\partial f_3^{*}}{\partial\sigma} & \frac{\partial f_3^{*}}{\partial\beta_0} & \frac{\partial f_3^{*}}{\partial\beta_1} \\ \frac{\partial f_4^{*}}{\partial\theta} & \frac{\partial f_4^{*}}{\partial\alpha} & \frac{\partial f_4^{*}}{\partial\sigma} & \frac{\partial f_4^{*}}{\partial\beta_0} & \frac{\partial f_4^{*}}{\partial\beta_1} \\ \frac{\partial f_5^{*}}{\partial\theta} & \frac{\partial f_5^{*}}{\partial\alpha} & \frac{\partial f_5^{*}}{\partial\sigma} & \frac{\partial f_5^{*}}{\partial\beta_0} & \frac{\partial f_5^{*}}{\partial\beta_1} \end{bmatrix} \tag{3.30}$$

The estimation technique considers an initial value to each parameter, then find the other values of the required estimated parameter within the new estimated value from the forms in ((3.23), (3.24), (3.25), (3.26), (3.28), (3.29)), this technique requires repeating the estimation over the forms above until the absolute value of the previous estimated parameter with the next estimated parameter is less than or equal a to a small value of epsilon. In fact, this value of epsilon represents the error ratio, and the also the stop condition that yields the optimal value of the estimated parameter.

3.2 Simulation Technique

In this part, we present a simulation technique that depends on generating data set from the generated distribution functions of the ODEL distribution and the EODEL distribution, respectively. The algorithm of simulation can be written as follows.

1. Generate data set from $U(0,1)$.
2. Generate the required data from the cdf of ODEL, or EODEL by setting

$$x=\sigma\left([(1+\lambda)-\theta\ln(-\ln u)]^{\frac{1}{\alpha}}-1\right),$$

$$x=\sigma\left([(1+\beta_0+\beta_1 Z_{i1})-\theta\ln(-\ln u)]^{\frac{1}{\alpha}}-1\right)$$

Where $z_i=i-\frac{(n+1)}{2}\,,\qquad i=1\ldots n$

3. Implementing the Newton-Raphson method in (2.17) with setting initials of each parameter.
4. Estimate the next parameter value within the previous value (initial value), by implementing the Newton-Raphson general form, that we mentioned in equations (3.23), (3.24), (3.25), (3.26), (3.27), and (3.28).
5. Repeal this process to find the required vale of each parameter.
6. Stop condition $\left|\tau_j^{\delta+1}-\tau^{\delta}\right|<\epsilon$ where ϵ is the ration of error.

The simulation technique consists of generating data set from continuous uniform distribution in order to analysis them within the ODEL-distribution, and the EODEL-distribution. This analysis is useful to find out which distribution is better. Analysis of data set that have been generated by the simulation technique is proposed to compare the analysis of the ODEL-distribution to the EODEL-distribution, and to specify which of them is better.

We have a real data set from Colorado Climate Center, Colorado state University (http://ulysses.atmos.colostate.edu). These data consist of 100 annual maximum precipitation (inch) for one rain gauge in Fort Collins, Colorado, from 1900 through 1999. The data set are:

Table 4.1: The data from Aarset [11].

239	232	434	85	302	174	170	121	193	71	112	218	71	249	240	152	185	213	306	212
168	148	116	132	132	144	183	223	96	269	134	95	110	203	321	129	199	354	169	443
298	97	116	146	84	230	138	170	117	162	96	95	285	121	161	102	463	108	184	146
115	132	125	156	124	189	193	71	176	115	271	187	215	151	136	162	241	215	98	94

105	93	354	60	151	160	219	142	117	98	348	176	103	103	181	135	87	223	297	183

Secondly, we separately present the MLE's of the parameters of ODEL-distribution alone, and then EODEL-distribution when $k = 1$. This estimation has been accomplished by using Matlab 2015. Table 4.2 shows the following values of the estimated parameters, that are

Table 4.2: MLEs of parameters, Log-likelihood

Model	MLEs Of Parameters
ODEL	$\hat{\theta} = 0.3557,\quad \hat{\alpha} = 0.8377,\quad \hat{\sigma} = 0.9949,\quad \hat{\lambda} = 5.6743$
EODEL	$\hat{\theta} = 0.1603,\quad \hat{\alpha} = 0.6440,\quad \hat{\sigma} = 0.9911,\quad \hat{\beta}_0 = 3.1865,\quad \hat{\beta}_1 = 0.0002,$

The estimated parameters in the ODEL-distribution and EODEL-distribution have approximately the same values. The impact of the linear model in the odd distribution can be recognized with the exceeded of the number of parameters in the EODEL-distribution within the analysis of data set.

Next, we evaluate the maximum log-likelihood functions (ℓ_1, ℓ_2) with the desired data set so that we can compute the values of AIC, BIC, CACI and HQIC. This provides the required comparison between the two distribution. In fact, Table 4.3 shows the values of each measure of quality, that are

Table 4.3: Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted

Model	$\hat{\ell}$	AIC	BIC	CACI	HQIC
ODEL	-19371.74	38751.48	38761.90	38751.90	38755.7
EODEL	-14606.66	29223.32	29236.35	29223.96	29228.59

The simulation part shows that the log-likelihood of the EODEL-distribution has a better value than the value of the log-likelihood of the ODEL-distribution. This fact led to conclude that the linear model is a good approach that can be used to enhance data analysis.

7. Conclusion

The generated odd distribution (ODEL-distribution) with its extension yields several modern characteristics. The simulation technique has a very good impact, it has a proper way to estimate the parameters of the ODEL-distribution, and the EODEL-distribution. The employing of Newton-Raphson method and the modified way that have been proposed was crucial in finding the estimated values of each parameter. The part of application and analysis of data set shows an enhancement on the log-likelihood value of the EODEL-distribution comparing with the ODEL-log-likelihood value. There are other results that have been obtained but we cannot find a wider range to present via this study.

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