Even Power Weighted Generalized Inverse Weibull Distribution

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Abstract.In this search a new even-power weighted generalized Inverse Weibulldistribution is derived,and some statistical properties of this distribution are discussed, such as [cumulative, probability generating, moment generating, reliability, and Entropy functions] and other properties,scale parameter for this distribution has been estimated by using two methods[Jackknife MaximumLikelihoodestimationand maximum likelihood],thena comparisons has been made between the results we obtained from simulation using MSE criteria to show the best estimator for the scale parameter.

1. Introduction

In some situations, it was noted that the classical distributions were not flexible for the data sets related to the field of biomedical, engineering, financial, environmental, computer science, Economy, and in other sciences [1-2] and [3]. Therefore continually needed to obtain a flexible model for applications in these areas.

There are many generalization of the inverse Weibull distribution in the literature, some shapesof the density and failure rate theoretically properties of three parameters inverse Weibull distribution and suggested the names complementary Weibull and reciprocal Weibull for (1993) [6] and Mudholkar and Kollia (1994) [7]. A three-parameter generalized inverse weibull distribution with decreasing and unimodal failure rate was introduced by Gusmão et. al. the distribution were studied byDrapella (2009) [8].In (2016) Khan and Robert Kinga introduces the four parameter new generalized inverse Weibull distribution and investigates thepotential usefulness of this model with application to reliability data from engineering studies.

A new class of continuous distributions based on generalized inverse Weibull has been introduced byHamza.s and ALNoor,n (2019).

Fisher at (1943) [4] proposed a new generalization of classical distribution called weighted distribution for any random variableassociated with probability function $f(x;\theta)$as follows:

$$f_w(x;\theta)=\frac{w(x)f(x;\theta)}{E(w(x))} \quad (1)$$

Under the condition $E(w(x))=\int_{-\infty}^{\infty} w(x)\, f(x;\theta)dx$, where w(x) and θ are the positive weighted function and parameter respectively

On the other hand, functions for the basic inverse Weibull model were discussed by Keller, Kamath (1982) [5].

Here, a new generalization of inverse Weibull distribution with even power-weighted function wasdrive, In addition some properties [functions and moments] with estimation of scale parameter are discussed. Then, simulation is performed to compare the performances of two estimators to show which is the best.

The remainder of this paper is organized as follows: : InIn Section2. we introduce an even power weighted generalized inverse Weibull distribution and its properties, in Section 3. estimation is done by using Maximum Likelihood and Jackknife Maximum Likelihood,where section4&5 includes simulations and conclusions.

2. Even Power Weighted distribution

The Even-Power weighted distribution is given by :[3]

$$f_w(x) = \frac{(w(x))^{2r} f(x)}{W_D} \qquad -\infty < x < +\infty, \qquad r \in z^+ \tag{2}$$

where, $W_D = E[(w(x))^{2r}] = \int_{-\infty}^{\infty} (w(x))^{2r} f(x)\, dx$

2.1 Even Power Weighted Generalized Inverse Weibull Distribution

In this section the probability density function pdf, cumulative distribution function cdf of (EPWGIW) distribution and other properties are obtained. the probability function(pdf) of the generalizedinverse Weibull distribution,Gusmão et. al. (2009) [8]as follows:

$$f(x; \alpha, \beta, \delta) = \delta\alpha\beta^{\alpha} x^{-(\alpha+1)} e^{-\delta\left(\frac{x}{\beta}\right)^{-\alpha}}, \qquad x, \alpha, \beta, \delta > 0$$

where α is a shape parameter and β, δ are scale parameters, and the cumulative distribution function (cdf) of generalizedinverse Weibull distribution is

$$F(x) = e^{-\delta\left(\frac{x}{\beta}\right)^{-\alpha}}$$

Here, assume that the shape parameter $\alpha$is known and equal 4, then the pdf becomes as:

$$f(x; \beta, \delta) = 4\delta\beta^4 x^{-5} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}, \qquad x, \beta, \delta > 0 \tag{3}$$

and the cumulative distribution function cdf as follows

$$F(x) = e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}$$

The weighted function used is w(x) = x, then the even poweras the following form:

$$w(x) = x^{2r}, \qquad r > 0 \tag{4}$$

we have

$$W_D = \int_0^{\infty} w(x) f(x; \beta, \delta)\, dx$$

Therefore by eq.'s (3&4)

$$W_D = \int_0^{\infty} 4\delta\beta^4 x^{2r-5} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} dx$$

Let $y = \delta\left(\frac{x}{\beta}\right)^{-4}$, then we get

$$W_D = \delta^{\frac{r}{2}} \beta^{2r} \int_0^{\infty} y^{-\frac{r}{2}} e^{-y} dy$$

This implies that

$$W_D = \delta^{\frac{r}{2}} \beta^{2r} \Gamma\left(1 - \frac{r}{2}\right)$$

As special case we choose $r = 1$, then

$$W_D = \delta^{\frac{1}{2}}\beta^2\sqrt{\pi}$$

then by equation (2), the pdf of even power weighted generalized inverse Weibull distribution denoted by (EPWGIW) can be written as follows:

$$f_w(x;\beta,\delta) = \frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} \quad x,\beta,\delta > 0 \qquad (5)$$

and the cdf of X~EPWGIW distribution is given as follows

$$F_w(x) = \int_0^x f_w(t;\beta,\delta)dt$$

Therefore,

$$F_w(x) = \int_0^x \frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2 t^{-3} e^{-\delta\left(\frac{t}{\beta}\right)^{-4}} \quad dt$$

Let $y = \delta\left(\frac{t}{\beta}\right)^{-4}$. Then

$$F_w(x) = \frac{1}{\sqrt{\pi}}\int_{\frac{\delta\beta^4}{x^4}}^{\infty} y^{-\frac{1}{2}} e^{-y} \quad dt$$

Rewriting above equation as follows

$$F_w(x) = \frac{1}{\sqrt{\pi}}\left\{\int_0^{\infty} y^{-\frac{1}{2}} e^{-y} \quad dt - \int_0^{\frac{\delta\beta^4}{x^4}} y^{-\frac{1}{2}} e^{-y} \quad dt\right\}$$

Then,

$F_w(x) = \frac{1}{\sqrt{\pi}}\ [\Gamma(1\backslash 2)\text{-}\ \gamma\left(\frac{1}{2},\frac{\delta\beta^4}{x^4}\right)\ ]$

$$F_w(x) = 1 - \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2},\frac{\delta\beta^4}{x^4}\right) \qquad (6)$$

Where, γ represents the incomplete gamma function.

2.1.1Some Important Properties

ConsiderX~EPWGIW(δ,β), then the moment generating function of x (denoted by$M_X(t)$) is given as follows

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} \frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} e^{tx}dx$$

rewriting the last equation as

$$M_X(t) = \int_0^\infty \frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} \sum_{k=1}^{\infty}\frac{(tx)^k}{k!}dx$$

Consequently,

$$M_X(t) = \sum_{k=0}^{\infty}\frac{4\sqrt{\delta}\beta^2(t)^k}{\sqrt{\pi}k!}\int_0^\infty x^{k-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}dx$$

by using the transform $y = \delta\left(\frac{x}{\beta}\right)^{-4}$, we get

$$M_X(t) = \sum_{k=0}^{\infty}\frac{\delta^{\frac{k}{4}}\beta^k}{\sqrt{\pi}k!}\int_0^\infty y^{-\frac{k-2}{4}} e^{-y}dy \tag{7}$$

We note that the themoment generating function for EPWGE distribution not exist.
Now, the p-moments for any probability distribution f(x) is obtained by using the following form

$$\mu_p = \int_{-\infty}^{\infty} x^p f(x)dx$$

Using pdf of the EPWGIW distribution in (5), we get on the p-moments of EPWGE distribution as

$$\mu_p = \int_0^\infty \frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2 x^{p-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}dx$$

Let $y = \delta\left(\frac{x}{\beta}\right)^{-4}$. Then

$$\mu_p = \frac{(\sqrt[4]{\delta}\beta)^p}{\sqrt{\pi}}\int_0^\infty y^{-\frac{p-2}{4}} e^{-y}dx$$

$$\mu_p = \frac{(\sqrt[4]{\delta}\beta)^p}{\sqrt{\pi}}\Gamma\left(\frac{-p+2}{4}\right) \tag{8}$$

Clearly, the integral in the right side of equation (8) is unknown when $p \geq 2$, therefore, the EPWGE distribution does not have finite moments of order greater than or equal to two.

$$\mu_1 = \frac{\sqrt[4]{\delta}\beta}{\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right) \tag{9}$$

2.1.1.1Entropy, Reliability function, Hazard rate, reversed hazard function and the probabilitygenerating function

A measure to quantify the uncertainty of an event was proposed by Shannon [10]. For any continuous random variable x associated with pdf f(x)is defined as follows:

$$H(X) = -E\big(\ln(f(x))\big)$$

Now, by equation (5), we have

$$H(X;\lambda) = -E\left\{\ln\left(\frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2\right) - 3\ln(x) - \delta\beta^4 x^{-4}\right\}, \quad x, \delta, \beta > 0$$

This implies

$$H(X;\lambda) = -\ln\left(\frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2\right) + 3E\big(\ln(x)\big) + \delta\beta^4 E(x^{-4}), \qquad x, \delta, \beta > 0$$

But by eq. (9)$E(x^{-4}) = \frac{\delta^{-1}\beta^{-4}}{\sqrt{\pi}}$, then

$$H(X;\lambda) = -\ln\left(\frac{4}{\sqrt{\pi}}\sqrt{\delta}\beta^2\right) + 3E\big(\ln(x)\big) + \frac{1}{\sqrt{\pi}}, \qquad x, \delta, \beta > 0 \tag{10}$$

In the right side of equation (10), Computing the expectation in a convenient and fast method needs to be based on numerical integration, therefore we prefer to use some methods of numerical integration such as Monte Carlo and importance sampling methods.

The reliability function $R_w(X)$ which also known as survival function is the probability of an item not failing prior to time t. The reliability function of a random variable x which associated with EPWGIW(δ, β) distribution is obtained as $R_w(x) = 1 - F_w(x)$ and by equation (6) it is given as follows

$$R_w(x) = \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right) \tag{11}$$

The hazard rate function which also known as force of mortality in actuarial statistics of a random variable x which associated with EPWGIW distribution is defined as

$$h_w(x) = \frac{f_w(x)}{R_w(x)}$$

by equations (5) and (11), it is obtained as follows

$$h_w(x) = \frac{4\sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}}{\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right)} \tag{12}$$

The reversed hazard function is given as

$$I_w(x) = \frac{f_w(x)}{F_w(x)}$$

and by equations (5,6), the reversed hazard can be written in the following form

$$I_w(x) = \frac{4\sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}}{\sqrt{\pi} - \gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right)} \tag{13}$$

Similarity the probabilitygenerating function of x~EPWGIW(δ, β) given as [5]

$$P_X(t) = E(e^{tx}) = \int_0^\infty \frac{4}{\sqrt{\pi}} \sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} e^{x \ln t} dx$$

rewriting the last equation

$$P_X(t) = \int_0^\infty \frac{4}{\sqrt{\pi}} \sqrt{\delta}\beta^2 x^{-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} \sum_{k=1}^{\infty} \frac{(x \ln t)^k}{k!} dx$$

Consequently,

$$P_X(t) = \sum_{k=0}^{\infty} \frac{4(\sqrt{\delta}\beta^2(\ln t))^k}{\sqrt{\pi}k!} \int_0^\infty x^{k-3} e^{-\delta\left(\frac{x}{\beta}\right)^{-4}} dx$$

Let $y = \delta\left(\frac{x}{\beta}\right)^{-4}$,we get

$$P_X(t) \sum_{k=0}^{\infty} \frac{4(\sqrt[4]{\delta}\beta^2(ln\ t))^k}{\sqrt{\pi}k!} \int_0^\infty y^{\frac{-k-2}{4}} dy \qquad (14)$$

We note that the integral in right side of above equation is not exist ; therefore, the EPWGE distribution has no Probability generating function

2.1.1.2Mode, Median and Limiting

The behavior of the density function in (5) is investigated when the variable x go to zero and infinite. Therefore, $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$are given in the following forms, respectively

$$\lim_{x\to 0} f_w(x) = \frac{4}{\sqrt{\pi}} \sqrt{\delta}\beta^2 \lim_{x\to 0}\left(x^{-3}\right) . \lim_{x\to 0}\left(e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}\right) = 0$$

$$\lim_{x\to\infty} f_w(x) = \frac{4}{\sqrt{\pi}} \sqrt{\delta}\beta^2 \lim_{x\to\infty}\left(x^{-3}\right) . \lim_{x\to\infty}\left(e^{-\delta\left(\frac{x}{\beta}\right)^{-4}}\right) = 0$$

Consequently, it is clear that the model has a unique mode. From equation (5), we have

$$\ln\left(f_w(x)\right) = \ln\left(\frac{4}{\sqrt{\pi}} \sqrt{\delta}\beta^2\right) - 3\ln(x) - \delta\left(\frac{x}{\beta}\right)^{-4}$$

differentiating both sides of an above equation with respect to x, we have

$$\frac{df_w}{dx} = \frac{-3}{x} + \frac{4\delta\beta^4}{x^5}$$

then

$$\frac{d^2 f}{dx^2} = -\left(\frac{20\delta\beta^4}{x^6} - \frac{3}{x^2}\right) < 0$$

Therefore, the value of x $(x \neq 0 , x \neq \infty)$ which satisfies the following equation represents the mode of EPWGIW distribution.

and

$$x = \left(\frac{4}{3}\delta\beta^4\right)^{\frac{1}{4}} \qquad (15)$$

Now,the value of x which is satisfies the equation:

$F_w(x) = \frac{1}{2}$, , represents the median of EPWGIW distribution,then by equation (6), we get:

$$1 - \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right) = \frac{1}{2}$$

$$\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right) = \frac{\sqrt{\pi}}{2}$$

then $\left(\gamma^{-1}\left(\frac{1}{2}, \frac{\sqrt{\pi}}{2}\right)\right) =, \frac{\delta\beta^4}{x^4}$

andthe median:

$$x = \delta\beta^4\left(\gamma^{-1}\left(\frac{1}{2}, \frac{\sqrt{\pi}}{2}\right)\right)^{\frac{-1}{4}} \quad (16)$$

3. Estimation

In this section, the estimation of a scale parameters δ of EPWGIW is discussed when β is knownandestimate β when δ is known. Let $x_1, x_2, \dots, x_n$ be the n-random sample from EPWGIW distribution.

3.1Maximum Likelihood Estimator (MLE)

Maximum Likelihood is a relatively simple method of constructing an estimator for an unknown parameter it was introduced by R. A. Fisher in 1912. Estimation is a method that determine values for the parameters of a model. The Likelihood function is given as follows

$$L = L(x_1, x_2, \dots, x_n; \lambda) = \left(\frac{4}{\sqrt{\pi}}\right)^n (\delta)^{\frac{n}{2}}\beta^{2n}\prod_{i=1}^{n} x_i^{-4} . e^{-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-4}}$$

Taking natural logarithm of above equation, we have

$$\ln L = n\ln\left(\frac{4}{\sqrt{\pi}}\right) + \frac{n}{2}\ln(\delta) + 2n\ln(\beta) - 4\sum_{i=1}^{n}\ln(x_i) - \delta\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-4}$$

differentiate both sides of above equation, we get

$$\frac{d\ln L}{d\delta} = \frac{n}{2\delta} - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-4}$$

$$\frac{d\ln L}{d\beta} = \frac{2n}{\beta} - 4\delta\beta^3\sum_{i=1}^{n}(x_i)^{-4}$$

The Maximum Likelihood estimator for the scale parameters $\delta$and $\beta$is obtained by equating the above equationsequal to zero.

Consequently,

$$\hat{\delta}_{MLE} = \frac{n}{2\beta^4\left(\sum_{i=1}^{n} x_i^{-4}\right)} \quad (17)$$

and

$$\hat{\beta}_{MLE} = \left(\frac{n}{2(\left(\sum_{i=1}^{n} x_i^{-4}\right)}\right)^{\frac{1}{4}} \quad (18)$$

3.1 Jackknife Maximum Likelihood Estimator (JMLE)

The Jackknife maximum Likelihood estimator (JMLE) to estimate the distribution parameter was proposedbyRezzokyin (2012).[12].

Theidea isthat if $\hat{\delta}_{MLE}(j)$ repressents the estimator of the maximum liklihood method resulting from applying the maximum liklihood method to all data except the value (t_i) then the Jackknife estimator of maximum liklihood method for parameter (δ) Then

$$\hat{\delta}_{JMLE} = n\hat{\delta}_{MLE} - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{\delta}_{MLE}(j)$$

Where, $\hat{\delta}_{MLE}(j) = \frac{n}{2\beta^4\left(\sum_{i=1,i\neq j}^{n} x_i^{-4}\right)}$ (19)

Similarity,

$$\hat{\beta}_{JMLE} = n\hat{\beta}_{MLE} - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{\beta}_{MLE}(j)$$

Where, $\hat{\beta}_{MLE}(j) = \left(\frac{n}{2\delta\left(\sum_{i=1,i\neq j}^{n} x_i^{-4}\right)}\right)^{\frac{1}{4}}$ (20)

4. Simulation

The use of simulation method to generate a certain distribution of data in order to find the best estimate [11]. Here, the results of numerical four experiments, based on Monte Carlo in MATLAB version 2019a to compare the performance of the two estimators with sample size (n=10,50,100) and default values for scale parameters $\delta_{\circ} = (0.5, 1.2, 2.5)$ and $\beta_{\circ} = (0.5, 1.2, 2.5)$. Then MSE $= \frac{1}{N}\sum_{i=1}^{N}(\hat{e}_i - e)^2$ for e= δ, β of parameter estimators in equations (17-20) with replications(500)times are given in tables below :we depend on equation (6) to generate the r.v. x as follow;

.

let F(x) =U , where Uis a randam variable on interval(0,1)

$1 - \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right)$ =U

$\gamma\left(\frac{1}{2}, \frac{\delta\beta^4}{x^4}\right) = \sqrt{\pi}$(1-U)

$$\gamma^{-1}\left(\frac{1}{2}, \sqrt{\pi}(1-U)\right) = \frac{\delta\beta^4}{x^4}$$

$$x_i = \left(\frac{\delta\beta^4}{\gamma^{-1}\left(\sqrt{\pi}(1-U_i), \frac{1}{2}\right)}\right)^{\frac{1}{4}}, i = 1,2,\ldots,n$$

Table 1. The MSE valuesfor estimate scale parameter $\delta$when $\beta =$

0.5

	$\delta_\circ = 0.5$		
N	JMLE	MLE	Best
10	0.004232	0.004231	MLE
50	0.000620	0.000640	JMLE
100	0.000306	0.000307	JMLE
	$\delta_\circ = 1.2$		
10	0.004300	0.004500	JMLE
50	0.000678	0.000694	JMLE
100	0.000301	0.000302	JMLE
	$\delta_\circ = 2.5$		
10	0.003500	0.003800	JMLE
50	0.000633	0.000626	MLE
100	0.000279	0.000287	JMLE

Table 2. The MSE valuesfor estimate scale parameter $\delta$when $\beta = 2.5$.

	$\delta_\circ = 0.5$		
n	JMLE	MLE	Best
10	0.084400	0.094700	JMLE
50	0.015900	0.016200	JMLE
100	0.008100	0.008300	JMLE
	$\delta_\circ = 1.2$		
10	0.088000	0.095800	JMLE
50	0.015445	0.015418	MLE
100	0.008200	0.008400	JMLE
	$\delta_\circ = 2.5$		
10	0.099500	0.102900	JMLE
50	0.014454	0.014100	MLE
100	0.008100	0.008300	JMLE

Table 3. The MSE values for estimate scale parameter $\beta$when$\delta = 0.5$.

	$\beta_\circ = 0.5$		
n	JMLE	MLE	Best
10	0.003800	0.004200	JMLE
50	0.626700	0.631700	JMLE
100	0.000301	0.000303	JMLE
	$\beta_\circ = 1.2$		
10	0.020400	0.021700	JMLE
50	0.003400	0.003500	JMLE
100	0.001800	0.001801	JMLE
	$\beta_\circ = 2.5$		
10	0.094600	0.098200	JMLE
50	0.017800	0.018500	JMLE

100	0.007400	0.007500	JMLE

Table 4. The MSE values for estimate scale parameter $\beta$when$\delta = 2.5$.

	$\beta_\circ = 0.5$		
n	JMLE	MLE	Best
10	0.004100	0.003800	MLE
50	0.000663	0.000674	JMLE
100	0.000360	0.000362	JMLE
	$\beta_\circ = 1.2$		
10	0.028700	0.030700	JMLE
50	0.003743	0.003700	MLE
100	0.001700	0.001701	JMLE
	$\beta_\circ = 2.5$		
10	0.102100	0.116500	JMLE
50	0.015300	0.015700	JMLE
100	0.007100	0.007200	JMLE

5. Conclusions

From tables above, weconcludethat as sample size increases, the MSE decrease that is quite inevitable and also verifies the consistency properties of all the estimates. The results show theJackknifemaximum Likelihood estimator is superior because it has lower MSE from the other. Furthermore, the Jackknifemaximum likelihood estimator better than maximum likelihood estimator to estimate the scale parameters of EPWGIW distribution.

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