Research Article

An Approximate Solution Of Fredholm Integral Equation Of The First Kind By The Regularization Method With Parallel Computing

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Abstract: In this paper, we study the regularization method for solving the Fredholm integral equation first kind. The discretization algorithm with two variables has applied to formulate the problem into a linear operator equation for the first kind. The parallel computing method has used to obtain the approximation solution by using a set of regularization parameters of the Tikhonov regularization method. The inverse initial value problem for the heat equation used as an example to test parallel computing and compared with sequential computing.

Keywords: regularization, Fredholm integral equation, ill-posed problem.

Introduction

Many papers have been used the discretization method as an important method of solving integral equations [1]–[4].The approximate solution error taking into account the discretization of an integral equation was estimated in [5][6].

In this paper, the finite-dimensional approximation method used to create the finite-dimensional operator for the integral equation which has two variables. The estimation error computed by using the general discrepancy principle method. Parallel computing implemented by taking the value of the regularization parameter from a set of parameters and obtained the approximation solutions independently.

1. Problem Statement

The following integral equation for the first kind has considered.

$$Au(s)=\int_a^b P(s,t)u(s)ds=f(t), c\le t\le d, \qquad (1)$$

where $u(s)\in L_2[a,b], f(t)\in L_2[c,d],$ the function $P(s,t)$ represent the kernel operator $A,$ where $P(s,t),P_t'(s,t)\in C^{1,1}([a,b]\times[c,d])$. We propose for $f(t)=f_0(t)$ there exist a true solution $u_0(s)$ for problem (1) in the set M_r

$$M_r=\left\{u(s):u(s),u'(s)\in L_2[a,b],u(a)=u(b)=0,\|u(s)\|_{L_2}^2\le r^2\right\}, \qquad (2)$$

The function $f_0(t)$ is unknown instead of we have $f_\delta\in L_2[c,d]$ and $\delta>0$ such that $\| f_\delta(t)-f_0(t)\|_{L_2}^2\le\delta^2$. For solving the problem (1) we need find the approximation solution $u_\delta(s)$ by using the given information $f_\delta(t),\delta$ and M_r. Then we estimate the deviation of the approximation solution $u_\delta(s)$ from the true solution $u_0(s)$ in the metric of space $L_2[a,b], \|u_\delta(s)-u_0(s)\|_{L_2}$

We need define an operator $B:L_2[a,b]\to L_2[a,b]$ by the following formula

$$u(s)=Bv(s)=\int_a^s v(\zeta)d\zeta, v(s),Bv(s)\in L_2[a,b], \qquad (3)$$

There is an operator named C which can be defined by

$$Cv(s)=ABv(s),\ v(s),\in L_2[a,b],\ Cv(s)\in L_2[c,d], \qquad (4)$$

from (3) and (4) we follow that $Cv(s)=\int_a^b K(s,t)v(s)ds,$ where $K(s,t)=-\int_a^b P(\varsigma,t)d\varsigma.$

The finite-dimensional operator $C_{n,m}$ has been defined for computing the numerical solution for problem (1), the C replaced with the operator $C_{n,m}$ and these operators satisfied the following relation

where the value of the $\eta_{n,m}$ define by

$$N(t)=\max_{a\le s\le b}\left|P(s,t)\right|,\ t\in[c,d], \quad (5)$$

and

$$N_1=\max\left\{\left|K_t'(s,t)\right|: a\le s\le b, c\le t\le d\right\}, \quad (6)$$

the $N(t)\in[c,d]$ and N_1 exist because the $P(s,t),P_t'(s,t)\in C([a,b]\times[c,d])$.

We need divide the intervals $[a,b]$ and $[c,d]$ into n and m equal parts respectively. Where interval $[a,b]$ divided by points $s_i=a+\frac{i(b-a)}{n}, i=0,1,\dots,n-1$, and interval $[c,d]$ divided by points $t_j=c+\frac{j(d-c)}{m}, j=0,1\ \dots,m-1$.

Now we need to define the following functions

$$\bar{K}_i(t)=K(s_i,t), \quad (7)$$

$$K_n(s,t)=\bar{K}_i(t); s_i\le s\le s_{i+1},\ t\in[c,d], i=0,1,\dots,n-1 \quad (8)$$

$$K_{n,m}(s,t)=\bar{K}_i(t_j); s_i\le s\le s_{i+1}, t_j\le t\le t_{j+1}, i=0,1,\dots,n-1, j=0,1,\dots,m-1, \quad (9)$$

By using the equations (7–9) we define the operators $C_{n,}$ and $C_{n,m}$

$$C_n v(s)=\int_a^b K_n(s,t)v(s)ds, t\in[c,d],$$

$$C_{n,m}v(s)=\int_a^b K_{n,m}(s,t)v(s)ds, t\in[c,d], \quad (10)$$

where $C_{n,}$ and $C_{n,m}$ map $L_2[a,b]$ into $L_2[c,d]$

Next step we need to estimate the $\left\|C_{n,m}-C\right\|,$ we will use the inequality relation

$$\left\|C_{n,m}-C\right\|\le\left\|C_{n,m}-C_n\right\|+\left\|C_n-C\right\|.$$

Since

$$\left|K_{n,m}(s,t)-K_n(s,t)\right|\le\left|\bar{K}_i(t)-\bar{K}_i(t_j)\right|, \quad (11)$$

for $s_i\le s\le s_{i+1}$ and $t_j\le t\le t_{j+1},\ i=0,1,\dots,n-1, j=0,1,\dots,m-1,$ from (6)

$\left|\bar{K}_i(t)-\bar{K}_i(t_j)\right|\le N_1\dfrac{d-c}{m},$ we find from (11) that

$$\left|K_{n,m}(s,t)-K_n(s,t)\right|\le N_1\frac{d-c}{m}, \quad (12)$$

By using the equality $\left\|C_{n,m}-C_n\right\|=\sup_{\|v\|\le1}\left\|C_{n,m}v-C_nv\right\|,$ we get

$$\left\|C_{n,m}-C_n\right\|^2=\sup_{\|v\|\le1}\int_c^d\left[\int_a^b\left|K_{n,m}(s,t)-K_n(s,t)\right|.\left|v(s)\right|ds\right]^2dt. \quad (13)$$

We derive from (12) and (13) the following

$$\left\|C_{n,m}-C_n\right\|^2 \le N_1{}^2\left(\frac{d-c}{m}\right)^2\int_c^d\left[\int_a^b |v(s)|ds\right]^2 dt. \tag{14}$$

Since $\int_a^b |v(s)|ds \le \sqrt{b-a}\,\|v(s)\|_{L_2}$, inequality (14) implies that

$$\left\|C_{n,m}-C_n\right\| \le \sqrt{(b-a)(d-c)}N_1\frac{d-c}{m}. \tag{15}$$

Now the term $\|C_n - C\|$ can be estimated.

Since $C_n v(s) - Cv(s) = \int_a^b (K(s,t)-K_n(s,t))v(s)ds$ and

$$\|C_n - C\|^2 = \sup\left\{\int_c^d\left[\int_a^b\left|K_{n,m}(s,t)-K_n(s,t)\right|.|v(s)|ds\right]^2 dt : \|v(s)\|\le 1\right\}.$$

Taking into account (5)(7) and (8) and the inequality

$$\int_a^b\left|K(s,t)-K_n(s,t)\right|.|v(s)|ds \le \int_a^b\left|K(s,t)-K(s_i,t)\right|.|v(s)|ds \le \frac{b-a}{n}N(t)\int_a^b|v(s)|ds,$$

we find the following

$$\left\|C_{n,m}v(s)-C_n v(s)\right\|^2 \le \left(\frac{b-a}{n}\right)^2\int_c^d N^2(t)\left[\int_a^b\left|v(s)ds\right|\right]^2 dt. \tag{16}$$

The $\|v(s)\|\le 1$ and $\int_a^b |v(s)|ds \le \frac{b-a}{n}\|v(s)\|$ with inequality (16) implies that

$$\|C_n - C\| \le \sqrt{(b-a)}\,\|N(t)\|_{L_2}\frac{b-a}{n}. \tag{17}$$

Thus from (15) and (17)

$$\eta_{n,m} = \sqrt{(b-a)(d-c)}N_1\frac{d-c}{m} + \sqrt{(b-a)}\,\|N(t)\|_{L_2}\frac{b-a}{n}. \tag{18}$$

2. Finite-dimensional of the Tikhonov regularization method

We define subspaces X_n and Y_m of spaces $L_2(a,b)$ and $L_2(c,d)$ respectively. Those subspaces consisting of all functions on intervals $[s_i, s_{i+1}), i = 0,1,...,n-1,$ for space $L_2(a,b)$ and $[t_j, t_{j+1}), j = 0,1,...,m-1,$ for space $L_2(c,d)$. We denote by G_n the metric projection operator where $G_n : L_2(a,b) \to X_n,$ and H_m the metric projection operator where $H_m : L_2(c,d) \to Y_m.$

The problem (1) reduce to linear operator problem first type.

$$C_{n,m}v(s) = f_\delta^m(t), \tag{19}$$

where $f_\delta^m(t) = H_m\left|f_\delta(t)\right|.$

The approximation solution for problem (1) can be obtained by using the generalized discrepancy method proposed in [7] and studied in [8]. The method reduce the problem (19) to the conditional extremum variational problem.

$$\inf\left\{\|v(s)\|^2 : v(s)\in X_n, \left\|C_{n,m}v(s)-f_\delta^m(t)\right\|\le\|v(s)\|\eta_{n,m}+\delta\right\}, \tag{20}$$

In [8] there was proved that it under the condition

$$\left\| f_\delta^m(t)\right\| = \|u_0'(s)\|\eta_{n.m}+\delta, \tag{21}$$

The variational problem (19) has a unique solution $v_{\delta,\eta_{n,m}}(s)$ such that

$$\left\|C_{n,m}v_{\delta,\eta_{n,m}}(s)-f_\delta^m(t)\right\| = \left\|v_{\delta,\eta_{n,m}}(s)\right\|\eta_{n,m}+\delta.$$

The conditional problem (19) is reduced to the unconditional problem by following from [8]

$$\inf\left\{\left\|C_{n,m}v(s)-f_\delta^m(t)\right\|^2+\alpha\|v(s)\|^2 : v(s)\in X_n\right\}, \alpha>0, \tag{22}$$

The (21) it is a finite-dimensional version of the Tikhonov regularization method [9].

There is a unique solution $v_\delta^\alpha(s)$ for problem (21) . this solution should satisfy the general discrepancy principle [10].

$$\left\|C_{n,m}v_\delta^\alpha(s)-f_\delta^m(t)\right\| = \left\|v_\delta^\alpha(s)\right\|\eta_{n,m}+\delta. \tag{23}$$

Under condition (20) the equation (22) has unique solution $v_\delta^\alpha(s)$ with respect of regularization parameter $\alpha(n,m)$. That condition known in [8] and by theorem defined $v_{\delta,\eta_{n,m}}(s)=v_\delta^{\alpha(n,m)}(s)$ where the approximation solution $u_{\delta,\eta_{n,m}}(s)=Bv_{\delta,\eta_{n,m}}(s)$.

For solving the problem (22) we get the equation

$$C_{n,m}^TC_{n,m}v(s)+\alpha v(s)=C_{n,m}^T f_\delta^m(t). \tag{24}$$

In spaces X_n and Y_m, the orthonormal bases have introduced $\{\varphi_i(s),\psi_j(t)\}$ by following

$$\varphi_i(s)=\begin{cases}\sqrt{\dfrac{n}{b-a}}; & s_i\le s<s_{i+1,}\\ 0; & s\notin\left[s_i,s_{i+1,}\right), i=0,1,...,n-1,\end{cases}$$

and

$$\psi_j(t)=\begin{cases}\sqrt{\dfrac{m}{c-d}}; & t_j\le t<t_{j+1,}\\ 0; & t\notin\left[t_j,t_{j+1}\right), j=0,1,...,m-1,\end{cases}$$

With these bases we define the isometric operators J_x and J_y where $J_x:R^n\to X_n$ and $J_y:R^m\to Y_m$ by following.

$$\begin{aligned} J_x\left[\bar{x}(s)\right]&=\sum_{i=0}^{n-1}x_i\varphi_i(s), \bar{x}=(x_0,x_1,....,x_{n-1}),\\ J_y\left[\bar{y}(t)\right]&=\sum_{j=0}^{m-1}y_j\psi_j(t), \bar{y}=(y_0,y_1,....,y_{m-1}),\end{aligned} \tag{25}$$

The problem (22)

$$\inf\left\{\left\|J_y^{-1}\left[C_{n,m}v(s)\right]-J_y^{-1}\left[f_\delta^m(t)\right]\right\|^2+\alpha\left\|J_x^{-1}\left[v(s)\right]\right\|^2 : J_x^{-1}\left[v(s)\right]\in X_n\right\},\alpha>0,$$

where

$$J_y^{-1}\left[f_\delta^m(t)\right]=\sqrt{\frac{m}{c-d}}\int_{t_j}^{t_{j+1}} f(t)dt$$

$$J_x^{-1}\left[v(s)\right]=\sqrt{\frac{n}{b-a}}\int_{s_i}^{s_{i+1}} v(s)ds$$

we can rewrite the equation (24) in matrix and vector form as the following

$$(\bar{C}_{j,i})^T\bar{C}_{j,i}\bar{v}_i+\alpha\bar{v}_i=(\bar{C}_{j,i})^T\bar{f}_j, i=0,1,...n-1,\ j=0,1,...,m-1,$$

Where

$$\bar{v}_i=J_x^{-1}\left[v(s)\right]=\sqrt{\frac{n}{b-a}}\begin{bmatrix}\bar{v}_0=\int_{s_0}^{s_1}v(s)ds\\ \bar{v}_1=\int_{s_1}^{s_2}v(s)ds\\ \vdots\\ \bar{v}_{n-1}=\int_{s_{n-1}}^{s_n}v(s)ds\end{bmatrix}$$

$$\bar{f}_j=J_y^{-1}\left[f_\delta^m(t)\right]=\sqrt{\frac{m}{c-d}}\begin{bmatrix}\bar{f}_0=\int_{t_0}^{t_1}f(t)dt\\ \bar{f}_1=\int_{t_1}^{t_2}f(t)dt\\ \vdots\\ \bar{f}_{m-1}=\int_{t_{m-1}}^{t_m}f(t)dt\end{bmatrix}$$

$$\bar{C}_{j,i}=\frac{d-c}{m}\sqrt{\frac{b-a}{n}}\begin{bmatrix}\bar{K}_{i=0}(t_{j=0}) & \bar{K}_{i=1}(t_{j=0}) & \cdots & \bar{K}_{i=n-1}(t_{j=0})\\ \bar{K}_{i=0}(t_{j=1}) & \bar{K}_{i=1}(t_{j=1}) & \cdots & \bar{K}_{i=n-1}(t_{j=1})\\ \vdots & \vdots & \ddots & \vdots\\ \bar{K}_{i=0}(t_{j=m-1}) & \bar{K}_{i=1}(t_{j=m-1}) & \cdots & \bar{K}_{i=n-1}(t_{j=m-1})\end{bmatrix}$$

$$(\bar{C}_{j,i})^T = \frac{d-c}{m}\sqrt{\frac{b-a}{n}}\begin{bmatrix} \bar{K}_{i=0}(t_{j=0}) & \bar{K}_{i=0}(t_{j=1}) & \dots & \bar{K}_{i=0}(t_{j=m-1}) \\ \bar{K}_{i=1}(t_{j=0}) & \bar{K}_{i=1}(t_{j=1}) & \dots & \bar{K}_{i=1}(t_{j=m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{K}_{i=n-1}(t_{j=0}) & \bar{K}_{i=n-1}(t_{j=1}) & \cdots & \bar{K}_{i=n-1}(t_{j=m-1}) \end{bmatrix}$$

3. Estimating the error of the approximate solution $u_{\delta,\eta_{n,m}}(s)$ to equation (1)

In order to estimate the approximation solution we define the following function

$$\omega(\tau,r) = \sup_u \left\{ \|u(s)\| : u(s) = Bv(s), \|v(s)\| \le r, \|Au(s)\| \le \tau \right\}, \tau, r > 0.$$

From the theorem, formulated in [11],it follows

Theorem 1. *let $u_{\delta,\eta_{n,m}}(s)$ approximate solution for equation (1), and $u_0(s)$ the exact solution, then*

$$\left\| u_{\delta,\eta_{n,m}}(s) - u_0(s) \right\| \le 2\omega\left(2r\eta_{n,m} + \delta, r\right)$$

4. Solving the inverse initial value problem for heat conduction problem

The inverse initial value problem for heat equation described by the following liner partial differential equation.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 \le x \le l \ , t \in (0,T], \tag{26}$$

$$u(0,t) = 0, t \in (0,T], \tag{27}$$

$$u(l,t) = 0 \ , t \in (0,T], \tag{28}$$

$$u(x,0) = u(x), 0 \le x \le l, \tag{29}$$

where the $u(0,t)$ and $u(l,t)$ are boundary conditions, $u(x)$ initial condition which is need to find. This problem solved in [12] by using the Tikhonov's regularization inversion method and it solved by Picard's method in [13].

4.1 Direct problem

In direct problem (26 - 29), the initial condition has been specified. In order to reduce this problem to a Fredholm integral equation first kind the separation of variables method used to get the Fourier series as the following:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{\frac{-(n\pi)^2 t}{l^2}} \sin(\frac{n\pi x}{l}), \tag{30}$$

$$u(x,0) = u(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{l}), \tag{31}$$

$$a_n = \frac{2}{l}\int_0^l u_0(x) \sin(\frac{n\pi x}{l}) dx, \tag{32}$$

From (30-32) we get

$$u(x,t)=\frac{2}{l}\int_0^l \sum_{n=1}^{\infty} e^{\frac{-(n\pi)^2 t}{l^2}} \sin(\frac{n\pi x}{l})\sin(\frac{n\pi y}{l})u(x)dx, \qquad (33)$$

The formula (33) can be rewriting such as integral equation first kind as following:

$$u(x,y,T)=\int_0^l P(x,y)u(x)dx \ ,0\le y\le l, \qquad (34)$$

$$Au(x)=\int_0^l P(x,y)u(x)dx=g(y),0\le y\le l, \qquad (35)$$

where the kernel $P(x,y)\in C([0,l]\times[0,l])$, $u(x)\in H_2^1[0,l]$ and $g(y)\in L_2[0,l],$. The kernel of the operator A is closed. This is the direct problem for heat conduction equation the initial temperature $u(x)$ known and need to find the temperature with specific time $g(y)$.

4.2 Inverse Problem

The inverse problem defined as finding the initial temperature $u(x)$. In order to estimate the ungiven initial temperature the measurement temperature given at specific time T over the space interval $0\le x\le l$.

$$u(x,\mathrm{y},T)=\mathrm{g}(y), \quad \mathrm{T}>0,0\le \mathrm{y}\le l, \qquad (36)$$

The measurement temperature contains some noise g_δ where δ > 0 and $\|g_\delta-g_0\|_{L_2}\le\delta$. Furthermore, the inverse operator A^{-1} is unbounded where $\|A^{-1}\|=\infty$, it means the solution typically poor approximated or unstable even the δ has a small value.

5. Parallel algorithm for selecting regularization parameter

The integral equation form for the inverse initial value problem will be

$$Au(x)=\int_0^l P(x,y)u(x)dx=g(y), \qquad (37)$$

Where the kernel $P(x,y)$ is an infinite series and we cannot handle infinite sum, so we need to finite the sum of series to 10 times

$$P(x,y)=\frac{2}{l}\sum_{n=1}^{10} e^{\frac{-(n\pi)^2 T}{l^2}} \sin(\frac{n\pi x}{l})\sin(\frac{n\pi y}{l}),\ T>0, \qquad (38)$$

For giving the approximate solution for $u(x)$ we can rewrite the problem as linear operator equation $C\overline{v}=\overline{g},$.

$$\overline{C}\begin{bmatrix}\overline{v}(x_0)\\ \overline{v}(x_2)\\ \vdots\\ \overline{v}(x_{n-1})\end{bmatrix}=\begin{bmatrix}\overline{g}(y_0)\\ \overline{g}(y_2)\\ \vdots\\ \overline{g}(y_{n-1})\end{bmatrix}, \qquad (39)$$

where

$$\bar{C} = \frac{l-0}{n} \begin{bmatrix} K(x_0, y_0) & K(x_0, y_1) & \dots & K(x_0, y_{n-1}) \\ K(x_1, y_0) & K(x_1, y_1) & \dots & K(x_1, y_{n-1}) \\ \vdots & \vdots & \dots & \vdots \\ K(x_{n-1}, y_0) & K(x_{n-1}, y_1) & \dots & K(x_{n-1}, y_{n-1}) \end{bmatrix}, \quad (40)$$

The bounded injective operator C is ill-conditioned that is mean any numerical attempt to solve the problem directly will be failed.

We created an algorithm can implement in parallel form to find best estimation solution by choosing a good regularization parameter depending on a finite-dimensional version of the Tikhonov regularization method as shown in (22). Selection parameter α will be based on the general discrepancy method equation (23). we can use the interval (0,1) and divide this interval to sequence of pattern for example $\alpha_k = \{\alpha_1 = 0.1, \alpha_2 = 0.01, ..., \alpha_k = 10^{-k}\}$ then we compute the approximation solution for each α_k in parallel computing by using the following

$$\bar{v}_{\alpha_k} = ((\bar{C})^T \bar{C} + \alpha_k I)^{-1} (\bar{C})^T \bar{f}, k = 1, 2, ... \quad (41)$$

$$\bar{u}_{\alpha_k} = B \bar{v}_{\alpha_k} \quad (42)$$

Where I is identity operator, the best result will be selected by (23). After that we can create new regularization parameters pattern start from the previous step to get more accrue approximation solution. The main goal of the parallel computing is to find the best solution with low time.

6. Numerical example

The initial temperature $u_0(x)$ will find by using the known function $u(x,T) = g_0(x)$ where $T = 0,1$, for checking the approximation solution we have the exact initial temperature $u_0(x) = \sin(x)$ as shown in fig. 1.

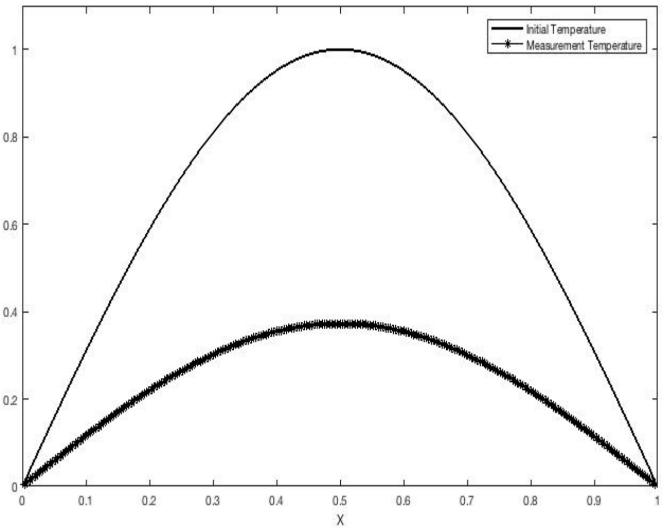

Fig. 1. Direct solution for measurement temperature $g_0(x)$

We can add a noise value $g_0(x) + Noise = g_\delta(x)$. By using the equation (41) we find the approximation solutions with parameters α . Selection parameter α will be based on the (23) we can use the set of regularization parameters to obtain the best estimated solution $\alpha_k = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, as shown in fig. 2.

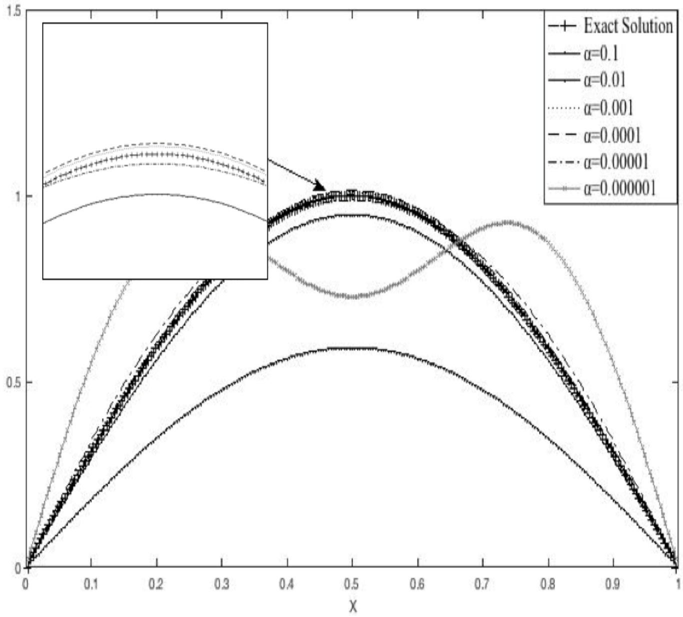

Fig. 2. Estimate solutions $\overline{u}_{\alpha_k}$ with $\alpha_k = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$

We can compare between two types of computing sequential computing and parallel computing with n number of equal-length subdivisions of interval [0,1] see fig 3.

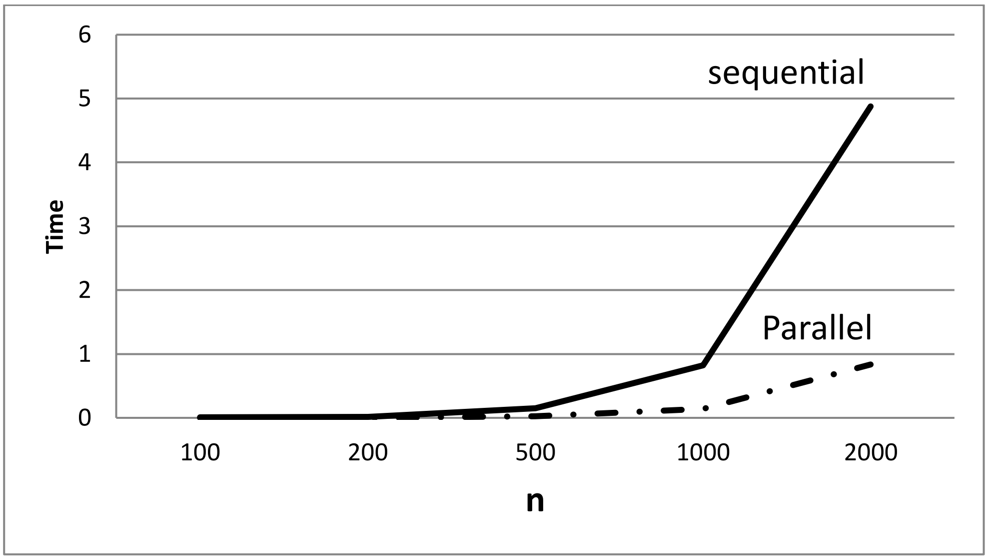

Fig. 3. Speedup of the parallel and sequential computing

We have prepare the following parameter set

$$\alpha_k = \{\alpha_1 = 0.1, \alpha_2 = 0.01, \alpha_3 = 0.001, \alpha_4 = 0.0001, \alpha_5 = 0.00001, \alpha_6 = 0.000001\},$$

then we used the sequential computing to find the best approximation solution. For parallel computing we define 6 workers and assigned for each of them a task to compute the approximation solution by using one parameter from parameter set α_k.

Conclusion

This work deals with the discretization method as base way for solving the Fredholm integral equation of the first kind. The discretization algorithm which explained in this work it is converted integral equation to linear operator equation and using the Tikhonov's regularization method for find the approximation solutions.

Regularization parameter α has been selected by general discrepancy principle method and used the parallel computing method to find approximation solution. The numerical analyses successfully apply to solve the inverse heat conducting. From the example, we noted that the algorithm was efficient to estimate the initial temperature depending on given measurement temperature with known noise level δ.

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