Research Article

METHOD OF REANALYSIS AFTER RECONSTITUTION OF THE DYNAMIC FLEXIBILITY MATRIX FOR A MODIFIED DISSIPATIVE SYSTEM

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ABSTRACT
In structural dynamics, several problems are solved with the help of formulations using matrices of frequency response functions. This article focuses on the exploitation and evaluation of these matrices. A technique of structural modifications, based on knowledge of the introduced modifications as well as the frequency response functions relating to the original structure, will first be described. Then, we will be interested in the evaluation of the used flexibility matrices. These can be either calculated from a mathematical model, or derived from experimental observations. In practice, only a limited number of the dynamic flexibility matrix columns can be measured. A technique to complete this matrix is proposed, following the description of the conventional techniques. The idea is combined with a procedure which makes it possible to choose, for the numerical tests, an optimal placement of the excitations. The proposed formulation is based on the use of clean solutions and is validated by a digital example.

Keywords: Frequency responses, structural modification, dynamic flexibility matrix, reconstruction, modal analysis.

1 INTRODUCTION

To optimize the calculations in structural dynamics, we are often faced with formulations using matrixes of frequency response functions (FRF), such as the problems of dynamic substructuring or structural modifications. [1, 2]. In practice, this resolution is based on the knowledge of the matrix H(ω) of the frequency response function (FRF).

This matrix can be estimated either from an analytical or a numerical simulation model, similar to the real model, or from experimental data. In the experimental case, the matrix H(ω), at each frequency in the analyzed band, is often evaluated either by reconstruction from the eight identified solutions of the system, which requires a prior modal identification [3], or by direct measurement of all its independent elements. The latter situation is rarely applied, because it is not economical, so only a very limited number of columns of the dynamic flexibility matrix can be measured, thus the other columns must be estimated.

In this work, we first develop a technique of structural modifications based on the knowledge of the frequency response functions with respect to the original structure and the introduced modifications. Then, after having exposed the conventional estimation techniques of the dynamic flexibility matrix, we propose a technique which allows us to evaluate the complete matrix without using a modal identification. A similar principle has already been proposed in [4, 5] and the idea is extended and combined with a procedure which makes it possible to choose, for numerical simulations, an optimal placement of the excitations [6].

An example of numerical simulation will be proposed to validate the proposed formulations, and to discuss the effects of the choice of the number and the positions of the exciters, used to measure the flexibility matrices, and the effect of the damping on the quality of the evaluation.

2 STRUCTURAL MODIFICATION PROBLEMS THROUGH TRANSFER FUNCTIONS

2.1 General Formulation

The modified structure can be represented by an assembly of two subsystems: the initial structure and an additional system consisting of the introduced modifications.

The equation representing the particular solution of the structure in its initial state, under a harmonic excitation, is expressed in matrix form as:

$$Z(s) = H(s)f, \quad s = j\omega \qquad (1)$$

Where $H(S) \in \mathbb{C}^{c,c}$ is the symmetrical FRF matrix of the initial structure (abbreviated IS), at the frequency ω, c is the number of degrees of freedom of capture (DOF) and $Z(s)$, $f \in \mathbb{C}^{c,1}$ represent the response vectors and the external force respectively.

To reduce the writing, we eliminate the s argument. The above equation is partitioned into the form:

$$\begin{pmatrix} z_i \\ z_a \end{pmatrix} = \begin{pmatrix} H_{ii} & H_{ia} \\ H_{ai} & H_{aa} \end{pmatrix} \begin{pmatrix} f_i \\ f_a \end{pmatrix} \qquad (2)$$

Where: the index a designates the degrees of freedom (abbreviated DOF) affected by the modification, and i designates the other DOF.

The relation of FRF of the IS subjected to the forces of connections f_{al}, due to the introduced modifications on the DOF of type a is written:

$$\check{f}_{al} = \Delta Z_{aa} \check{\hat{z}}_a \in C^{a,1} \qquad (3)$$

Where: $\check{\hat{z}}_a$ is the displacement of the DOF of the additional system on the points of connection with the IS $\check{f}_{al}$ represents the external forces exerted by the IS on the introduced modification.

$$\begin{pmatrix} \hat{z}_i \\ \hat{z}_a \end{pmatrix} = \begin{pmatrix} H_{ii} & H_{ia} \\ H_{ai} & H_{aa} \end{pmatrix} \begin{pmatrix} \hat{f}_i \\ \hat{f}_a \end{pmatrix} \qquad (4)$$

Where: $\hat{f}_i = f_i \in \mathbb{C}^{c-a,1}$ et $\hat{f}_a = f_a + f_{al} \in \mathbb{C}^{a,1}$

The additional system, consisting of a few known parametric modifications which do not modify the order of the system, is represented by the dynamic stiffness matrix:

$$\Delta Z_{aa} = [\Delta K_{aa} + s\Delta B_{aa} + s^2 \Delta M_{aa}] \in \mathbb{C}^{a,a} \qquad (5)$$

Where ΔK_{aa}, ΔM_{aa}, $\Delta B_{aa} \in \mathbb{R}^{a,1}$ are the symmetrical matrices of rigidity, mass and damping of the structural modification, respectively.

The matrices ΔK_{aa}, ΔM_{aa} and ΔB_{aa} have the following general form:

$$\Delta K = \begin{pmatrix} 0 & 0 \\ 0 & \Delta K_{aa} \end{pmatrix}$$
$$\Delta M = \begin{pmatrix} 0 & 0 \\ 0 & \Delta M_{aa} \end{pmatrix}$$
$$\Delta B = \begin{pmatrix} 0 & 0 \\ 0 & \Delta B_{aa} \end{pmatrix}$$

Connection conditions:

$$\check{\hat{z}}_a = \hat{z}_a; \quad \check{f}_{al} + f_{al} = 0 \qquad (6)$$

After using equations (3) and (6), equation (4) can be formulated as:

$$\begin{pmatrix} \hat{z}_i \\ \hat{z}_a \end{pmatrix} = \begin{pmatrix} H_{ii} - H_{ia}\Delta Z_{aa} G H_{ai} & H_{ia}(I_a - \Delta Z_{aa} G H_{aa}) \\ G H_{ai} & G H_{aa} \end{pmatrix} \begin{pmatrix} f_i \\ f_a \end{pmatrix} \qquad (7)$$

Where $G = [I_a + H_{aa}\Delta Z_{aa}]^{-1}$

Using equation (7), we can express the forced responses of the MS, without resorting to an exact but expensive reanalysis, using only the dynamic flexibility matrix of the SI and the dynamic stiffness matrix of the introduced modification. The modal parameters of the EM are then accessible by applying a method of modal identification on the preceding frequency responses. In order to assess the FRF of the EM from (7), we need to determine the matrix *G(s)* at each frequency ω. This evaluation cost depends on the number *a* of DOF affected by structural changes.

2.1.1 Program Code

The rearrangement code of the dynamic flexibility matrix:

```
function Mr=rearrangement(matrice)
matricenouelle=matrice;
rep=input('entrez le nombre de lignes/colonnes que vous voulez supprimer:\n');
for i=0:rep-1
num=input('entrez les numéros de lignes/colonnes que vous voulez réaranger:\n');
```

```
element(i+1)=num;
num=num-i;
disp(num);
matricenouelle(num,:)=[];
matricenouelle(:,num)=[];
end
matricenouelle=padarray(matricenouelle,[size(matrice,1)-size(matricenouelle,1),size(matrice,1)-
size(matricenouelle,1)],0,'post');
h=0;
for k=rep:-1:1
matricenouelle(size(matrice,1)-h,:)=matrice(element(k),:);
matricenouelle(:,size(matrice,1)-h)=matrice(:,element(k));
h=h+1;
end
Mr=matricenouelle;
end
```

2.2 Case Of DOF Connection To Earth

For the problems concerning the attached the DOF to the ground, in the simplest case, we choose for the disturbance matrices $\Delta M_{aa} = 0$ and ΔK_{aa} as a diagonal matrix with very large diagonal elements. Then, the perturbation of the rigidity connects in a quasi-rigid way the DOFs to the fixed reference. Equation (7) reduces to:

$$\hat{z}_i = \widehat{H} f_i \tag{8}$$

Where $\widehat{H} = H_{ii} - H_{ia}\Delta K_{aa} G H_{ai}$;

$$= [I_a + H_{aa}\Delta Z_{aa}]^{-1}$$

If we take ΔK_{aa} in the following form:

$\Delta K_{aa} = kI_a$, k is a positive scalar and et I_a is the unit matrix of a order.

The matrice G and $\widehat{H}$ become:

$$G = \frac{1}{k}[\frac{1}{k}I_a + H_{aa}]^{-1}$$

$$\widehat{H} = H_{ii} - H_{ia}[\frac{1}{k}I_a + H_{aa}]^{-1}H_{ai}$$

And for k tending to infinity, $\widehat{H}$ is written as:

$$\widehat{H} = H_{ii} - H_{ia}[H_{aa}]^{-1}H_{ai} \tag{9}$$

In this formulation, the introduction of structural modifications is avoided, but we are still faced with the inversion of the sub-matrix H_{aa} of equal order to the number of fixed DOFs.

We can find the same formulation as (9), but established in a different way, by using (4) and by imposing the constraint $\hat{z}_a = 0$.

3 FRF MATRIX EVALUATION

To solve the problems of structural modifications defined in point (7), for example, we need to know the dynamic flexibility matrix of the IS which can be estimated in different ways.

3.1 Estimation From An Updated Finite Element Model

In the dynamics of mechanical structures, a continuous system is often discretized and represented by models made up of a limited number n of FRFs [7, 8]. A first way to determine the FRF matrix $H(\omega) \in \mathbb{C}^{n,n}$, at a frequency ω, is through a calculation from an available finite element model. If one notes M, B et K, respectively the matrices of mass, damping and rigidity of the structure, the matrix FRF is then calculated by the following relation:

$$H(\omega) = (K + j\omega B - \omega^2 M)^{-1} \tag{10}$$

This may be a very intensive computation in the case of component models comprising a large number of DOFs and / or a wide range of excitation frequencies. After all, the dynamic stiffness matrix needs to be reversed for every discrete frequency in the frequency range of interest to us.

3.2 Estimation Using Experimental Measurements

In the case where the data come from the measurements, one is often forced to operate with a reduced sub-matrix $H_{cc} \in \mathbb{C}^{c,c}$ where c $(c << n)$ represents the limited number of sensors placed on the tested structure.

The elements of $H_{cc}(\omega)$ are generally evaluated

either by reconstruction from the eigen solutions identified, or by direct measurement of its c (c + 1)/2 independent elements.

3.2.1 Reconstitution Using The Identified Own Solutions

A second way to determine the FRFs of a damped structure consists in using an FRF synthesis based on a finite number of eigenvectors and eigenfrequencies of the structure. If we consider a structure with n DOF whose behavior is represented on the basis of its $2n$ complex modes, the relation between the synthesized FRF matrix $H(\omega)$ and the eigenvectors is expressed by.

$$H(\omega) = Y(j\omega I - S)^{-1}Y^T + \bar{Y}(j\omega I - \bar{S})^{-1}\bar{Y}^T \tag{11}$$

Where: $Y \in C^{n,n}$; $S \in \mathbb{C}^{n,n}$ and $H(\omega) \in \mathbb{C}^{n,n}$ represent respectively the modal base, the spectral matrix and the dynamic flexibility, with the pulsation ω, of the dissipative structure.

In a given frequency band, we can, with sufficient precision, express (8) approximately by:

$$H(\omega) \cong H^s(\omega) + H^d(\omega) \tag{12}$$

Where $H^s(\omega)$; $H^d(\omega) \in C^{n,n}$ represent respectively the static part of $H(\omega)$ relating to the unidentified modes and the part of $H(\omega)$ relating to the identified modes.

$$H^d(\omega) = Y_1(j\omega I_m - S_1)^{-1}Y_1^T + \bar{Y}_1(j\omega I_m - \bar{S}_1)^{-1}\bar{Y}_1^T \tag{13}$$

Where : $Y_1 \in \mathbb{C}^{n,m}$; $S_1 \in \mathbb{C}^{m,m}$ represents the identified modal sub-base, of the dissipative structure.

The introduction of the static part, $H^s(\omega)$, of $H(\omega)$ is introduced in order to partially compensate for the unidentified $n - m$ modes. This compensation has an important role in the regions outside the resonances of $H^d(\omega)$ regions where the static contributions of the truncated modes play a preponderant role.

In practical cases, only the sub-matrix $H_{cc} \in C^{c,c}$ calculated or measured $(c << n)$ is used frequently exploited. The equation (4.2) is then written as:

$$H_{cc}(\omega) \cong H_{cc}^s(\omega) + H_{cc}^d(\omega) \tag{14}$$

With:

$$H_{cc}^d(\omega) = Y_{1c}(j\omega I_m - S_1)^{-1}Y_{1c}^T + \bar{Y}_{1c}(j\omega I_m - \bar{S}_1)^{-1}\bar{Y}_{1c}^T \; Y_{1c} \in C^{c,m} (m < c) \tag{15}$$

is the modal sub-base identified on the c sensors.

The construction of $H_{cc}(\omega)$ requires the identification of the matrices: Y_{1c}, S_1 and $H_{cc}^s \in C^{c,c}$. To identify Y_{1c} and S_1 a few p $(p < c)$ columns or a few rows of $H_{cc}(\omega)$ are sufficient (in the limited case a column or a row is sufficient). As an example, one can quote as reference of modal identification methods: the method known as of linear smoothing [1] and the total method [2].

The problem is that for the matrix of static residue H_{cc}^s we can identify only p columns. The missing information can induce some effects on the unknown columns of $H_{cc}(\omega)$, especially in the regions where the residual terms have a preponderant role. To get around this problem and that of the extraction of the clean solutions we are interested in the direct reconstruction of the dynamic flexibility $H_{cc} \in C^{c,c}$ from the knowledge of a submatrix $H_1 \in C^{c,p}$ of H_{cc}.

3.2.2 Direct Evaluation Of FRF Matrices

For this purpose, the contributions of all structural modes are taken into account. The set of knowledge of $H_{cc}(\omega)$ requires c sensors and c excitations. Usually, for economic reasons, only a limited number p of linearly independent excitation configurations is available.

Problem: Knowing p $(p < c)$ columns of $H_{cc}(\omega)$ denoted by the submatrix $H_1(\omega) \in C^{c,p}$, we need to estimate (at best) the remaining $c - p$ columns without performing a modal identification.

We describe below a technique that helps to resolve this problem. We can find references to a similar method [4, 5].

Pour préciser les inconnues du problème, la matrice FRF $H_{cc}(\omega)$ is divided into sub-matrices as follows:

$$H_{cc} = [H_1 \quad H_2] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{16}$$

Where: $H_1 \in \mathbb{C}^{c,p}$ is the known part of H_{cc}, $H_{11} \in \mathbb{C}^{p,p}$ a square sub-matrix of H_1et $H_2 \in \mathbb{C}^{c,c-p}$ is the unknown part of H_{cc}.

We only consider the cases where the matrix $H_{cc}(\omega)$ of the FRF is asymmetric:

$H_{12} = H_{21}^T, H_{11} = H_{11}^T, H_{22} = H_{22}^T$ (17)

In this case, the number of unknown elements of the rectangular matrix H_2 is contained in the matrix H_{22} reduces to $(c-p)*(c-p+1)/2$

2.2.5 Evaluation By Spectral Factorization Of The Square Sub-Matrix H_{22}

The eigenvalues γ_i and the eigenvectors $\varphi_i(i = 1,...,p)$ of the matrix H_{11} are defined by the problem of the eigenvalues [12]:

$$(H_{11} - \gamma_i I_p)\varphi_i = 0, \quad i = 1, \ldots, p \tag{18}$$

One can then write the complex symmetric matrix H_{11} in the form:

$$H_{11} = \phi_{11}\Gamma\phi_{11} \tag{19}$$

Where: $\Gamma, \phi_{11} \in C^{p,p}$ are the diagonal matrix of the eigenvalues and the modal matrix of the eigenvectors of H_{11}, respectively. These eigenvectors are normalized so that:

$$\phi_{11}\phi_{11}^T = \phi_{11}^T\phi_{11} = I_p \tag{20}$$

The factorization (19) is valid for the matrices having distinct eigenvalues and possibly for the matrices having multiple eigenvalues.

To estimate the set of the FRF matrix, let us find the matrix $\phi_{21} \in C^{c-p,p}$ such that:

$$\begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} = \phi_1\Gamma\phi_{11}^T, \phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} \tag{21}$$

We can conclude that :

$$H_{cc}^{rec} = \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} \Gamma (\phi_{11}^T \quad \phi_{21}^T) \tag{22}$$

In general, the eigenvalues of a matrix do not give precise information on its rank. If we wish to control the rank of the matrix H11, it is preferable to use a decomposition in singular values [5, 13].

4 RESULTATS AND DISCUSSION

To illustrate the procedure relating to the evaluation of the FRF, we consider the following examples:

4.1 Exemple 1

We consider a first example of a shock absorber mass system with the following parameters:

K = [80 -40 0 0 0
-40 80 -40 0 0
0 -40 80 -40 0
0 0 -40 80 -40
0 0 0 -40 40];

M = [5 0 0 0 0
0 5 0 0 0
0 0 5 0 0
0 0 0 5 0
0 0 0 0 5];

B = [0.6000 -0.3000 0 0 0
-0.3000 0.6000 -0.3000 0 0
0 -0.3000 0.6000 -0.3000 0
0 0 -0.3000 0.6000 -0.3000
0 0 0 -0.3000 0.3000];

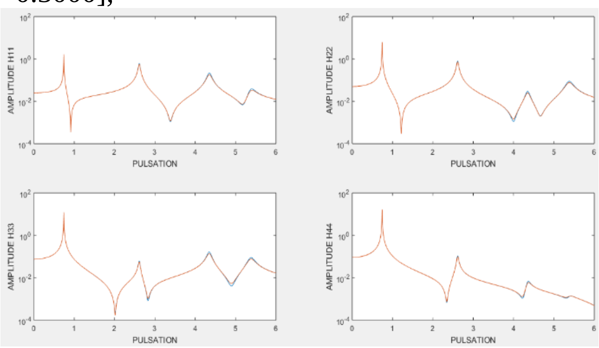

Figure 1: Graphs between the direct calculation and the method of equation (7).

4.2 Exemple 2

We consider a second example shown in Figure 2. The structure is modeled using a finite element code. This model is discretized into 42 beam elements, it contains 43 unconstrained nodes with 3 DOF per node. The grid has the following physical and geometric characteristics:
Its characteristics are as follows:
Young's modulus E = 0.499 = 10^9 N m^{-2};
the density = 7800 kg m^{-3};
the moment of inertia I = 0:279 = $10^{-4}m^4$;
the cross-section A = $0.001m^2$;
the length L = 1.5m;

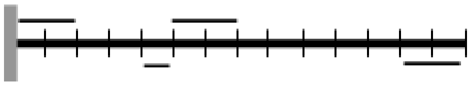

Figure 2: Free recessed beam divided into 42 elements.

A proportional damping ($BM^{-1}K = KM^{-1}B$) is introduced and the "exact" FRF matrix H(ω) is calculated, at each frequency ω in the analyzed band, by using the clean modes of the dissipative structure. We note that $a_i = |Re(s_i)|/Im(s_i)$ is the i^{th} modal damping factor; $s_i = -a_i\omega_i + j\omega_i$ is the i^{th} eigenvalue of the structure. The frequency band considered [0, 300 Hz] contains the first 10 natural frequencies of the structure (see table 1).

Table 1: The proper frequencies of the initial structure by finite element method.

Mode number	Frequencies (Hz)
1	10.5
2	42.15
3	65.84
4	126.53
5	184.37
6	211.08

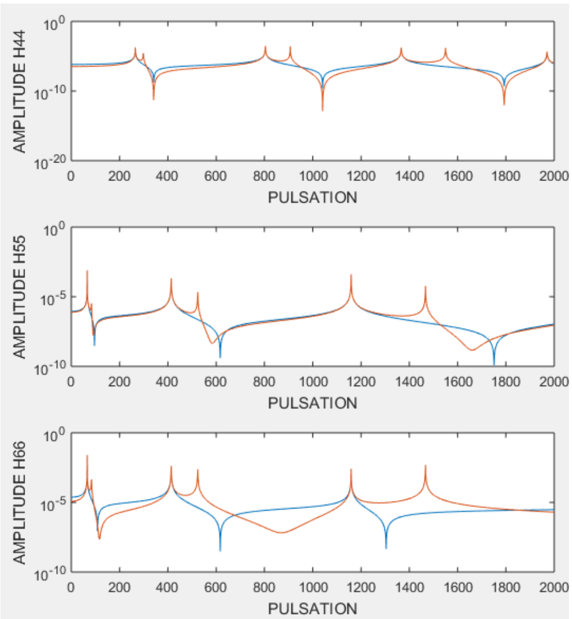

Figure 1: Exact FRF to calculate and the one to calculate with the approximate method.

4 CONCLUSIONS

The objective was to contribute to the resolution of certain problems of dynamic structures established from the FRF matrices. For this, in section 2, we presented a formulation dealing with the reanalysis of the problems of modified structures, and discussed the case where some FRFs can be rigidly connected to the ground. In general, the quality of the frequency responses of the modified structure depends on the quality of the estimation of the flexibility matrices of the original structure. To do this, we have proposed a method, based on a spectral decomposition of a square sub-matrix of FRF; in some regions of low amplitude.

Through numerical simulations, we have seen that the quality of the estimation of the FRF Hcc(ω) matrix depends on several factors. In the case where a degradation in the quality of the estimate is observed, even with a better choice of the positions of the exciters, an increase in the number of exciters can correct this defect. The increase in damping also makes it possible to improve the quality of the estimate and to attenuate the parasitic peaks which appear in the spectrum.

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