EVALUATION OF ANALYTICAL SOLUTION FOR FRACTIONAL DIFFUSION EQUATIONS USING FIXED POINT THEOREMS AND ENHANCED ADOMIAN DECOMPOSITION METHOD

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ABSTRACT:
In recent years, the diffusion equations and fixed-point theorems are considered as a main role to solve different mathematical analysis sensing. Hence in this paper, enhanced Adomian decomposition method is developed to solve the considered problems with efficient solutions. Here, the fractional diffusion equations are considered with variable coefficients to formulate the problems. The fractional integral or derivative functions are utilized with the consideration of caputo definition. This proposed approach is utilized to analysis the analytical solutions to compute the optimal solutions. Additionally, optimal solutions are obtained with the consideration of three fixed point theorems such as Arzela Ascoli theorem, Schaefer fixed point theorem and Banach fixed point theorem. The formulated problems are solved by using the enhanced Adomian decomposition method as well as fixed point theorems. In last, the numerical analysis of the considered problems with the solutions of the fixed-point theorems are presented. The proposed method is working based on iteration process to achieve the best solutions related with the considered problems. Finally, the optimal solutions of the considered problems are analyzed and validated with the proof analysis.

Keywords: fixed point theorems, fractional integral and derivative, Banach fixed point theorem, Adomian decomposition method and numerical analysis

1. INTRODUCTION

In recent years, general diffusion equations have been increased in various applications such as mathematical model of spatial heterogeneous medium and mathematical modelling of processes in transport applications [1]. The diffusion equations are utilized in many different applications such as optimal calculation of natural diffusion distance, multiscale analysis of complex structure, data representation and dimensionality reduction. Different kinds of discussion algorithm are available in mathematics such as partial differential equations and fractional differential equations. The partial differential equations are utilized in the many applications of science and engendering with fractional order consideration [2]. The utilization of the fractional computations has been considered as important thing to design the enhanced mathematical design of various real time applications. Many authors are discussed the application and theoretical development of the fractional computations [3]. From the theoretical computations, the fractional differential equations are more efficient to present the natural conditions and which have been utilized in various domain of applications in mathematics. The fractional differential equations are advantages contrasted with the conventional differential equations [4].

Fixed point theorems are one of the enhanced subfields of nonlinear fractional analysis. So, the fixed-point theorems are utilized to solve the nonlinear sciences and mathematics problems. The Banach fixed point theory is developed in 1922 year and which a complete metric space every contraction has a fixed point [5]. This fixed-point theorem has the properties of existence and uniqueness of a fixed point and constructing the fixed point. This property is completely different from the different fixed-point theorems. Hence, the Banach fixed point theorem has been utilized by authors to analysis the different problems. The fractional diffusion equations are solved with the consideration of fixed-point theorems [6]. To solve the problem, the caputo fractional derivation to achieve high optimal outcomes which also suitable for fractional differential equation. This caputo fractional derivative properties are attractive which compared with different derivative functions such as Erdelyi-Kober derivation, Miller-Ross derivative, Marchaud derivative, Liouville derivative Hadamard derivative, Sonin Letnikov derivative and Grunwald-Letnikov derivative [7].

Generally, it is very complex to compute the analytic solution of fractional diffusion equations with fixed point theorems. Various numerical methods are available to solve fractional diffusion equations and achieved accurate results. The fractional diffusion equations and fixed-point theorems are solved by utilizing different methods such as Adomian decomposition method (ADM) [8], matrix method, variational iteration method (VIM) [9], homotopy analysis method (HAM), homotopy perturbation method (HPM) [10] and so on. These numerical methods are designed to achieve accurate results and needed the execution time in computation process. These methods have different definitions which described by different authors to solve the fractional operators in fractional diffusion equations. For understanding purpose, the Riemann-Liouville derivative, Grunwald-Letnikov derivative and fractional conformable derivative operator definitions are given by different authors which representing the fractions operators in fractional diffusion equations [11]. These definitions are an approximation of fractional diffusion equations but, the exact definition of fractional diffusion equations is not available. From the derivatives, the caputo fractional derivative and ADM is a significant method to solve the fractional diffusion equation problem because of its properties. This ADM and caputo derivative, fixed point theorems are compared with the conventional functions, which is an efficient method.

The remaining part of the paper is organized as follows, section 2 provides the detail review of the analytical solution problems, section 3 provides the detail description about the fractional diffusion equations and Banach fixed point theorems. The section 4 provides the result analysis of the mathematical analysis of diffusion equations. The section 5 concludes the paper.

2. LITERATURE REVIEW

Many different methods are available to solve the fractional diffusion equations which are developed by authors. Some of the methods are reviewed in this section.

Honey Teestani *et al.*, [12] have introduced the mathematical arrangement of multi-dimensional fractional difference conditions using fractional-Lucas functions (FLF) and development strategy. To begin with, FLFs and their properties were provided. At that time, the area introduced the mathematical strategy, as indicated by the pseudo-operational network of subsidiaries and modified operational lattes. Similarly, for the computational method, estimate the upper limit of errors. So, take note of some kind of issues and clarify the conspiracy that was introduced. Our computational results reveal that the technique introduced is surprising and is subject to non-linear multi-demand partial conditions, diffusion conditions with time-fractional convection-variable coefficients, and time-interval fractional diffusion conditions with variable coefficients.

S.S. Al-Zahrani *et al.*, [13] have introduced a high-order timing effort, which uses Fourier in another world space, and the fourth order diagonal in the corner indicates the matrix significant potential for multi-dimensional space-area address response propagation conditions. The next time venture plan was developed based on the pending time difference approach with the ultimate goal, which mitigates the resolution of a large non-straightforward structure at each time point while maintaining the solidity of the plan. In some other mathematical schemes for these conditions the fragmentary administrator's territory triggers full and dense networks. The project had the option to overcome such a computational failure due to the fully biased portrayal of the fragmented manager. It additionally strongly integrates into various spatial measurements. The reliability of the project was talked about by examining the expansion picture and planning its security areas, which indicates the optimal nature of the technique. The assembly inquiry was carried out properly to show that the plan was accurate in the fourth request timetable. Mathematical analyzes of the response scatter structure with application to the design arrangement were spoken to show the impact of fractional demand on space-area fractional diffusion conditions and to recognize the reliability of the scheme.

Lily Wei *et al.*, [14] have introduced a mathematical strategy for overcoming the expressed diffusion state, which manifests itself in the number of super low diffusion operations found in some real problems, whose arrangement rotates in a loop mode when the infinity was observed. Looking at the local discontinuous Galerkin method near space, creating a completely unique project proves that the project is truly sustainable and connected to demand. Mathematical models were processed to show the assembly request and the shocking mathematical function of the techniques introduced. It is worth noting that the project proposed in this study can be extended to deal with problems in many dimensional areas, however, the calculation may be too large. In the future, develop high-demand mathematical techniques for the time-space fractional difference level of the assigned request and the two-dimensional case.

Xue-LeiLin *et al.,* [15] have developed a discretization plot for the multi-dimensional time-space fractional diffusion state with different coefficients and the associated quick solution, in which the intermediate and spatial subdivisions of the moving Grünwald equation are subdivided separately. A separation and conquest method are used for the massive direct structure that always collects the distinct levels, thus solving the progress of the multi-dimensional straight structure identified with spatial individuality. The pre-conditioned summary remaining strategy was used to deal with the spatial direct structure that comes due to spatial individuality. Uniqueness has undoubtedly proven to be consistent and cohesive in the sense of infinity for common instability coefficients. Mathematical results were calculated to show the productivity of the proposed strategy.

Xiangcheng Zheng *et al.,* [16] developed the Rankel-Nicholson finite volume estimate for non-linearly assigned demand space-fractional diffusion conditions in three space measurements. Furthermore, the subsequent nonlinear mathematical structure with the Kronecker-item type coefficient phase was discovered. The finite block plot was undoubtedly consistent with the second claim accuracy relative to the minimum demand accuracy in space with respect to the progressive measures of the planned and distributed demands and the unique standard. At each time point, Newton's operational strategy was used as a linear solution to deal with successive nonlinear polynomial mathematical structures. Furthermore, during each Newton's rotation, an efficient biconjugate gradient stabilized method (BiCGSTAB) was developed, in which both the LAD stock and the structural vector amplification are efficiently guided by the duplicate substructure of the coefficient networks. It has been demonstrated that the BiCGSTAB strategy requires the efficiency of O (N) and the computational cost of O (NlogN) for an emphasis, while no accuracy is lost in the paradox and Gaussian final technique. Since then, an efficient limited volume strategy has been developed, and it is well-suited for many purposes, especially multi-dimensional issues, large-scale demonstrations and restructuring. Mathematical analyzes were introduced to confirm the hypothetical outcomes and to show the definitive potential of the production strategy.

3. BASIC STUDY ABOUT PRELIMINARIES

In this section, some general concepts of fractional derivatives and fractional integrals based on their properties. Here, the caputo function is utilized which very familiar function in the area of applied mathematics. Additionally, describe some lemmas and theorems that are utilized to prove the uniqueness and existence scenarios of specified problems.

Definition 1:

In the fractional functions, the gamma function is defined as extension of fractional functions which is real numbers and mathematically formulated as below,

$$\Gamma(\theta) = \int_o^{\infty} \tau^{\theta-1} \, exp(-\tau) d\tau, \theta > 0 \qquad (1)$$

$$\Gamma(\theta+1) = \theta\Gamma(\theta) \qquad (2)$$

Where, Gamma function can be described by $\Gamma(.)$.

Definition 2:

The real function is mathematically presented as follows,

$$G(Y), Y > 0 \qquad (3)$$

Real function is defining to be in the space S^{θ}. This real function is check with the condition of $If\ \theta\epsilon r$ which condition is met which is exist a real number $V > \theta$. Similarly, it is check with the condition of $G(Y) = Y^V G^1(Y)$, where, $G^1(Y)\epsilon C[0, \infty]$. This condition, the space is denoted as follows,

$$C_\theta^N \ if \ G^N, N\epsilon N \cup \{0\} \qquad (4)$$

Definition 3:

The fractional integral operator in Riemann Liouville of order is $\sigma \geq 0$. The function is denoted as follows,

$$G\epsilon C^\theta \geq -1 \qquad (5)$$

The abovementioned function is denoted by,

$$M^\sigma YG(Y) = \frac{1}{\Gamma(\sigma)}\int_0^Y (Y-\xi)^{\sigma-1} G(\xi)d\xi, \quad \sigma > 0, Y > 0 \qquad (6)$$

$$M^0 YG(Y) = G(Y) \qquad (7)$$

Based on the fractional operator $(M^0\xi)$, some of the properties are formulated. The condition is presented as,

$$for \ G\epsilon C^\theta \geq -1 \ ; \ \xi > -1, \sigma, \omega \geq 0 \qquad (8)$$

Related to this above-mentioned condition, the properties are presented follows,

$$M_Y^\sigma M_Y^\omega G(Y) = M_Y^{\sigma+\omega} G(Y) \qquad (9)$$

$$M_Y^\sigma M_Y^\omega G(Y) = M_Y^\sigma M_Y^\omega G(Y) \qquad (10)$$

$$M_Y^\sigma M_Y^\omega = \frac{\Gamma(\xi+1)}{\Gamma(\sigma+\xi+1)} Y^{\sigma+\xi} \qquad (11)$$

The disadvantages of the Riemann-Liouville derivative can be removed by caputo derivative combined with that. The enhanced version of the Riemann-Liouville derivate is named as caputo derivate.

Definition 4:

Caputo sense of the fractional derivation can be formulated as below,

$$D_Y^\sigma G(Y) = M_Y^{N-\sigma} D_Y^N = \frac{1}{\Gamma(N-\sigma)}\int_0^Y (Y-\xi)^{N-\sigma-1} G^N(\xi)\, d\xi \qquad (12)$$

For $N-1 < \sigma \leq N, N\epsilon\mathbb{N}, Y > 0, G\epsilon C_{-1}^N$ and $D_Y^\sigma a = 0$.

The linear operators are considered as caputo fractional derivatives which formulated as,

$$D_Y^\sigma(aT(Y) + bU(Y)) = ad_Y^\sigma T(Y) + bD_Y^\sigma U(Y) \qquad (13)$$

Where, b and a are considered as constants.

Definition 5:

The caputo time fractional derivative operator of order with smallest integer N and the integer higher than $\sigma > 0$. This function is mathematically formulated as follows,

$$D_\xi^\sigma W(Y,\xi) = \frac{\partial^\sigma W(Y,\xi)}{\partial \xi^\sigma} = \begin{cases} \dfrac{1}{\Gamma(N-\sigma)}\displaystyle\int_0^Y (\xi - T)^{N-\sigma-1} \frac{\partial^\sigma W(Y,T)}{\partial T^\sigma} dT; & N-1 < \sigma < N \\ \dfrac{\partial^N W(Y,\xi)}{\partial \xi^\sigma} & \sigma = N\epsilon\mathbb{N} \end{cases} \qquad (14)$$

Definition 6:

The fractional operator Riesz is mathematically formulated with the consideration of $S-1 < \sigma \leq S$ and the interval $0 \leq Y \leq P$ which presented below,

$$\frac{\partial^\sigma}{\partial |Y|^\sigma} M(Y,\theta) = -H_\sigma (0D_Y^\sigma + YD_L^\sigma) M(Y,\theta) \qquad (15)$$

Where, $H_\sigma = \frac{1}{2\cos\left(\frac{\pi\sigma}{2}\right)}, \sigma \neq 1$.

$$0D_Y^\sigma M(Y,\theta) = \frac{1}{\Gamma(S-\sigma)}\int_0^Y \frac{M^S(\xi,\theta)}{(Y-\xi)^{\sigma+1-S}} d\xi \qquad (16)$$

$$YD_Y^\sigma M(Y,\theta) = \frac{(-1)^S}{\Gamma(S-\sigma)}\int_0^P \frac{M^S(\xi,\theta)}{(\xi - Y)^{\sigma+1-S}} d\xi \qquad (17)$$

Lemma 1:

The infinite domain is a function of $M(Y)$ and $-\infty < Y < \infty$ this infinite function can be mathematically formulated as follows,

$$-(-\Delta)^{\frac{\alpha}{2}}M(Y) = -\frac{1}{2\cos\left(\frac{\pi\sigma}{2}\right)}(-\infty D_Y^\sigma + YD_\infty^\sigma)M(Y) \qquad (18)$$

$$= \frac{\partial^\sigma}{\partial|Y|^\sigma}M(Y) \qquad (19)$$

The above-mentioned fraction parameters imply that the below mathematical analysis which formulated below,

$$(-\infty D_Y^\sigma + YD_\infty^\sigma)M(Y) = -2\cos\left(\frac{\pi\sigma}{2}\right)\frac{\partial^\sigma}{\partial|Y|^\sigma}M(Y) \qquad (20)$$

$$S-1 < \sigma \le S \quad (21)$$

Based on equation (21), the formulation is presented below,

$$-\infty D_Y^\sigma M(Y,\theta) = \frac{1}{\Gamma(S-\sigma)}\int_{-\infty}^{\infty}\frac{M^S(\xi,\theta)}{(Y-\xi)^{\sigma+1-S}}d\xi \qquad (21)$$

$$YD_Y^\sigma M(Y,\theta) = \frac{(-1)^S}{\Gamma(S-\sigma)}\int_Y^{\infty}\frac{M^S(\xi,\theta)}{(\xi-Y)^{\sigma+1-S}}d\xi \qquad (22)$$

Definition 7:

The analytic mathematical formulation of the Riesz fractional derivations are presented follows. Which consider the function,

$$O^R(\eta) = \eta^R(1-\eta)^R, \eta\epsilon[0,1] \qquad Where, R = 1,2,3,\ldots \qquad (23)$$

$$\frac{\partial^P O^R(\eta)}{\partial|\eta|^\rho} = -\Psi^P \times \sum_{P=0}^{R}\frac{(-1)^P R!\,(R+P)!}{P!\,(R-P)!\,\Gamma(R+P+1-\rho)}[\eta^{R+P-\rho} + (1-\eta)^{R+P-\rho}] \qquad (24)$$

Where, $\Psi^P = \frac{1}{2\cos\left(\frac{\pi\sigma}{2}\right)}, \rho \neq 1$.

The lemma functions are utilized to find out the solutions for considered problems which helps the problem-solving issues. The lemma functions are presented below,

Lemma 2:

This lemma function is considered the condition which presented below,

$$For\ N-1 < \sigma \le N, N\epsilon\mathbb{N}\ and\ , G\epsilon C_{-1}^N\ and, \theta \ge -1 \qquad (25)$$

after that,

$$D_Y^\sigma M_Y^\sigma G(Y) = G(Y) \qquad (26)$$

$$M_Y^\sigma D_Y^\sigma G(Y) = G(Y) - \sum_{Q=0}^{N-1} G^Q(0^+)\frac{Y^Q}{Q!}, Y > 0 \qquad (27)$$

Definition 8:

This is defined as operator which completely continuous operation. If it is continuous and maps considered as pre-compact sets from the bounded sets.

Definition 9:

In this definition, consider $(Y, ||.||)$ is a normed space. The mapping in contraction of Y is denoted as $P: Y \to Y$ which compensated for each $Y^1, Y^2 \epsilon Y$. The formulation is presented as follows,

$$\left||P(Y^1) - P(Y^2)|\right| \le \delta\left||Y^1 - Y^2|\right| \qquad (28)$$

Based on equation 28, some of the real values are computed $0 \le \delta < 1$. The different lemma functions are achieved by utilizing the caputo derivations functions. The lemma functions are important role in the analytical analysis of fractional diffusion algorithms.

Lemma 3:

In this lemma function, $\sigma > 0$, after that general result to the homogeneous equation,

$$D_{O^+}^\sigma \psi\,(Y) = 0 \qquad (29)$$

The homogeneous equation can be formulated as follows,

$$\psi\,(Y) = A^0 + A^1Y + A^2Y^2 + A^3Y^3 + \cdots + A^{S-1}Y^{S-1}, A^M\epsilon\Re, M = 1,2,..,S-1(S = [\sigma] + 1 \qquad (30)$$

Lemma 4:

In this lemma function, consider $\sigma > 0$. This function is presented follows,

$$M_{O^+}^{\sigma} D_{O^+}^{\sigma} \psi\,(Y) = \psi\,(Y) + A^0 + A^1 Y + A^2 Y^2 + A^3 Y^3 + \cdots + A^{S-1} Y^{S-1}, A^M \epsilon \Re, M = 1,2,..,S-1(S = [\sigma]+1 \qquad (31)$$

To solve the problems, the three fixed point theorems are consdiered which explained follows,

Theorem 1:

Arzela Ascoli theorem.

In this theorem [17], the compact metric space is considered as Y. Thr sup norm metric can be considered as $C(Y, \mathrm{K})$. After that, the set is denoted by $\mathrm{A} \subset C(Y)$ which compace. Here, A is a equicontinous and closed.

Theorem 2:

Schaefers fixed point theorem

In this algortihm [18], the continous operation can be denoted as $A: Y \to Y$. The set is denoted as $S(A) = \{Y \in N: Y = C * P(Y)\}$. Here, $C * \in [0,1]$. After that, which named as fixed points in P.

Theorem 2:

Banach fixed point theorem

In this theorem [19], each contraction mapping on a whole metric space consisting of specific fixed point.

4. DESCRIPTION ABOUT ENHANCED ADOMIAN DECOMPOSITION METHOD

In this proposed methodology, fractional diffusion equations are considered with the different coefficients [20]. The fractional diffusion equations are solved with the consideration of enhanced adomian decomposition method. The different coefficients with fractional diffusion equation is mathematically formulated as follows,

$$\frac{\partial M(\omega^1, \omega^2, \dots, \omega^S, \theta)}{\partial \theta} = \sum_{I=1}^{M} \Big(H^{I+}(\omega^I)_{-\infty} D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + N(\omega^1, \omega^2, \dots, \omega^S, \theta), (\omega^1, \omega^2, \dots, \omega^S, \theta) \epsilon \mathbb{R}^M, \theta \in (0, \Theta) \Big) \qquad (32)$$

The initial conditions of the problem is formulated as follows,

$$M(\omega^1, \omega^2, \dots, \omega^S, 0) = \psi M(\omega^1, \omega^2, \dots, \omega^S), M(\omega^1, \omega^2, \dots, \omega^S, \theta) \epsilon \sum, M\,(\omega^1, \omega^2, \dots, \omega^S, 0) = 0, (\omega^1, \omega^2, \dots, \omega^S), \Re^S \backslash \sum, \theta \in [0, \theta] \qquad (33)$$

Where, $M(\omega^1, \omega^2, \dots, \omega^S, 0)$ can be described as forcing term, H^{I+} and $H^{I-}, I = 1,2,\dots,M$, $N-1 < \sigma \le N, -\infty < \omega^{iP} < \omega^{iR} < \infty, with \sum = \Pi_{I=1}^{M} \left(\omega^{IP}, \omega^{iR}\right)$ can be described as positive functions. The physical and well-posedness of the problem is mentioned in equation 32. $(\omega^I)_{\omega^I} D_{\infty}^{\sigma}$ and $(\omega^I)_{-\infty} D_{\omega^I}^{\sigma}$ are the operators which also utilized in equation (32). This operators are present order is right and left which caputo fractional derivatives of $M(\omega^1, \omega^2, \dots, \omega^S, 0)$. The caputo fractional derivatives are formulated as follows,

$$-\infty D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S) = \frac{1}{\Gamma(N-\sigma)} \int_{-\infty}^{\omega^I} \frac{S^N(\omega^1, \omega^2, \dots, \omega^{I-1}, \xi, \dots, \omega^S)}{(\omega^I - \xi)^{\sigma-1}} d\xi \qquad (34)$$

$$\omega^I D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S) = \frac{1}{\Gamma(N-\sigma)} \int_{\omega^I}^{\infty} \frac{S^N(\omega^1, \omega^2, \dots, \omega^{I-1}, \xi, \dots, \omega^S)}{(\xi - \omega^I)^{\sigma-1}} d\xi \qquad (35)$$

Where, $M(\omega^1, \omega^2, \dots, \omega^S)$ can be considered as the problem, $-\infty D_{\omega^I}^{\sigma} = \omega^{IP} D_{\omega^I}^{\sigma}$ and $\omega^I D_{\infty}^{\sigma} = \omega^I D_{\omega^I R}^{\sigma}$ can be considered as the provided solution. From the equation 32, the operator form can be mathematically formulated as follows,

$$M_\theta M(\omega^1,\omega^2,\dots,\omega^S,\theta) = \sum_{I=0}^{M}\Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma}M(\omega^1,\omega^2,\dots,\omega^S,\theta) + H^{I-}(\omega^I)_{\omega^I}D_{\infty}^{\sigma}M(\omega^1,\omega^2,\dots,\omega^S,\theta) + N(\omega^1,\omega^2,\dots,\omega^S,\theta),(\omega^1,\omega^2,\dots,\omega^S,\theta)\Big) \quad (36)$$

From the equation, the inverse operator $M_\theta$of D_θ which applied and formulated as follows,

$$M(\omega^1,\omega^2,\dots,\omega^S,\theta) = \psi M(\omega^1,\omega^2,\dots,\omega^S) + \sum_{I=0}^{M}\Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma}M(\omega^1,\omega^2,\dots,\omega^S,\theta) + H^{I-}(\omega^I)_{\omega^I}D_{\infty}^{\sigma}M(\omega^1,\omega^2,\dots,\omega^S,\theta) + N(\omega^1,\omega^2,\dots,\omega^S,\theta),(\omega^1,\omega^2,\dots,\omega^S,\theta)\Big) \quad (37)$$

The recursion formulation of the enhanced adomian decomposition method from the equation (37) can be formulated as follows,

$$M_\theta(\omega^1,\omega^2,\dots,\omega^S,\theta) = \psi M(\omega^1,\omega^2,\dots,\omega^S) + M_\theta\big(N(\omega^1,\omega^2,\dots,\omega^S,\theta)\big) \quad (38)$$

And,

$$M_P(\omega^1,\omega^2,\dots,\omega^S,\theta) = \sum_{I=0}^{M}\Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma}M_P(\omega^1,\omega^2,\dots,\omega^S,\theta) + H^{I-}(\omega^I)_{\omega^I}D_{\infty}^{\sigma}M_P(\omega^1,\omega^2,\dots,\omega^S,\theta)\Big) \quad (39)$$

Where, $\rho = 0,1,2 \dots,$. The initial iteration of the equation (38) can be divided into different components which presented as follows,

$$M_0 = B_0 + B_1 + B_2 + \cdots + B_M = B \quad (40)$$

Where, $B_0, B_1, B_2, \dots, B_M$ can be described as terms which achieved from integrating the source parameters and mentioned conditions.

From the above equation, any parameter is considered, the M_0 should be satisfies the equation 32. Once achieves the best solutions based on their conditions, the process can be terminated. If M_0 does not satisfied means, select the next terms and continue the process. After that, select the next term and process was continued. If does not get best solution means, the B can be considered as best solution. This steps are continued upto B_M term. Hence, it is named as enhanced adomian decomposition method. This proposed method is applicable to the formulated problem, after that achieved solution is an analytical results to the problem. The main limitation of the method is initial term zero of the series that compensates the specified isses and regarding boundary and initial conditions. So, the initial step is get the semi analytical solution with its next step. After that, the enhanced adomian decomposition method is applied in formulated problem and achieved the best solution without consideration of discretization and linearization. Hence, the enhanced adomian decomposition method is a efficient and powerful to solve the problem in compared with other conventional numerical methods without consideration of adomian polynomial, discretization and linearization.

5. NUMERICAL RESULTS OF FRACTIONAL DIFFUSION EQUATIONS

In this section, the demonstration of the uniqueness and existence of the problem solution which presented by the equation 32 and 33 with the consideration of fixed point theorems. Here, the banach is denoted by $C(\eta, R)$ with the continous function of space $\sum\times(o,\theta) = \eta$ into $\Re$ which endowered with the normalization $||.||_\infty$ which defined as $||S||_\infty = \sup\{|M(\omega^1,\omega^2,\dots,\omega^S,\theta)|; (\omega^1,\omega^2,\dots,\omega^S,\theta) \in \eta\}$.

Definition 10: Banach fixed point theorem

A function $M(\omega^1,\omega^2,\dots,\omega^S,\theta) \epsilon C(\eta, R)$ is defined as a solution of the problem is 32 and 33. If $M(\omega^1,\omega^2,\dots,\omega^S,\theta)$ compensates the fractional diffusion equations with the related on initial and boundary conditions of variable coefficients. The below mentioned assumptions can be utilized to validate the existence of solution and uniqueness.

The continous parameter of the functions are denoted by $-\infty D_{\omega^I}^{\sigma} M$ and $\omega^I D_{\infty}^{\sigma}$, and, non negative constants are denoted by H^{IR}, H^{IL}, A^{IR} and A^{IL}, $I = 0,1,2,\dots M$ such that,

$$\left|-\infty D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S)\right| \leq H^{IL} M(\omega^1, \omega^2, \dots, \omega^S) \qquad (41)$$

$$\left|\omega^I D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S)\right| \leq H^{IR} M(\omega^1, \omega^2, \dots, \omega^S) \qquad (42)$$

Additionaly,

$$\left|-\infty D_{\omega^I}^{\sigma}(M1 - M2)\right| = \left|\omega^{IL} D_{\infty}^{\sigma}(M1 - M2)\right| \leq A^{IL}(M1 - M2) \qquad (43)$$

$$\left|\omega^{IL} D_{\infty}^{\sigma}(M1 - M2)\right| = \left|\omega^I D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S)\right| \leq A^{IR}(M1 - M2), \forall(\omega^1, \omega^2, \dots, \omega^S, \theta)\epsilon\eta \qquad (44)$$

The function (R^2) can be represented as $\psi: C(\eta) \to C(\eta)$ which continuous function and limit $K^1 > 0$. This function is mathematically presented as follows,

$$\left|\psi(\omega^1, \omega^2, \dots, \omega^S)\right| \leq K^1, \forall(\omega^1, \omega^2, \dots, \omega^S)\epsilon\eta \qquad (45)$$

The function (R^2) can be represented as $N: C(\eta) \to C(\eta)$ which continuous function and limit $K^2 > 0$. This function is mathematically presented as follows,

$$\left|N(\omega^1, \omega^2, \dots, \omega^S)\right| \leq K^1, \forall(\omega^1, \omega^2, \dots, \omega^S, \theta)\epsilon\eta \qquad (46)$$

The function (S^4), $-\infty D_{\grave{\omega_I}}^{\sigma}, M, -\infty D_{\ddot{\omega_I}}^{\sigma} M, \grave{\omega_I} D_{\infty}^{\sigma} M, \ddot{\omega_I} D_{\infty}^{\sigma} M$ are described as continuous function and non constant negatives are $G_{IP}, G_{IR}, I = 0,1,2, \dots, M$ which formulated as follows,

$$\left|-\infty D_{\grave{\omega_I}}^{\sigma} M\left(\grave{\omega_1}, \grave{\omega_2}, \dots, \grave{\omega_S}, \grave{\theta}\right) - -\infty D_{\grave{\omega_I}}^{\sigma} M\left(\ddot{\omega_1}, \ddot{\omega_2}, \dots, \ddot{\omega_S}, \grave{\theta}\right)\right| \leq \sum_{I=0}^{S} G_{IP} \left|\grave{\omega_I} - \ddot{\omega_I}\right| + R^1 \left|\grave{\theta} - \ddot{\theta}\right| \qquad (47)$$

$$\left|\grave{\omega_I} D_{\infty}^{\sigma} M\left(\grave{\omega_1}, \grave{\omega_2}, \dots, \grave{\omega_S}, \grave{\theta}\right) - \grave{\omega_I} D_{\infty}^{\sigma} M\left(\ddot{\omega_1}, \ddot{\omega_2}, \dots, \ddot{\omega_S}, \grave{\theta}\right)\right| \leq \sum_{I=0}^{S} G_{IR} \left|\grave{\omega_I} - \ddot{\omega_I}\right| + R^2 \left|\grave{\theta} - \ddot{\theta}\right| \qquad (48)$$

The function (S^5), $H^{I+}(\omega^I)$, $H^{I-}(\omega^I)$ can be considered as continuous function and the non negative constants are $G_{IP}, G_{IR}, I = 0,1,2, \dots, M$ which presented as follows,

$$\left|H^{I+}(\omega^I)\right| \leq B^{IP} \qquad (49)$$

$$\left|H^{I-}(\omega^I)\right| \leq B^{IR} \qquad (50)$$

This is the first solution which obtained by using Banach fixed point theorem.

Theorem 4:

Schaefer fixed point theorem

In this theorem, the hypotheses (R^1) is hold based on below condition,

$$\left(\Theta \times \sum_{I=1}^{S} (A^{IP} B^{IP} + A^{IR} B^{IR})\right) < 1 \qquad (51)$$

After that, the mentioned problem 32 and 33 have a specific solution related on $C(\eta, \Re)$.

Proof of theorem 4:

In this theorem, the specified problem is chaned into a fixed point issues. The operator can be denoted as $\Lambda: C(\eta, \Re) \to (\eta, \Re)$ which formulated as follows,

$$\begin{aligned} \Lambda M(\omega^1, \omega^2, \dots, \omega^S, \theta) &= \psi(\omega^1, \omega^2, \dots, \omega^S) \\ &+ M_\theta \sum_{I=0}^{M} \Big(H^{I+}(\omega^I)_{-\infty} D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) \\ &+ N(\omega^1, \omega^2, \dots, \omega^S, \theta)\Big) \qquad (52) \end{aligned}$$

The solution of the mentioned problem is denoted as Λ (fixed point operator)which is solved with the help of banach contraction problem. The banach contraction problem is proved the problem by fixed point analysis. Intially, the fixed point operator can be proved as contraction. Let, $M_1, M_2 \epsilon C(\eta, \Re)$. After that, each fraction set is denoted by $(\omega^1, \omega^2, \dots, \omega^S, \theta)\epsilon\eta$. The mathematial forulation of banach contraction is presented follows,

$$|\Lambda M_1 - \Lambda M_2| = \Bigg|\psi(\omega^1, \omega^2, \dots, \omega^S) + M_\theta \sum_{I=0}^{M} \Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + N(\omega^1, \omega^2, \dots, \omega^S, \theta)\Big) - \Bigg(\psi(\omega^1, \omega^2, \dots, \omega^S) + M_\theta \sum_{I=0}^{M} \Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma} M(\omega^1, \omega^2, \dots, \omega^S, \theta) + N(\omega^1, \omega^2, \dots, \omega^S, \theta)\Big)\Bigg)\Bigg| \quad (53)$$

$$= \Bigg|M_\theta \sum_{I=0}^{M} \Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma} \left(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\right) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma}\big(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\big)\Big)\Bigg|$$

$$\leq \quad M_\theta \sum_{I=0}^{M} \Big(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma} \left(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\right) + H^{I-}(\omega^I)_{\omega^I} D_{\infty}^{\sigma}\big(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\big)\Big)$$

$$\leq \theta \sum_{I=1}^{M} (|H^{I+}(\omega^I)| \ A^{IP} + |H^{I-}(\omega^I)|\big|\big(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\big)$$

$$\leq \Theta \ \sum_{I=1}^{M} (|H^{I+}(\omega^I)| \ A^{IP} + |H^{I-}(\omega^I)||sup\big(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\big)$$

$$\leq \Theta \sum_{I=1}^{M} (|H^{I+}(\omega^I)| \ A^{IP} + |H^{I-}(\omega^I)||\big(M_1(\omega^1, \omega^2, \dots, \omega^S, \theta) - M_2(\omega^1, \omega^2, \dots, \omega^S, \theta)\big)\infty \quad (54)$$

This is the second results of the specified problem which is achived with the help of Schaefer,s fixed point theorem.

Theorem 5: Arzela Ascoli theorem.

In this theorem, the problem is solved by utilizing the fixed-point theorem. Here, Assumption is $R^1 - R^4$ hold and the solution is denoted by $C(\eta, \Re)$. The fixed-point theorem is computed based on four steps which validated in this section.

Step 1:

In this step, the fixed-point operator is considered as continuous. Here, the sequences are considered as S_μ and it replaced with $M_\mu \to Sin\ C(\eta, \Re)$. After that, the sequence set is denoted as $(\omega^1, \omega^2, \dots, \omega^S, \theta)$

$$\left|\Lambda M_\mu - \Lambda M\right| = \left| M_\theta \sum_{I=0}^{M} \left(H^{I+}(\omega^I)_{-\infty} D^{\sigma}_{\omega^I} \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right) \right. \right.$$
$$\left. \left. + H^{I-}(\omega^I)_{\omega^I} D^{\sigma}_{\infty} \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right) \right) \right|$$
$$\leq \quad M_\theta \sum_{I=0}^{M} \left(H^{I+}(\omega^I)_{-\infty} D^{\sigma}_{\omega^I} \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right) \right.$$
$$\left. + H^{I-}(\omega^I)_{\omega^I} D^{\sigma}_{\infty} \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right) \right)$$
$$\leq \theta \sum_{I=1}^{M} (|H^{I+}(\omega^I)|\ A^{IP} + |H^{I-}(\omega^I)|| \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right)$$
$$\leq \Theta \ \sum_{I=1}^{M} (|H^{I+}(\omega^I)|\ A^{IP}$$
$$+ |H^{I-}(\omega^I)|| sup \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right)$$
$$\leq \Theta \sum_{I=1}^{M} (|H^{I+}(\omega^I)|\ A^{IP}$$
$$+ |H^{I-}(\omega^I)|| \left(M_\mu(\omega^1, \omega^2, \dots, \omega^S, \theta) - M(\omega^1, \omega^2, \dots, \omega^S, \theta) \right) \infty \quad (55)$$

Where, Λ can be considered as continuous function and $\mu \to \infty$ and $\left|\Lambda M_\mu - \Lambda M\right|\infty$.

Step 2: The bounded sets $C(\eta, \mathfrak{R})$. are divided from the map Λ

In this step, some of the assumption is considered which presented follows,

$$\left||M|\right|\infty = \{Sup|M|(\omega^1, \omega^2, \dots, \omega^S, \theta)\epsilon\psi, \left||M|\right|\infty \leq \epsilon\ ans\ \pi^\epsilon M\epsilon C(\eta, \mathfrak{R}), M\ \epsilon\ \pi^\epsilon \quad (56)$$

$$
\begin{aligned}
&\Lambda M(\omega^1,\omega^2,\dots,\omega^S,\theta)\\
&= \Bigg|\psi(\omega^1,\omega^2,\dots,\omega^S,\theta)M_\theta\sum_{I=0}^{M}\Big(H^{I+}(\omega^I)_{-\infty}D^{\sigma}_{\omega^I}\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)\\
&- M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\\
&+ H^{I-}(\omega^I)_{\omega^I}D^{\sigma}_{\infty}\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\Big)\Bigg|\\
&\le\ M_\theta\sum_{I=0}^{M}\Big(H^{I+}(\omega^I)_{-\infty}D^{\sigma}_{\omega^I}\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\\
&+ H^{I-}(\omega^I)_{\omega^I}D^{\sigma}_{\infty}\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\Big)\\
&\le \theta\sum_{I=1}^{M}(|H^{I+}(\omega^I)|\ A^{IP}\\
&+|H^{I-}(\omega^I)|\Bigg|\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\\
&\le\ K^1\sum_{I=1}^{M}(|H^{I+}(\omega^I)|\ A^{IP}+|H^{I-}(\omega^I)|\Bigg|\big(M(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\big)\\
&\le K^2\sum_{I=1}^{M}(|H^{I+}(\omega^I)|\ A^{IP}\\
&+|H^{I-}(\omega^I)||\Big(M_\mu(\omega^1,\omega^2,\dots,\omega^S,\theta)-M(\omega^1,\omega^2,\dots,\omega^S,\theta)\Big)\infty+K^2\\
&\le\ K^1\Theta\times\sum_{I=1}^{M}(H^{IP}B^{IP}+H^{IR}B^{IR})\big||M(\omega^1,\omega^2,\dots,\omega^S,\theta)|\big|\infty+K^2\Theta \quad (57)
\end{aligned}
$$

Therefore,

$$\Lambda M(\omega^1,\omega^2,\dots,\omega^S,\theta)\le K^1\Theta\times\sum_{I=1}^{M}(H^{IP}B^{IP}+H^{IR}B^{IR})\big||M(\omega^1,\omega^2,\dots,\omega^S,\theta)|\big|\infty+K^2\Theta=C \qquad (58)$$

Hence, the fraction diffusion operation can be divided into bounded sets in $C(\eta,\mathfrak{R})$.

Step 3:

The fraction diffusion operator maps can be sets into equicontinous sets of $C(\eta,\mathfrak{R})$. In this consider, $\theta^{`}<\theta^{``}$, $\omega_I^{`}<\omega_I^{``}$, $I=1,2,\dots,M, X\epsilon\pi_\epsilon,(\omega^1,\omega^2,\dots,\omega^S,\theta)\epsilon\psi$.

$$\left|\Lambda M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`}) - \Lambda M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right|$$
$$= \left| M_\theta \sum_{I=0}^{M} \left(H^{I+}(\omega_1^{`})_{-\infty}D_{\omega^I}^{\sigma} \left( M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`}) - M(\omega^1,\omega^2,\dots,\omega^S,\theta)\right)\right.\right.$$
$$\left.\left. + H^{I-}(\omega_1^{`})_{\omega^I}D_{\infty}^{\sigma} \left( M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`}) - M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`})\right)\right)\right|$$
$$\leq \quad M_\theta \sum_{I=0}^{M} \left(H^{I+}(\omega_1^{``})_{-\infty}D_{\omega^I}^{\sigma} \left( M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`}) - M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right)\right.$$
$$\left. + H^{I-}(\omega_1^{``})_{\omega^I}D_{\infty}^{\sigma} \left( M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`}) - M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right)\right)$$
$$\leq \theta \sum_{I=1}^{M} (|H^{I+}(\omega^I)|\ A^{IP} + \left|H^{I-}(\omega_1^{``})\right|| \left(\mathrm{M}(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`}) - M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`})\right)$$
$$\leq \Theta \ \sum_{I=1}^{M} (\left|H^{I+}(\omega_1^{``})\right|\ A^{IP}$$
$$+ \left|H^{I-}(\omega_1^{``})\right|| sup \left( M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`}) - M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right)$$
$$\leq \Theta \sum_{I=1}^{M} (\left|H^{I+}(\omega_1^{``})\right|\ A^{IP}$$
$$+ \left|H^{I-}(\omega_1^{``})\right|| \left( M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`}) - M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right) \infty \quad (59)$$

From equation, $\omega_I^{`} \to \omega_I^{``}, I = 1,2,\dots,M, \left|\left|\Lambda M(\omega_1^{`},\omega_2^{`},\dots,\omega_S^{`},\theta^{`}) - \Lambda M(\omega_1^{``},\omega_2^{``},\dots,\omega_S^{``},\theta^{`})\right|\right|\infty \to 0, \theta^{`} \to \theta^{``}$. The inequality of the parameters is considered as zero. The direct consideration of the Arzela Ascoli fixed point theorems conclude as $\Lambda: (\eta,\mathfrak{R}) \to C(\eta,\mathfrak{R})$ which is continuous function.

Step 4:

In this step, the theorem assumptions are presented follows,

$$X = \{M \in C(\eta,\mathfrak{R}); M =\in \Lambda(\mathrm{M}) for\ 0 < \epsilon < 1\} \quad (60)$$

Here, $M \in P$, then formulations are presented as follows,

$$|M(\omega^1,\omega^2,\dots,\omega^S,\theta)| = |\epsilon\Lambda\mathrm{M}|\psi(\omega^1,\omega^2,\dots,\omega^S,\theta)|$$
$$\leq \epsilon \times (\omega^1,\omega^2,\dots,\omega^S)$$
$$+ M_\theta \sum_{I=0}^{M} \left(H^{I+}(\omega^I)_{-\infty}D_{\omega^I}^{\sigma} M(\omega^1,\omega^2,\dots,\omega^S,\theta) + H^{I-}(\omega^I)_{\omega^I}D_{\infty}^{\sigma} M(\omega^1,\omega^2,\dots,\omega^S,\theta)\right.$$
$$\left. + N(\omega^1,\omega^2,\dots,\omega^S,\theta)\right)| \qquad (62)$$

From the equation 58, the bleow mathematical formulation is achieved,

$$\Lambda M(\omega^1,\omega^2,\dots,\omega^S,\theta) \leq \in \times K^1\Theta \times \sum_{I=1}^{M} (H^{IP}B^{IP} + H^{IR}B^{IR})\left||M(\omega^1,\omega^2,\dots,\omega^S,\theta)|\right|\infty + K^2\Theta$$
$$= C \qquad (63)$$

Therefore, $\left||M|\right|\infty < \infty$. Based on the formulation, P is bounded. From the consideration of schafer fixed point theorem, the M can be deduced and which is the results of the problem $C(\eta,\mathfrak{R})$.

6. CONCLUSION

In this paper, enhanced Adomian decomposition method has been developed to solve the considered problems with efficient solutions. Here, the fractional diffusion equations have been considered with variable coefficients to formulate the problems. The fractional integral or derivative functions are utilized with the consideration of caputo definition. This proposed approach is utilized to analysis the analytical solutions to compute the optimal solutions. Additionally, optimal solutions are obtained with the consideration of three fixed point theorems such as Arzela Ascoli theorem, Schaefer fixed point theorem and Banach fixed point theorem. The formulated problems are solved by using the enhanced Adomian decomposition method as well as fixed point theorems. In last, the numerical analysis of the considered problems with the solutions of the fixed-point theorems are presented. The proposed method is working based on iteration process to achieve the best solutions related with the considered problems. Finally, the optimal solutions of the considered problems are analyzed and validated with the proof analysis. In future, various fixed-point theorems with diffusion equations will be considered to solve the problems in different applications.

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