Research Article

t −pebbling property satisfies the $C_5 \times C_5 \times C_5$ Graph

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Abstract

A graph pebbling is mathematical game play on the vertices of graph with respect to the pebbling steps. The pebbling step is the shifting of two pebbles and attachment of one pebble on the specified arbitrary vertex. The t-pebbling number of the graph, $f_t(G)$ is the maximum of $f_v(G,v)$ over all the vertices of G where, $f(G,v)$ is the pebbling number of a vertex v. We propose the t −pebbling number of $C_5 \times C_5 \times C_5$, where $C_5 \times C_5 \times C_5$ graph is with 125 vertices.

Keywords: Pebbling Number, Cycle Graphs, Graham's Conjectures, t-pebbling property.

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1. Introduction

In literature, graph pebbling was introduced by F. R. K. Chung in 1989 [5]. Graph pebbling comes from the foundations of graph theory, number theory and combinatorial theory. Pebbles are represented by positive integers on the vertices of graph. Now, pebbling number is depicted as a new parameter which evaluates the graph on computer and widely used in the field of animations. For an example, consider the pebbles as fuel tankers or containers, then the loss of the pebble during a move is the cost of transportation and shipment of fuel takes place between the initial and final destination or vertices [7].

Consider a graph which is set of vertices and set of edges where incidence relationship is preserved. Pebbling step is the subtraction of two pebbles from an arbitrary vertex and addition of one pebble on its adjacent vertex. Graph pebbling is the collections of pebbling steps on the n number of vertices on graph with different configurations of pebbles which is known as $f(G)$ or $\pi(G)$ or the pebbling number of graphs G [6]. For an example,

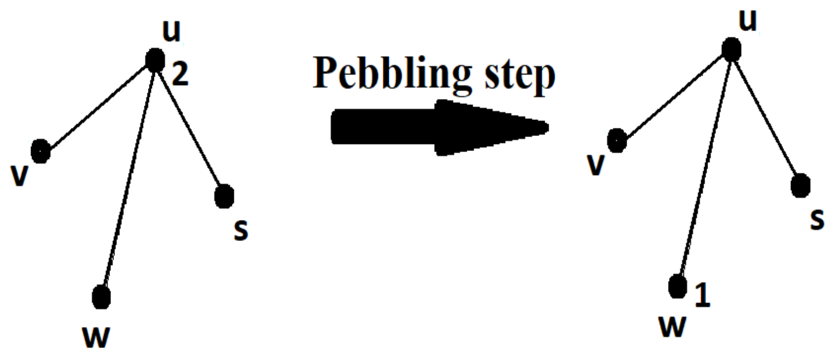

Figure 1. Pebbling step on graph G

The pebbling number of graphs is equal to number of vertices of graph is known as demonic. $p(v)$ represents the number of pebbles on vertex v.

Cartesian product of any two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is defined by direct product of $G \times H$ where vertex set and edges set are follows [8]:

$$V_{G\times H} = V_G \times V_H = \{(x_i, x_j): x_i \in V_G, x_j \in V_H\},$$

$E_{G\times H} = \left\{\left((x_i, x_j)(x_i', x_j')\right): x_i = x_i' \ and \ (x_j, x_j') \in E_H, or \ (x_i, x_i') \in E_G \ and \ x_j = x_j'\right\}$

1.1. t −Pebbling property:

In a graph G, the t-pebbling number of a vertex v in G is the smallest number $f_t(G, v)$ with the property that is from every distribution of pebbles in G, it is possible to move t pebbles to v by a finite number of pebbling moves [3, 4].

$f_t(G)$: The t-pebbling number of the graph G, is the maximum of $f_v(G, v)$ over all the vertices of G.

1.2. Two-pebbling property:

A graph G satisfies the two-pebbling property if we can sent to pebbles on any specific but arbitrary vertex of G from every placement [3, 4].

Cycle graph C_n on n vertices $\{x_1, x_2, \dots \dots \dots, x_n\}$ is a closed path where every vertex x_i is adjacent to x_{i+1} and x_{i-1} for $1 \leq i \leq n$ [2, 3, 4].

1.3. Preposition [4, 6, 8]:

The pebbling number of odd and even undirected cycles, for $k \geq 2$

$$f(C_{2k}) = 2^k \qquad and \qquad f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$$

In particular, the pebbling number of C_5 is that is $f(C_5) = 5$. It is demonic.

1.4. Preposition [4]:

The t-pebbling number of a cycle is given by

$$f_t(C_{2n}) = t.2^n \quad \text{and} \quad f_t(C_{2n+1}) = \frac{2^{n+2} - (-1)^n}{3} + (t-1).2^n$$

In particular, the two-pebbling number of C_5 is that is $f_2(C_5) = 9$.

2. Graham's Conjecture [2, 4, 8]

Graham's conjecture is stated from cartesian products of two graphs. It is true for any graphs, G and H, $f(G \times H) \leq f(G).f(H)$. Any graph which satisfies the two-pebbling property (also known as weak-two-pebbling property) also satisfy the weak-two-pebbling property. Trees, all graphs of diameter two satisfies 2 −pebbling property.

Also prove that graham's conjecture holds foe these case:

- When G is an even cycle and H satisfies the two-pebbling property.
- When G and H both are odd cycles and one of them has at least 15 vertices.

Some important measures of Graham's conjecture which holds on the graph families with the aspect of t-pebbling and 2-pebbling property:

- Suppose that G satisfies the two-pebbling property. Then $f\left(K_{m,n} \times G\right) \leq f\left(K_{m,n}\right).f(G)$ [13].
- $f(G \times K_t) \leq t.f(G)$

If $f(G \times K_t) = t.f(G)$, then $G \times K_t$ satisfies 2-pebbling property see [13].

- If $m, n \geq 5 \ \ and \ \ |\mathrm{n} - \mathrm{m}| \geq 2$, then see [8]

$$f\left(M(C_{2n}) \times M(C_{2m})\right) \leq f\left(M(C_{2n})\right).f\left(M(C_{2m})\right)$$

- If $n \geq 2$, then see [8]

$$f_t\left(M(C_{2n})\right) \leq t.2^{n+1} + 2n - 2$$

- $f(G \times T) \leq f(G).f(T)$, when G has 2-pebbling property and T is any Tree [9].
- Graham's conjecture holds when G and H are thorn graphs of the complete graphs with every $p_i > 1 \ \ (i = 1, 2, \dots \dots, n)$ [10].
- G satisfies the t-pebbling property. Then, $\alpha(G)$ is the α −pebbling number of graph G,

$\alpha\left(C_{p_j} \times \dots \dots \dots \times C_{p_2} \times C_{p_1} \times G\right) \leq \alpha\left(C_{p_j}\right) \dots \dots \dots \alpha(C_{p_2})\alpha(C_{p_1})\alpha(G)$ when none of the cycles is C_5 [4].

- $\pi(G \times H) \leq \pi(G).\pi(H)$ for any connected graphs G and H, where G and H have 2-pebbling property and shows that $\pi(L \times T) \leq \pi(L).\pi(T)$ and $\pi(L \times k_n) \leq \pi(L).\pi(k_n)$ where, T is any Tree, k_n is complete graph and L is Lemke graph [11].
- Herscovici conjectured that $\pi_{st}(G \times H) \leq \pi_s(G).\pi_t(H)$,

In particular, $\pi_{st}(T \times G) \leq \pi_s(T).\pi_t(G)$, $\pi_{st}(k_n \times G) \leq \pi_s(k_n).\pi_t(G)$ and $\pi_{st}(C_{2n} \times G) \leq \pi_s(C_{2n}).\pi_t(G)$ where, G has 2-pebbling property, T is Tree, k_n is complete graph and C_{2n} is cycle on $2n$ vertices [12].

2.1. Conjecture:

The t −pebbling property satisfies the Graham's conjectures that is

$$f_t(G \times H) \leq f_t(G).f_t(H)$$

Proof: Using the Herscovici conjecture, $\pi_{st}(G \times H) \leq \pi_s(G).\pi_t(H)$ see [4].

If the representation of Herscovici conjecture changes, if $s = t$ and $st = tt = t$ (using the Boolean algebra as $a^2 = a$), then

$$\pi_t(G \times H) \leq \pi_t(G).\pi_t(H)$$

2.2. Illustrative Example:

For an example, $G = \{a, b\}$ and $H = \{1, 2, 3\}$,

$$G \times H = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

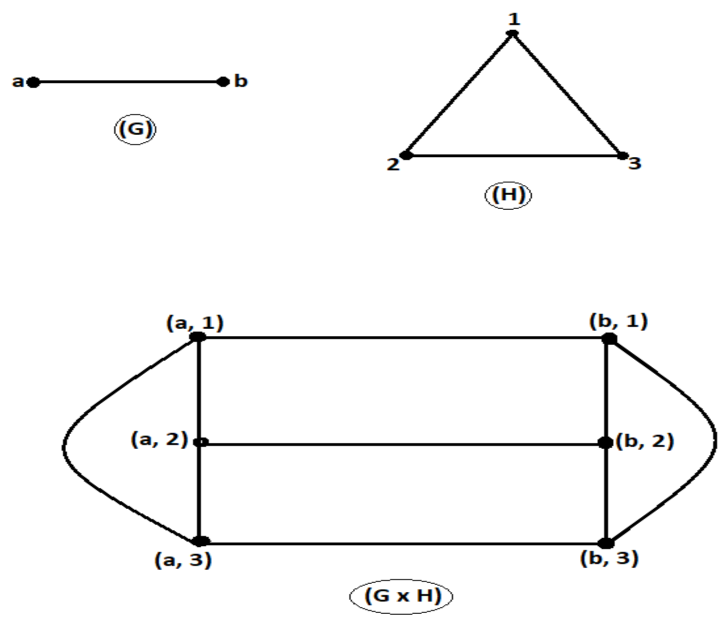

Figure 1. $G \times H$ Graph

Here, $G = P_2$ and $H = C_3$ where, G and H are different graphs.

If $f(G) = 2$ and $f(H) = 3$, then pebbling number of the $G \times H$ graph is

$$f(G \times H) \leq f(G).f(H) \leq 2.3 \leq 6$$
$$f(G \times H) = 6$$

If $f_2(G) = 4$ and $f_2(H) = 5$, then pebbling number of the $G \times H$ graph is

$$f_2(G \times H) \leq f_2(G).f_2(H) \leq 4.5 \leq 20$$
$$f_2(G \times H) = 9$$

If $f_3(G) = 6$ and $f_3(H) = 7$, then pebbling number of the $G \times H$ graph is

$$f_3(G \times H) \leq f_3(G).f_3(H) \leq 6.7 \leq 42$$
$$f_3(G \times H) = 13$$

If $f_t(G) = t.2^{n-1}$ and $f_t(H) = 2(t) + 1$, then pebbling number of the $G \times H$ graph is

$$f_t(G \times H) \leq f_t(G).f_t(H) \leq (t.2^{n-1}).(2(t) + 1)$$
$$\text{Or } f_t(P_2 \times C_3) \leq f_t(P_2).f_t(C_3) \leq (t.2^{n-1}).(2(t) + 1)$$

So, t-pebbling property hold's on the Graham's conjectures. Therefore, t-pebbling is applicable on Graham's conjecture for any connected G and H graphs.

If $G = H = C_3$ where, G and H are same graphs. Then,

$$f_t(C_3 \times C_3) \leq f_t(C_3).f_t(C_3) \leq (2(t) + 1).(2(t) + 1) = (2(t) + 1)^2$$

3. t-pebbling number of $C_5 \times C_5 \times C_5$

Here, determines the t-pebbling number of $C_5 \times C_5 \times C_5$. Let us assume the target vertex be $((x_3, x_3), x_3)$. Using the symmetrical approach on $C_5 \times C_5 \times C_5$. There are 25 columns and 5 rows in $C_5 \times C_5 \times C_5$ graph which is a regular graph of degree 28.

Let $V(C_5) = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$ and

$$V(C_5 \times C_5 \times C_5) = \{\left(\left(x_i, x_j\right), x_k\right) : 1 \le i, j, k \le 5\}$$

3.1. Preposition [3]:

The pebbling number of $C_5 \times C_5$ is 25 i.e. $f(C_5 \times C_5) = 25$. It is demonic.

3.2. Preposition [4]:

If $G = C_5 \times C_5$ by showing that $C_5 \times C_5$ satisfies Chung's 2-pebbling property and establishing bounds for $f_t(C_5 \times C_5)$.

$$f_t(C_5 \times C_5) = 16(t) + 9 \qquad \text{for } t \ge 1$$

3.3. Preposition [1]:

The pebbling number of $C_5 \times C_5 \times C_5$ is 125 i.e. $f(C_5 \times C_5 \times C_5) = 125$. It is demonic.

3.4. Theorem

The t-pebbling number of $C_5 \times C_5 \times C_5$ is less than or equal to

$$f_t(C_5 \times C_5 \times C_5) \le 64t^2 + 52t + 9 \qquad \text{for } t \ge 1$$

Proof: Using the Graham's conjectures on the cycle graph C_5 with five vertices that is $G = H = C_5$ and using the conjecture 2.1.

From preposition 1.4, the t-pebbling number of C_5 is $f_t(C_5) = 4t + 1$ where, C_5 satisfies the two-pebbling property see [3].

$$\begin{aligned} f_t(C_5) &= \frac{16 - (-1)^2}{3} + (t-1).\, 2^2 \\ &= 5 + 4(t-1) \\ &= 4 + 1 + 4(t-1) \\ &= 4(1 + t - 1) + 1 \\ &= 4t + 1 \qquad \ldots \ldots \ldots (1) \end{aligned}$$

From preposition 3.2, the t-pebbling number of $C_5 \times C_5$ is

$$f_t(C_5 \times C_5) = 16(t) + 9 \qquad \text{for } t \ge 1 \qquad \ldots \ldots \ldots (2)$$

where, $C_5 \times C_5$ satisfies the two-pebbling property.

Using the associative property, $(a \times b) \times c = a \times (b \times c)$.

Also, using the conjecture 2.1, from graham's conjecture where $G = C_5 \times C_5$ and $H = C_5$ or $G = C_5$ and $H = C_5 \times C_5$.

Then, from equation (1) and (2),

$$\begin{aligned} f_t((C_5 \times C_5) \times C_5) &\le f_t(C_5 \times C_5).\, f_t(C_5) \\ &\le (16(t) + 9).\,(4(t) + 1) \\ &\le 16.4.\,(t^2) + 16(t) + 9.4.\,(t) + 9 \\ &\le 64t^2 + 16(t) + 36(t) + 9 \\ &\le 64t^2 + 52t + 9 \end{aligned}$$

Or $f_t(C_5 \times (C_5 \times C_5)) \le 64t^2 + 52t + 9$

In particular, when $t = 1$, $f(C_5 \times (C_5 \times C_5)) \le 64(1)^2 + 52(1) + 9 = 125$.

4. Conclusion

We calculate the t-pebbling number of $C_5 \times C_5 \times C_5$. Cycle graphs record the all type of movements of path of an operator. Pebbling represent pebbles which are widely used in transportation of discrete items or objects, number theory and animations.

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