Research Article

A Review Study of New Numerical Methods for Solving Differential Equations with Impetus of Inter Disciplinary Applications with MATLAB

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Abstract

In this paper, we present a review study of new numerical methods to solve ordinary differential equations in both linear and non-linear cases with impetus of inter disciplinary applications with MATLAB. We use and apply Daftardar - Gejji technique on theta-method to derive a new family of numerical method in form of review study. It is shown that the method may be formulated in an equivalent way as a Runge-Kutta method. The Stability of the method is analyzed.

Keywords- Ordinary Differential Equations, Numerical Methods, Iterative Method.

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1. Introduction

Numerical methods are one of the main techniques used for solving differential equations. For many years, the construction and stable numerical methods for the solutions of ordinary differential equations (ODE) with initial value problems has been considered widely and with great new contributions. Recently, the method proposed by Daftardar – Gejji and Jafari (DJM) is powerful technique for solving a wide range of non-linear equations [1, 2]. In this paper, review studies employ the (DJM) to construct a new family of numerical scheme for solving ordinary differential equations and discuss error, stability and convergence of the proposed methods[3,4].

1.1 An Iterative Method

Consider the following general functional equation

$$u = f + N(u) \qquad \ldots\ldots\ldots 1$$

Where N is a non-linear operator from a Banach Space $B \to B$ and f is a known function.
u is assumed to be a solution of (1) having the series form

$$u = \sum_{i=0}^{\infty} u_i \qquad \ldots\ldots\ldots 2$$

The nonlinear operator N is decomposed as

$$N(u) = N(u_0) + [N(u_0 + u_1) - N(u_0)] + [N(u_0 + u_1 + u_2) - N(u_0 + u_1)] + \ldots\ldots \qquad \ldots 3$$

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \qquad \ldots\ldots 4$$

From equation (2) and (4), equation (1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \qquad \ldots\ldots 5$$

We define the recurrence relation

$$u_0 = f$$
$$u_1 = N(u_0)$$
$$u_{m+1} = N(u_0 + \ldots\ldots\ldots + u_m) - N(u_0 + \ldots\ldots\ldots + u_{m-1}), m = 1,2, \ldots\ldots\ldots$$

Then, $(u_0 + \dots\dots\dots.. + u_{m+1}) = N(u_0 + \dots\dots\dots.. + u_m), m = 1,2,\dots\dots\dots$
and $u = f + \sum_{i=1}^{\infty} u_i$.

1.2 Daftardar – Gejji and Jafari Method

Daftardar -Gejji and Jafari method (DJM) was first introduced by Daftardar -Gejji and Jafar in 2006, it has been proved that this method is a better technique for solving different kinds of non-linear equations [5, 6]. DJM has been used to create a new predictor – corrector method. DJM will be discussed, which successfully used to solve differential equations and non-linear equations in the form

$$u = f + L(u) + N(u) \qquad \dots\dots.6$$

Where L and N are linear and nonlinear operators respectively and f is given function . u is assumed to be a solution of equation (1) having the series form

$$u = \sum_{i=0}^{\infty} u_i \qquad \dots\dots.7$$

The nonlinear operator N is decomposed as

$$N(u) = N(u_0) + [N(u_0 + u_1) - N(u_0)] + [N(u_0 + u_1 + u_2) - N(u_0 + u_1)] + \dots\dots\dots\dots\dots$$

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]$$

Since L represents a linear operator

$$\sum_{i=0}^{\infty} L(u_i) = L\left(\sum_{i=0}^{\infty} u_i\right) \qquad \dots\dots.8$$

We define the recurrence relation

$$u_0 = f \qquad \dots\dots.9$$

$$u_1 = L(u_0) + N(u_0) \qquad \dots\dots.10$$

$$u_{m+1} = L(u_m) + N(u_0 + \dots\dots\dots.. + u_m) - N(u_0 + \dots\dots\dots.. + u_{m-1}), m = 1,2,\dots\dots\dots$$

$$u_{m+1} = L(u_m) + N(u_m) \qquad \dots\dots.11$$

We may write

$$\sum_{i=0}^{m+1} u_i = \sum_{i=0}^{m} L(u_i) + N\left(\sum_{i=0}^{m} u_i\right) \qquad \dots\dots.12$$

$$\sum_{i=0}^{m+1} u_i = L\sum_{i=0}^{m}(u_i) + N\sum_{i=0}^{m} u_i\,, m = 1,2,\dots\dots\dots\dots.$$

So that

$$\sum_{i=0}^{\infty} u_i = f + L\sum_{i=0}^{\infty} u_i + N\sum_{i=0}^{\infty} u_i \qquad \dots\dots.13$$

From the equation above, it is clear that $\sum_{i=0}^{\infty} u_i$ is the solution for equation (6).
The K-term series solution, which is given by

$$u_i = \sum_{i=0}^{k-1} u_i \qquad \dots\dots.14$$

represents an approximate solution for equation (13).

1.3 New Family of Numerical Methods

Consider the initial value problem

$$\frac{dy}{dx} = f(x,y) \qquad \dots\dots\dots 15$$

$$y^i(x_0) = \eta\,, i = 1,2,\dots\dots\dots..n$$

Where $y: [a,b] \to R^n, \eta \in R^n, f: [a,b] \times R^n \to R^n$
To obtain the numerical solution of initial value problem (15), we take partition of the interval $[a,b]$ as

$$a = x_0 < x_1 < x_2 < \dots\dots\dots\dots < x_n = b$$

These points are called the mesh points.
A sufficiently small spacing between the points is given by

$$h = x_j - x_{j-1}, j = 1,2, \dots\dots\dots n \qquad \dots\dots\dots 16$$

Which is called the step length. we also have

$$x_j = x_0 + jh\ , j = 1,2, \dots\dots\dots n. \qquad \dots\dots\dots .17$$

If y_j is an approximation to (x_j) , then implicit trapezium formula is given by

$$y_{j+1} = y_j + \frac{h}{2}\left[f\left(x_j, y_j\right) + f(x_{j+1}, y_{j+1})\right] \dots\dots\dots .18$$

Equation (18) is of the form (6), where

$$u = y_{j+1}$$
$$f = y_j + \frac{h}{2} f\left(x_j, y_j\right)$$
$$N(u) = \frac{h}{2} f\left(x_{j+1}, y_{j+1}\right)$$

Applying DJM on equation (18), we obtain 3-term solution as

$$u = u_0 + u_1 + u_2$$
$$u = u_0 + N(u_0) + [N(u_0 + u_1) - N(u_0)]$$
$$u = u_0 + N(u_0 + u_1)$$
$$u = u_0 + N\left(u_0 + N(u_0)\right)$$

That is

$$y_{j+1} = y_j + \frac{h}{2}f\left(x_j, y_j\right) + N(y_j + \frac{h}{2} f\left(x_j, y_j\right) + N\left(y_j + \frac{h}{2} f\left(x_j, y_j\right)\right),$$
$$j = 0,1,2, \dots \dots. \qquad \dots\dots..19$$

$$y_{j+1} = y_j + \frac{h}{2}f\left(x_j, y_j\right) + \frac{h}{2} f\left(x_{j+1}, y_{j+1} + \frac{h}{2}f\left(x_j, y_j\right)\right) + \frac{h}{2}f\left(x_{j+1}, y_{j+1} + \frac{h}{2}f\left(x_j, y_j\right)\right),$$
$$j = 0,1,2, \dots \dots. \qquad \dots\dots.20$$

if we set

$$k_1 = f\left(x_j, y_j\right) \qquad \dots\dots.21$$
$$k_2 = f\left(x_{j+1}, y_j + \frac{h}{2}k_1\right) \qquad \dots\dots.22$$
$$k_3 = f\left(x_{j+1}, y_j + \frac{h}{2}k_1 + \frac{h}{2}k_2\right) \qquad \dots\dots.23$$

then equation (20) becomes

$$y_{j+1} = y_j + \frac{h}{2}k_1 + \frac{h}{2}k_3 \qquad \dots\dots.24$$

1.4 Non Runge-Kutta Method

If we write NNM as

$$y_{j+1} = y_j + h(\ b_1k_1, b_2k_2, b_3k_3) \qquad \dots\dots.25$$

$$K_i = f(x_j + c_ih, y_j + h(\ a_{i1}k_1, a_{i2}k_2, a_{i3}k_3)), i = 1,2,3 \qquad \dots\dots.26$$

Then

$$b_1 = \frac{1}{2}, b_2 = 0, b_3 = \frac{1}{2}\ ;\ c_1 = 0, \quad c_2 = c_3 = 1$$
$$a_{11} = a_{12} = a_{13} = 0\ , a_{21} = \frac{1}{2}\ , a_{22} = a_{23} = 0\ , a_{31} = \frac{1}{2}\ , a_{32} = \frac{1}{2}\ , a_{33} = 0$$

Thus the table for the NNM is

$$0 \ \big|\ 0 \quad 0 \quad 0$$

$$\begin{array}{c|ccc} 1 & 1/2 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 1/2 & 0 & 1/2 \end{array}$$

for Runge-Kutta method, it is necessary

$$\sum_{j=1}^{3} a_{ij} = c_i^{[5]}$$

from the above table $a_{21} + a_{22} + a_{23} = 1/2 \neq c_2$. This shows that the NNM is different from Runge- Kutte method. Now let us consider the famous family of methods, called by θ methods which has the following formula

$$y_{j+1} = y_j + h\left[\theta f(x_j, y_j) + (1+\theta) f(x_{j+1}, y_{j+1})\right], \theta = \in [0,1] \qquad \ldots\ldots\ldots 27$$

Where

$$h = x_j - x_{j-1}$$

and

$$x_j = x_0 + jh, \qquad j = 1,2,\ldots\ldots n$$

We can take different value of θ in formula (27) to generate many of methods

For example

$\theta = 1$, Explicit Euler method

$\theta = 1/2$, Implicit Trapezoidal method

$\theta = 0$, Implicit Euler method

We can write formula (27) as the form of (6) by consider

$$u = y_{j+1}$$
$$f = y_j + h\theta(x_j, y_j)$$
$$N(u) = h(1-\theta) f(x_{j+1}, y_{j+1})$$

Now let us apply [DJM] on equation (27) to get 3-term solution as

$$u = u_0 + u_1 + u_2$$
$$u = u_0 + N(u_0) + N(u_0 + u_1) - N(u_0)$$
$$u = u_0 + N(u_0 + u_1)$$
$$u = u_0 + N(u_0 + N(u_0))$$

Which is

$$y_{j+1} = y_j + h\theta f(x_j, y_j) + N\left(y_j + h\theta f(x_j, y_j)\right) + N\left(y_j + h\theta f(x_j, y_j)\right), \quad j = 0,1,\ldots. \qquad \ldots\ldots 28$$

$$y_{j+1} = y_j + h\theta f(x_j, y_j) + h(1-\theta) f\left(x_{j+1}, y_j + h\theta f(x_j, y_j)\right) + h(1-\theta) f\left(x_{j+1}, y_j + h\theta f(x_j, y_j)\right) \qquad \ldots\ldots 29$$

Therefore, we obtain a new family of θ methods. However, the new family can be formulated in an equivalent way as Runge-Kutta method as follow

$$\left.\begin{array}{l} k_1 = f(x_j, y_j) \\ k_2 = f(x_{j+1}, y_j + h\theta k_1) \\ k_3 = f(x_{j+1}, y_j + h\theta k_1 + h(1-\theta) k_2) \end{array}\right\} \qquad \ldots\ldots\ldots 30$$

Where

$$y_{j+1} = y_j + h\theta k_1 + h(1-\theta) k_3 \qquad \ldots\ldots 31$$

Now, to obtain some examples for the new family we choose some different values of θ in equation (30) as follow

For $= 0$, we get

$$\left.\begin{array}{l} k_1 = f(x_j, y_j) \\ k_2 = f(x_{j+1}, y_j) \\ k_3 = f(x_{j+1}, y_j + hk_2) \end{array}\right\} \qquad \ldots\ldots\ldots 32$$

Where $y_{j+1} = y_j + hk_3$

For $= \frac{1}{2}$, we get

$$\left.\begin{aligned} k_1 &= f(x_j, y_j) \\ k_2 &= f\left(x_{j+1}, y_j + \tfrac{h}{2}k_1\right) \\ k_3 &= f(x_{j+1}, y_j + \frac{h}{2}k_1 + \frac{h}{2}k_2) \end{aligned}\right\} \quad \dots\dots\dots 33$$

Where $y_{j+1} = y_j + \frac{h}{2}k_1 + \frac{h}{2}k_3$

For $= \frac{3}{4}$, we get

$$\left.\begin{aligned} k_1 &= f(x_j, y_j) \\ k_2 &= f\left(x_{j+1}, y_j + \tfrac{3h}{4}k_1\right) \\ k_3 &= f(x_{j+1}, y_j + \frac{3h}{4}k_1 + \frac{h}{4}k_2) \end{aligned}\right\} \quad \dots\dots\dots 34$$

Where $y_{j+1} = y_j + \frac{3h}{4}k_1 + \frac{h}{4}k_3$

For $= 1$, we get

$$\left.\begin{aligned} k_1 &= f(x_j, y_j) \\ k_2 &= f\left(x_{j+1}, y_j + hk_1\right) \\ k_3 &= f(x_{j+1}, y_j + hk_1) \end{aligned}\right\} \quad \dots\dots\dots 35$$

Where $y_{j+1} = y_j + h\theta k_1$

1.5 Theorem: The new family defined by (30) and (32) are second order if $\theta = \frac{1}{2}$ and first order for any another choice of θ.

Proof: The Taylor series expansion of y_{j+1} may be written as

$$y(x_{j+1}) = y_j + hf + \frac{1}{2}h^2 f f_y + \frac{1}{6}h^3\left(f f_y^2 + f^2 f_{yy}\right) + O(h^2) \quad \dots\dots\dots 36$$

Notice that for simplicity of the algebra f have been considered as a function of y only, without loss of generality. This will considerably reduce the Taylor series expansions of $k_i, i = 1,2,3$ in (30) to the following.

$$\left.\begin{aligned} k_1 &= f \\ k_2 &= f + \theta h f f_y + \frac{1}{2}\theta^2 h^2 f^2 f_{yy} + \frac{1}{6}\theta^3 h^3 f^3 f_{yyy} + \quad \dots\dots\dots. \\ k_3 &= f + f_y(\theta f h + (1-\theta)h\left(f + \theta f h f_y + \tfrac{1}{2}\theta^2 h^2 f^2 f_{yy} + \tfrac{1}{6}\theta^3 h^3 f^3 f_{yyy}\right))^2 + \quad .... \end{aligned}\right\} \quad \dots 37$$

Traditionally, the equation (37) would be substituted in (32) to obtain expression of y_{j+1}. Since the error of the method can be measured using the expression

$$T_{j+1} = y(x_{j+1}) - y_{j+1}$$

Therefore ,

$$T_{j+1} = \left(\theta - \frac{1}{2}\right) f h^2 f_y + \left(\frac{1}{6} - \theta + 2\theta^2 - \theta^3\right) f h^3 f_y^2 + \left(\frac{\theta}{2} - \frac{1}{3}\right) f^2 h^3 f_{yy} + \quad \dots\dots\dots. \quad \dots\dots\dots 38$$

Clearly, by choosing $\theta = \frac{1}{2}$ we get

$$T_{j+1} = \frac{1}{24} f h^3 f_y^2 - \frac{1}{12} f^2 h^3 f_{yy} + O(h^4) \quad \dots\dots\dots 39$$

Which is mean the method is second order, otherwise its first order.

Definition – A scheme is said to be consistent if the difference of the computation formula exactly approximates the differential equation it tends to solve.

1.6 Theorem: The new family of modified θ method is consistent.

Proof: Subtract y_j on the both side of (31) and we have

$$y_{j+1} - y_j = h\,(\theta k_1 + (1-\theta)k_3) \quad \dots\dots..40$$

Dividing all through by h and taking limit as h tend to zero on both sides, we have

$$\lim_{h\to 0}\frac{y_{j+1}-y_j}{h} = \lim_{h\to 0}(\theta k_1 + (1-\theta)k_3) = f(x_j, y_j) \qquad \ldots\ldots.41$$

Hence the method is consistent.

1.7 The stability function for the new modification methods

In order to validate the stability of the method, the equation (30) and (32) are substituted in the simple test equation.

$$y' = \lambda y, \lambda \epsilon C, Re(\lambda) < 0 \qquad \ldots\ldots.42$$

We get

$$k_1 = \lambda y_j$$
$$k_2 = \lambda y_j(1+\theta\lambda h)$$
$$k_3 = \lambda y_j\big(1+\theta\lambda h + \lambda h(1-\theta)(1+\theta\lambda h)\big) \qquad \ldots\ldots..43$$

substituting (43) in (32) and letting $z = h\lambda$, the simplified equation is obtained as follows

$$y_{j+1} = y_j(1+z+z^2-\theta z^2-\theta z^3+2\theta^2 z^3-\theta^3 z^3) \qquad \ldots\ldots..44$$

or in more simplified form

$$y_{j+1} = y_j R(z)$$
$$R(z) = (1+z+z^2-\theta z^2-\theta z^3+2\theta^2 z^3-\theta^3 z^3)$$

2. Illustration and MATLAB Code

We consider an example. To find the approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x$ and $y = 1$ when $x = 0$.

2.1 Runge-Kutta Fourth order method

$k_1 = 0$
$k_2 = 0.24$
$k_3 = 0.244$
$k_4 = 0.2888$
solution is $y = 1.2428$

2.2 New family of Theta Method

(i) $\theta = 0$
$k_1 = 1$
$k_2 = 1.2$
$k_3 = 1.44$
solution is $y = 1.288$

(ii) $\theta = \frac{1}{2}$
$k_1 = 1$
$k_2 = 1.3$
$k_3 = 1.43$
solution is $y = 1.2430$

(iii) $\theta = \frac{3}{4}$
$k_1 = 1$
$k_2 = 1.35$
$k_3 = 1.4175$
solution is $y = 1.220875$

(iv) $\theta = 1$
$k_1 = 1$

$k_2 = 1.4$
$k_3 = 1.44$
solution is $y = 1.2$

2.3 MATLAB Code

```
function dy = myfunRK(t,y)
 dy = t+y;
 end
```

Runge-Kutta fourth order Method

```
% solve ODE -IVP using rk4 standard method
% y' = t+y
% y(0) = 1
t0 = 0;
y0 = 1;
tEnd = 2;
h = 0.2;
N = (tEnd-t0)/h;
 %% initializing solution
T = [t0:h:tEnd]';
Y = zeros(N+1,1);
Y(1) = y0;

% solving using rk4 method
for i = 1:N
    k1 = myfunRK(T(i),Y(i));
    k2 = myfunRK(T(i)+h/2,Y(i)+h*k1/2);
    k3 = myfunRK(T(i)+h/2,Y(i)+h*k2/2);
    k4 = myfunRK(T(i)+h,Y(i)+h*k3);
    Y(i+1) = Y(i) + h/6*(k1+2*k2+2*k3+k4);
end

%% plot results and obtain errors
plot(T,Y);
Ytrue = exp(-T.^2);
err = abs(Ytrue-Y);
```

Theta Method

```
% solve ODE -IVP using theta method
% y' = t+y
% y(0) = 1
theta=0.5;
t0 = 0;
y0 = 1;
tEnd = 2;
h = 0.2;
N = (tEnd-t0)/h;
 %% initializing solution
T = [t0:h:tEnd]';
Y = zeros(N+1,1);
Y(1) = y0;
 % solving using rk4 method
for i = 1:N
    k1 = myfunRK(T(i),Y(i));
```

```
   k2 = myfunRK(T(i)+h,Y(i)+h*theta*k1);
   k3 = myfunRK(T(i)+h,Y(i)+h*theta*k1+h*(1-theta)*k2);
   Y(i+1) = Y(i) + h*theta*k1+h*(1-theta)*k3;
end
 %% plot results and obtain errors
plot(T,Y);
Ytrue = exp(-T.^2);
err = abs(Ytrue-Y);
```

3. Applications and Concluding Remarks

Differential equations are a fundamental ingredient for mathematical modelling of almost all modern science and technology phenomena. They are persuaded by problems which arise in diverse fields such as artificial intelligence, engineering, earth sciences, economics, biology, bioinformatics, fluid dynamics, physics, differential geometry, control theory, materials science, and quantum mechanics. In order to see the nature of the background of these phenomena, we have to solve differential equations. To solve these natural phenomena the nonlinear differential equation are practically very important. With the advancements in the science and technology, a number of phenomenon could not be well approximated by the classical differential equations [1, 2]. To reach the approximate the exact solution of such nonlinear phenomenon needs modifications e.g. Linearization method, decomposition method, homotopy perturbation method in the available methods. Theta method are widely used for solving initial value ODE and PDE [1, 4]. The theta method is appropriately used in image processing, to forecast the spread of pandemic, to explain natural phenomenon like recurrence of ice age, analysis of ECG, propagation of blood pressure in blood vessels or distribution of drugs in blood etc.

In this paper we present a review compare study of new numerical methods to solve ordinary differential equations in both linear and non-linear cases with impetus of inter disciplinary applications with MATLAB. In this paper, the review study about new family of numerical methods has been successfully examined, analyzed the order consistency and the stability for the new family.

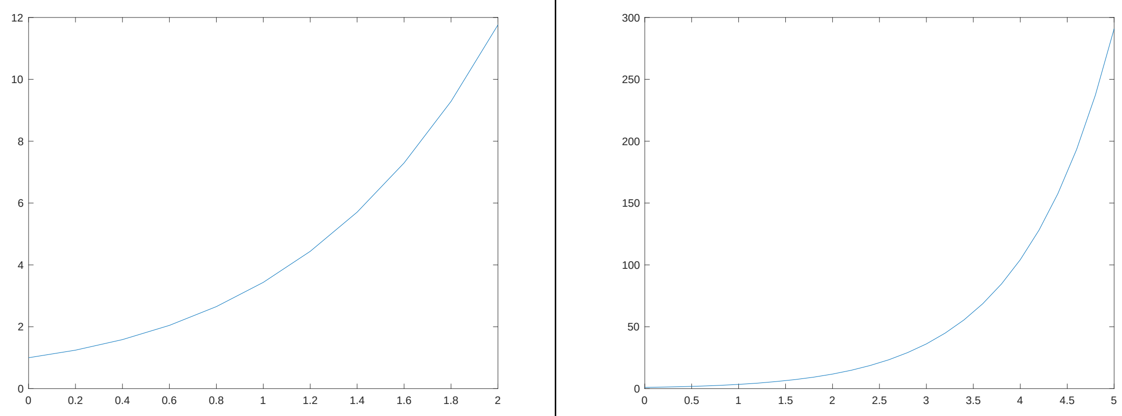

Fig1. Numerical result plot (a) Runge-Kutta fourth order (b) Theta method

References:

1. Barclay, G. J., Griffiths, D. F., & Higham, D. J. (2000). Theta method dynamics. *LMS Journal of Computation and Mathematics, 3, 27-43*.
2. Li, W., & Pang, Y. (2020). Application of Adomian decomposition method to nonlinear systems. *Advances in Difference Equations, 2020(1), 1-17.*
3. Ababneh, O. Y. (2019). New numerical methods for solving differential equations. *J. Adv. Math., 16, 8384-8390.*

4. Patade, J., & Bhalekar, S. (2015). A new numerical method based on Daftardar-Gejji and Jafari technique for solving differential equations. *World J Modell Simul, 11(4), 256-271*.
5. Bhalekar, S., & Daftardar-Gejji, V. (2011). Convergence of the new iterative method. *International Journal of Differential Equations, 2011*.
6. Daftardar-Gejji, V., & Jafari, H. (2006). An iterative method for solving nonlinear functional equations. *Journal of Mathematical Analysis and Applications, 316(2), 753-763*.