T-ABSO and strongly T-ABSO Fuzzy second submodules and Related Concepts

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Abstract: In this search, we present the concepts T-ABSO fuzzy second submodules and strongly T-ABSO fuzzy second submodules, as well as some basic properties and characterizations of these concepts under the categories of multiplication fuzzy modules, cocyclic fuzzy modules, and comultiplication fuzzy modules. We also address the relationship among T-ABSO fuzzy second submodules, strongly T-ABSO fuzzy second submodules, and quasi T-ABSO fuzzy second submodules. Also, we study these concepts with other related fuzzy submodules.

Keywords: T-ABSO Fuzzy Second Submodules, Strongly T-ABSO Fuzzy Second Submodules, Quasi T-ABSO Fuzzy Second Submodules, Cocyclic Fuzzy Modules and Comultiplication Fuzzy Modules.

1. Introduction

In this study, M is a unitary R-module, and R is a commutative ring with identity. Zadeh [1], proposed the concept of fuzzy (in short F.) sets in 1965. Rosenfeld introduced the concept of F. groups in 1971, [2]. Deniz S. et al. presented the concept of a 2-absorbing F. ideal in [3]. which is a generalization of prime F. ideal. Rabi [4] introduced the definition of the prime F. submodule (in short F. submd.). Hatam first proposed the definition of quasi-prime F.submd. in 2001 [5]. Wafaa investigated and introduced the T-ABSO F. submds definition in 2019, [6]. H.Ansari Toroghy introduced the dual notion of F.prime (that is,F. second) submds in the year 2019, [7].

There are two sections to this paper. Section (1) investigates and presents the definition of T-ABSO F.second submd. and the properties that are required, as well as some propositions, theorems, and examples. In section (2), we look at the concepts of strongly T-ABSO F.second submd., and relationship its concept with T-ABSO F.second submd., and quasi T-ABSO F.second submd.

2. Concepts Basic

Definition 2.1: Let *S* be a non-empty set and L be an interval [0,1] of the real line (real number). A F. set *A* in *S* (F. subset of *S*) is a function from *S* into L, [1].

Definition 2.2: Let x_u: *S*→ L be a F. set in *S*, where $x \in S, u \in L$, define by
$x_u(y) = \begin{cases} u \ if \ x = y \\ 0 \ if \ x \neq y \end{cases}$, x_u is called F. singleton in *S*, [8].
If *x* =0 and *u*=1, then $0_1(y) = \begin{cases} 1 \ if \ y = 0 \\ 0 \ if \ y \neq 0 \end{cases}$, [9].

Definition 2.3: A F. subset *K* of a ring R is called F. ideal of R, if ∀ *x*, *y* ∈ R:

1. $K(x\text{-}y) \geq \min\{K(x), K(y)\}$
2. $K(xy) \geq \max\{K(x), K(y)\}$, [10].

Definition 2.4: Let *M* be an R-module (in short mod.). F. set *Y* of *M* is called F. md. of an R-md *M* if

1. $Y(x\text{-}y) \geq \min\{Y(x), Y(y)\}$, for all $x, y \in M$.
2. $Y(rx) \geq Y(x)$, for all $x \in M$, $r \in R$
3. $X(0)=1$ (0 is the zero element of M), [10].

Definition 2.5: Let *Y* and *A* be two F. mds of an R-md. *M*. *A* is called F. submd. of *Y* if $A \subseteq Y$, [11].

Proposition 2.6: Let *A* be F. set of *M*. Then the level subset A_u, ∀*u*∈L is a submd. of *M* iff *A* is F. submd. of F. md. of an R-md. *M*, [12].

Definition 2.7: Let *A* and *B* be two F. submds of F. md. *Y*. The residual quotient of *A* and *B* denoted by (*A*:*B*) is the F. subset of R defined by:

$(A:_R B)(r) = \sup\{v \in L: r_v . B \subseteq A\}$ for all $r \in R$.That $(A:_R B) = \{r_v: r_v . B \subseteq A\, ; r_v$ is a F. singleton of $R\}$. If B=<x_k>, then $(A:_R$<x_k>$)=\{r_v: r_v . x_k \subseteq A\, ; r_v$ is F. singleton of R }, [10]".

Definition 2.8: Let A be a proper F. submd. of F. md. Y. The F. annihilator of A denoted by *F-annA* is defined by:

(*F-annA*)(r)=sup{v: v ∈L,r_vA ⊆ 0_1}, for all $r \in R$, [11]".

Note that: $\mathrm{F}-\mathrm{ann}A = (0_1:_R A)$, hence $(F - annY)_v \subseteq annY_v$, [5]".

Proposition 2.9: If Y is F. md. of an R-md. M, then *F-annY* is F. ideal of R, [11]".

Definition 2.10: A F. ideal Ĥ of a ring R is called prime F. ideal if Ĥ is a non-empty and for all a_s, b_l F. singletons of R such that $a_s b_l \subseteq$ Ĥ implies that either $a_s \subseteq$ Ĥ or $b_l \subseteq$ Ĥ, ∀s,l ∈ L, [13].

Definition 2.11: Let Ĥ be a non-empty F. ideal of R. Then Ĥ is called 2-absorbing F. ideal if for any F. singletons a_s, b_l, r_k of R such that $a_s b_l\, r_k \subseteq$ Ĥ implies that either $a_s b_l \subseteq$ Ĥ or $a_s\, r_k \subseteq$ Ĥ or $b_l\, r_k \subseteq$ Ĥ, [3].

Definition 2.12: A F. md. Y of an R-md. M is called a multiplication F. md. if for each non-empty F. submd. A of Y there exists a F. ideal Ĥ of R such that A=ĤY,[5]

Definition 2.13: Let Y be F. md. of an R-md.M, let A≠ 0_1 is called F.second submd. if ∀ r ∈ R we have 1_r.A=A or 1_r.A=0_1 where 1_r is F.ideal of R, [7].

Definition 2.14: F.md.Y of an R-md. M is called a comultiplication F.md. if A= F-$ann_Y F - ann_R A$ for each F.submd. A of Y [6].

3. T-ABSO F. Second Submds.

In this section, we will provide some definitions, remarks, examples, theorems, and propositions.

Definition 3. 1: Let Y be F.md.of an R-md.M. A proper submd. A of Y is said to be completely irreducible (in short irred.) F.submd.if A =$\bigcap_{i\in I} A_i$,where $\{A_i\}_{i\in I}$ is a family of F.submds.of Y, implies that A=$A_i$for some i∈I. It is easy to see that every F.submd.of Y is an intersection of completely irred.F.submd.of Y.

Theorem 3.2: For F.submd. A of Y F.md. of an R-md.M the following statements are equivalent:

a) A is F. second submd. of Y.
b) A ≠ 0_1 and 1_r.A⊆ K, where r ∈R and K is F. submd.of Y denotes either 1_r.A=0_1 or A⊆ K.
c) A≠ 0_1 and 1_r.A⊆ H, where r∈ R and H is a completely irred. F. submd.of Y implies either 1_r.A=0_1 or A⊆ H

Proof

a) ⟹(b) A is F. second submd. then 1_r.A=A or 1_r.A=0_1, ∀ r∈ R, hence1_r.A⊆ K since A⊆ K
b) ⟹ (c) Every F. submd. of Y is an intersection of completely irred. F. submd. of Y. A is F. second submd. Then 1_r.A=A, 1_r.A ⊆ H → A ⊆ H.
c) (c)⟹ (a) Suppose that r∈ R and 1_r.A≠ 0_1. If 1_r.A ⊆ H for some completely irred. F. submd. H of Y by assumption A ⊆ H. Hence 1_r.A ⊆ A.

Definition 3.3: Let A ≠ 0_1, A be called Prime (Strongly prime) F. second submd. if F.singleton a_s of R and B be completely irred. F. submd (B be F.submd.). a_s A⊆ B, then A⊆B or a_s ⊆ F-ann (A).

Definition 3.4: Let A≠ 0_1 be F.submd. of F.md of Y of an R-md.M. A is called T- ABSO F. second submd. if whenever F.singletons a_s, b_l of R, B is completely irred.

f.submd. and a_s b_l A ⊆ B then either a_s A⊆ B or b_l A ⊆ B or a_s b_l ⊆ F-ann (A).

Definition 3.5: Let $A \neq 0_1$ be F.submd. of F.md of Y of an R-md.M. A is called quasi-prime (strongly quasi-prime) F.second submd. if whenever F.singletons a_s, b_l of R, B is completely irred. F. submd.(B is F. submd.) and $a_s\, b_l\, A \subseteq B$ then either $a_s\, A \subseteq B$ or $b_l\, A \subseteq B$.

The following proposition specicates of T-ABSO F. second submd. in terms of its level subm.

Proposition 3.6: Let $A \neq 0$ be F.submd. of F.md of Y of an R-md.M. Then A is T-ABSO F. second submd. iff the level submd. $A_u, A_u \neq 0$ is T- ABSO F. second submd.of Y_u for all $u \in L$.

Proof: $\Longrightarrow$) Let ab $A_u \subseteq B_u$ for every a, b$\in$ R and $A_u \neq 0$ be submd. of Y_u, B_u be completely irred. submd. of Y_u we have a b y $\in B_u$ for all y $\in A_u$, then B(aby)$\geq$ u.So $(aby)_u \subseteq$ B,implies that $a_s b_k y_l \subseteq B, \forall\, y_l \in A$ where u= min{s,k,l}, hence $a_s b_k\, A \subseteq B$.

Since A is T-ABSO F. second submd. then either $a_s\, A \subseteq B$ or $b_k\, A \subseteq B$ or $a_s\, b_k \subseteq$ F-ann (A). Hence $a_s y_l \subseteq B$ or $b_k y_l \subseteq B$ or $a_s b_k \subseteq$ F-ann $((y_l))$ so that $(ay)_u \subseteq B$ or $(by)_u \subseteq B$ or$(ab)_u \subseteq$ F-ann(y_l). Thus either ay $\in B_u$ or by $\in B_u$ or ab $\in$ ann ((y)), $\forall$ y $\in A_u$ so a $A_u \subseteq B_u$ or b $A_u \subseteq B_u$ or ab $\in$ ann(A_u). Therefore A_u is T- ABSO second submd.of Y_u.

$\Longleftarrow$)Let $a_s b_k\, A \subseteq B$ for all F.singletons a_s, b_k of R and B be completely irred. F. submd. of Y. Subsequently $a_s\, b_k\, y_l \subseteq B, \forall\, y_l \in A$,so $(aby)_u \subseteq B$ where u = min {s,k,l}, hence B(aby)$\geq$ u,then a b y $\in B_u$, $\forall$ y $\in A_u$ indicates ab $A_u \subseteq B_u$,but A_u is T- ABSO second submd.of Y_u,so that either a $A_u \subseteq B_u$ or b $A_u \subseteq B_u$ or ab $\in$ ann(A_u) subsequently ay $\in B_u$ or by$\in B_u$ or ab $\in$ ann((y)), $\forall$ y $\in A_u$ hence either $(ay)_u \subseteq B$ or $(by)_u \subseteq B$ or$(ab)_u \subseteq$ F-ann$((y_l))$ so either $a_s\, A \subseteq B$ or $b_k\, A \subseteq B$ or $a_s\, b_k \subseteq$ F-ann (A). Thus A is T- ABSO F. second submd.of Y.

Remarks and Examples 3.7

1. Every prime F. second submd. is T-ABSO F.second submd.

Proof: Let A be prime F. second submd. of Y F.md of an R-md M, let $a_s\, b_l\, A \subseteq B$ where a_s, b_l are F.singletons of R, B is completely irred. F.submd. $a_s(b_l\, A) \subseteq B$, but A is prime F.second submd. hence $b_l\, A \subseteq B$ or $a_s \subseteq$ F-ann (A), so that A is T-ABSO F. second submd. But, the converse incorrect in general for example:

Let $Y\colon Z_6 \to L$ where $Y(y) = \begin{cases} 1 \text{ if } y \in Z_6 \\ 0 \ \text{o.w.} \end{cases}$

It is evident Y F.md. of Z_6 as Z-md.

Let $A\colon Z_6 \to L$ where $A(y) = \begin{cases} u \text{ if } y \in Z_6 \\ 0 \ \text{o.w.} \end{cases}$

It is evident A is F.submd. of Y.

Now, $A_u = Z_6$ is T-ABSO second submd. of $Y_u = Z_6$ as Z-md.

Since $2.3Z_6 \subseteq (2) \to 2.Z_6 \subseteq (2)$ or $2.3 \in$ ann (Z_6) = 6Z. But $A_u = Z_6$ is not prim second submd. since $2.Z_6 \subseteq (2)$ but $Z_6 \nsubseteq (2)$ and $2 \notin F - \text{ann}(Z_6) = 6Z$. So that A is T-ABSO F.second submd., but it is not prim F.second submd.

2. It is evident every quasi-prime F.second subm. is T-ABSO F. second submd.
3. Let A, B be two F. submds. non zero F.submds.of F.mds. X and Y resp.of an R-md. M, and B$\subset$A. If A is T-ABSO F.second submd. of Y then it is not necessary that B is T-ABSO F.second submd. for example:

Let $X\colon Z_{10} \to L$ where $X(y) = \begin{cases} 1 \text{ if } y \in Z_{10} \\ 0 \ \text{o.w.} \end{cases}$,

$Y\colon Z_8 \to L$ where $Y(y) = \begin{cases} 1 \text{ if } y \in Z_8 \\ 0 \ \text{o.w.} \end{cases}$

It is evident X is F.md. of Z_{10} as Z- md. and Y is F.md. of Z_8 as Z-md.

Let A: $Z_{10} \to L$ where A(y) = $\begin{cases} u \text{ if } y \in Z_{10}, \forall u \in L \\ 0 \text{ o.w.} \end{cases}$

B: $Z_8 \to L$ where B(y) = $\begin{cases} u \text{ if } y \in Z_8, \forall u \in L \\ 0 \text{ o.w.} \end{cases}$

it is evident A is F.submd. of X and B F.submd. of Y.

Now, $A_u = Z_{10}$ as Z-md. and $B_u = Z_8$ as Z-md. where $B_u \subseteq A_u$ and A_u is T-ABSO second submd., but B_u is not T-ABSO second submd. since $2.2Z_8 \subseteq (\bar{4})$, but $2Z_8 \nsubseteq (\bar{4})$ and $2.2 \notin \text{ann}(Z_8) = 8Z$.

4. Let A and B be F. submds. of F.md. Y of an R-md. M and A⊂B. If A is T-ABSO F. second submd. of Y then A is T-ABSO F. second submd. of B.

Proof: If B=Y then don′t need to proved

Let a_s b_l A ⊆H, a_s, b_l are F.singletons of R, H be a completely irred.F.submd. of B. Since B is F.second submd. of Y then H is F. submd. of Y, H is a completely irred. F. submd. of Y, we have either a_s A⊆ H or b_l A ⊆ H or a_s b_l ⊆ F-ann (A). (Since A is T-ABSO F. second submd. of Y) so that A is T-ABSO F. second submd. of B.

5. Every non zero F.submd. of F.md. Y of an R-md M define as follows is T-ABSO F. second submd. for example:

Let Y: $Z_n \to L$ where $Y(y) = \begin{cases} 1 \text{ if } y \in Z_n, n = p \text{ or } n = pq \\ 0 \text{ o.w.} \end{cases}$

Where p,q are prime integers. It is evident Y is F.md. of Z_n as Z- md.

Let A: $Z_n \to L$ where $A(y) = \begin{cases} u \;\; y \in z_p \text{ or } z_{pq} \\ 0 \text{ o.w.} \end{cases}$

It is evident A is F.submd. of Y.

Now, A_u =Z_p or Z_{pq} is T-ABSO second submd. So that A is T-ABSO F. second. submd. by Proposition(3.6)

6. Every non zero F.submd. of F.md. Y of an R-md M define as follows is not T-ABSO F. second submd.of Y:

Let Y: Z → L where $Y(y) = \begin{cases} 1 \text{ if } y \in Z \\ 0 \text{ o.w.} \end{cases}$

It is evident Y is F.md. of Z as Z- md.

Let A:Z → L where $A(y) = \begin{cases} u \;\; y \in 2Z \\ 0 \text{ o.w.} \end{cases}$

It is evident A is F.submd. of Y.

Now, A_u=2Z is not T-ABSO second submd.of Y_u =Z as Z-md., since 2.2.2Z⊆8Z where 8Z is a completely irred. submd of Y_u =Z as Z-md., but 2.2Z⊈ 8Z and 2.2∉ ann(2Z)=(0).

So that A is not T-ABSO F. second submd.

7. The sum of two T-ABSO F. second submds. of Y F. md. of an R-md. M, is T- ABSO F. second submd. of Y.

Proof: Let A,B two T-ABSO F. second submds. Assume that a_s, b_l are F. singletons of R, H is a completely irred. F.submd. of F.md. Yof an R-md. M, such that $a_s\ b_l(A+B) \subseteq H \rightarrow (a_s\ b_l\ A + a_s\ b_l B) \subseteq H$, so that $a_s\ b_l\ A \subseteq H$ and $a_s\ b_l B \subseteq H$.But A and B are T-ABSO F.second submds. of Y. Thus either $a_s\ A \subseteq H$ or $b_l\ A \subseteq H$ or $a_s\ b_l \subseteq$ F-ann (A) and either $a_s B \subseteq H$ or $b_l B \subseteq H$ or $a_s\ b_l \subseteq$ F-ann (B), so that either $a_s\ (A+B) \subseteq H$ or $b_l(A+B) \subseteq H$ or $a_s\ b_l \subseteq$ F-ann (A+B). thus A+B is T-ABSO F. second submd.

Theorem 3.8: Let Y be F.md. of an R-md. M.If either A is F.second submd. of Y or A is a sum of two F.second submd. of Y then A is T-ABSO F.second submd.of Y.

Proof: The first assertion is clear. To see the F.second submd. assertion, let A_1 and A_2 be two F.second submds.of F.md. Y, we show that $A_1 + A_2$ is T-ABSO F. second submd. of Y. Assume that F. singletons a_s, b_l of R,H is a completely irred. F.submd. of Y and $a_s\ b_l(A_1 + A_2) \subseteq H$. Since A_1 is F. second submd.$a_s\ b_l A_1 = 0_1$ or $A_1 \subseteq H$ by theorem (3.2). Similarly $a_s\ b_l A_2 = 0_1$ or $A_2 \subseteq H$. If $a_s\ b_l A_1 = 0_1 = a_s\ b_l A_2$ (resp. $A_1 \subseteq H$ and $A_2 \subseteq H$). then we are done. Now let $a_s\ b_l\ A_1 \subseteq 0_1$ and $A_2 \subseteq H$, then $a_s\ A_1 \subseteq 0_1$ or $b_l A_1 \subseteq 0_1$ because F-ann(A_1) is a prime F.ideal of R. If $a_s\ A_1 \subseteq 0_1$, then $a_s(A_1 + A_2) \subseteq a_s\ A_1 + A_2 \subseteq A_2 \subseteq H$ Similarly if $b_l A_1 = 0_1$, we get $b_l(A_1 + A_2) \subseteq H$ as desired.

Proposition 3.9: Let K be F.ideal of R and A be T-ABSO F.second submd.of F.md.Y of an R-md. M. If $a_s\ KA \subseteq H$ for a_s f.singleton of R and H is a completely irred.F.submd. of Y, then either $a_s\ A \subseteq H$ or $KA \subseteq H$ or $a_s\ K \subseteq$ F-ann(A).

Proof: Let $a_s\ A \not\subseteq H$ and $a_s\ K \not\subseteq$ F-ann(A). Then there exists $b_l \subseteq K$, so $a_s\ b_l A \neq 0_1$. Now as A is T-ABSO F.second submd. of Y, $b_l a_s\ A \subseteq H$ implies that $b_l\ A \subseteq H$.

we show that $KA \subseteq H$, let r_i be an arbitrary F.singleton of K. Then $(b_l + r_i)\ a_s\ A \subseteq H$.

Hence either $(b_l + r_i)\ A \subseteq H$ or $(b_l + r_i)\ a_s \subseteq$ F -ann(A). If $(b_l + r_i)\ A \subseteq H$, then since $b_l\ A \subseteq H$ we have $r_i\ A \subseteq H$. If $(b_l + r_i)\ a_s \subseteq$ F-ann(A) then $r_i\ a_s \not\subseteq$ F-ann(A), but $r_i\ a_s A \subseteq H$. Thus $r_i\ A \subseteq H$. Hence we conclude that $KA \subseteq H$.

Proposition 3.10: Let K and N be two F.ideals of R and A be T-ABSO F.second submd. of F. md. Y of an R-md. M. If H is a completely irred. F.submd. of Y and $KNA \subseteq H$, then either $KA \subseteq H$ or $NA \subseteq H$ or $KN \subseteq$ F-ann(A).

Proof: Let $KA \not\subseteq H$ and $NA \not\subseteq H$.We show that $KN \subseteq$ F-ann(A). Assume that $C_i \subseteq K$ and $d_r \subseteq N$. By assumption there exists $a_s \subseteq K$ such that $a_s\ A \not\subseteq H$, but $a_s\ NA \subseteq H$.

Now Proposition (3.9) shows that $a_s N \subseteq F - \mathrm{ann}(A)$ and so$(K \backslash (H:_R A))\ N \subseteq$ F-ann(A). Similarly there exists $b_l \subseteq (N \setminus (H:_R A))$ such that $K\ b_l \subseteq$F-ann(A) and also $K\ (N \setminus (H:_R A)) \subseteq$F-ann(A). Thus we have $a_s\ b_l \subseteq$ F-ann (A), $a_s\ d_r \subseteq$ F-ann (A) and $c_i\ b_l \subseteq$ F-ann (A). As $(a_s + c_i) \subseteq K$ and $(b_l + d_r) \subseteq N$, we have $(a_s + c_i)\ (b_l + d_r)A \subseteq H$. Since A is T-ABSO F.second submd. Therefore $(a_s + c_i)\ A \subseteq H$ or $(b_l + d_r)A \subseteq H$ or $(a_s + c_i)\ (b_l + d_r) \subseteq$ F-ann (A). If $(a_s + c_i)\ A \subseteq H$ then $c_i\ A \not\subseteq H$.Hence $c_i \subseteq K \setminus (H:_R A)$, which implies that $c_i d_r \subseteq$ F-ann (A). Similarly if $(b_l + d_r)A \subseteq H$,we can deduce that $c_i d_r \subseteq$ F-ann (A). At last if $(a_s + c_i)\ (b_l + d_r) \subseteq$ F-ann (A). Then $(a_s\ b_l + a_s\ d_r + c_i\ b_l + c_i d_r) \subseteq$ F-ann (A), so that $c_i d_r \subseteq$ F-ann (A) therefore $KN \subseteq$ F-ann(A).

Corollary 3.11: Let Y be F.md. of an R-md. M, and A be T-ABSO F.second submd. of Y. Then KA is T-ABSO F. second submd. of Y,for all F.ideals K of R with $K \not\subseteq$ F-ann(A).

Proof: Let K be F.ideal of R with $K \not\subseteq$ F-ann(A), a_s, b_l be F.singletons of R, H be a completely irred. F.submd. of Y and $a_s\ b_l KA \subseteq H$, then $a_s\ A \subseteq H$ or $b_l KA \subseteq H$ or $a_s\ b_l \subseteq$F-ann(KA) = $(0_1: KA)$ (i.e. $a_s\ b_l KA \subseteq 0_1$ by Proposition (3.9)

If $a_s\ b_l KA \subseteq H$ or. $a_s\ b_l KA \subseteq 0_1$, then we are done.
If $a_s\ A \subseteq H$, then $a_s\ KA \subseteq a_s\ A$ implies that $a_s\ KA \subseteq H$ it is required.

Corollary 3.12: Let Y be a multiplication F.md.of an R-md. M, then every F.submd. $A \neq 0_1$ of Y is T-ABSO F.second submd.

Proof: This follows from Corollary (3.11)

The following example shows that the condition Y is a mulutiplication F. md. cannot delete.

Example 3.13: Let Y: $Z_{p^\infty} \to L$ where Y(y)= $\begin{cases} 1 & y \in Z_{p^\infty} \\ 0 & o.w. \end{cases}$

where p is any prime integer. It is evident Y F.md. of Z-md. Z_{p^∞}

Let A: $Z_{p^\infty} \to L$ where A(y)= $\begin{cases} u & y \in \left(\frac{1}{p^3} + Z\right) \\ 0 & o.w. \end{cases}$

it is evident A F.submd. of Y.

Now, $A_u = \langle \frac{1}{p^3} + Z \rangle$ is submd. of $Y_u = Z_{p^\infty}$ as Z-md., A_u is not T-ABSO second submd. since $P^2 \langle \frac{1}{p^3} + Z \rangle \subseteq \langle \frac{1}{p} + Z \rangle$ but $p\langle \frac{1}{p^3} + Z \rangle \nsubseteq \langle \frac{1}{p} + Z \rangle$ and $P^2 \nsubseteq$ ann($\langle \frac{1}{p^3} + Z \rangle$)=(0)

so that A is not T-ABSO F.second submd. of Y by Proposition (3.6)

Definition 3.14: A F.md.Y of an R-md. M is said to be a cocyclic F.md. if F-soc(Y) is large and simple F.submd. of Y.[Here F-soc(Y) denotes the sum of all minimal F. submds.of Y]

Note that: H is a completely irred. F.submd. of Y iff Y/ H is a cocyclic F.md.

Lemma 3.15: Let H be a completely irred. F.submd. of Y F.md. of an R-md. M and a_s be F.singleton of R then $(H:_Y a_s)$ is a completely irred. F.submd. of Y.

Proof: This follows from the fact that F.submd. H of Y is a completely irred. F.submd. of Y iff Y/H is a cocyclic F.md.and that Y/ $(H:_Y a_s) \cong (a_s Y+H)$ /H, we use the following basic fact without comment.

Proposition 3.16: Let A be T-ABSO F. second submd.of F.md. Y of an R-md. M. Then we have the following:

a. If H is a completely irred. F.submd. of Y such that A⊈H, then $(H:_R A)$ is T-ABSO F.ideal of R.
b. If Y is a cocyclic F.md.,then F-ann(A) is T-ABSO F.ideal of R.
c. If F. singleton a_s of R, then a_s^n A =a_s^{n+1} A, ∀ n≥2.
d. If F-ann(A) is a prime F.ideal of R then $(H:_R A)$ is a prime F.ideal of R for all completely irred. F.submd H of Y such that A⊈H.

Proof: a) Since A ⊈ H, we have $(H:_R A)$ is proper F.ideal of R, let F.singletons a_s, b_l, c_i of R and $a_s b_l c_i \subseteq (H:_R A)$. Then $a_s b_l A \subseteq (H:_Y c_i)$ thus $a_s A \subseteq (H:_Y c_i)$ or $b_l A \subseteq (H:_Y c_i)$ or $a_s b_l A \subseteq 0_1$Since A is T-ABSO f.submd. $(H:_Y (c_i))$ is completely irred.F.submd. of Y by Lemma (3.15).Therefore $a_s c_i \subseteq (H:_R A)$ or $b_l c_i \subseteq (H:_R A)$ or $a_s b_l \subseteq (H:_R A)$.

b) Since Y is a cocyclic F.md.the zero F.submd. 0_1 of Y is completely irred.F.submd. of Y. Thus F-ann(A) is T-ABSO F.ideal of R by part (a).

c) It is enough to show that $a_s^2 A = a_s^3 A$. It is clear that $a_s^3 A \subseteq a_s^2 A$. Let H be completely irred.F.submd. of Y such that $a_s^3 A \subseteq H$ then $a_s^2 A \subseteq (H:_R a_s)$. Since A is T-ABSO F.second submd. and $(H:_R a_s)$ is a completely irred.F.submd.of Y by Lemma (3.15) $a_s A \subseteq (H:_R a_s)$ or $a_s^2 A \subseteq 0_1$. Therefore $a_s^2 A \subseteq H$ this implies that $a_s^2 A \subseteq a_s^3 A$.

d) Let F.singletons a_s, b_l of R, H be a completely irred.F.submd. of Y such that A⊈H and $a_s b_l \subseteq (H:_R A)$ then $a_s A \subseteq H$ or $b_l A \subseteq H$ or $a_s b_l A \subseteq 0_1$. Since A is T-ABSO F.second submd. If $a_s b_l A \subseteq 0_1$, then by assumption $a_s A \subseteq 0_1$ or $b_l A \subseteq 0_1$. Thus is any case we get that $a_s A \subseteq H$ or $b_l A \subseteq H$.

Theorem 3.17: Let A be T-ABSO F.second submd. of Y F.md. of an R-md. M.Then we have the following:

a. If $\sqrt{F-\text{ann}(A)}$ = P for some prime F.ideal P of R and H is a completely irred. F.submd. of Y such that A⊈H, then $\sqrt{(H:_R A)}$ is a prime F.ideal of R containing P.
b. If $\sqrt{F-\text{ann}(A)}$ = P∩ Q for some prime F.ideals P and Q of R, H is a completely irred.F.submd. of Y such that A⊈H and P ⊆ $\sqrt{(H:_R A)}$ then $\sqrt{(H:_R A)}$ is a prime F. ideal of R.

Proof: a) Assume that F.singletons a_s, b_l of R and $a_s b_l \subseteq \sqrt{(H:_R A)}$. Then there is a positive integer t such that $a_s^t b_l^t A \subseteq H$. By hypotheses, A is T-ABSO F.second submd. of Y, thus $a_s^t A \subseteq H$ or $b_l^t A \subseteq H$ or $a_s^t b_l^t \subseteq F - ann(A)$. If either $a_s^t A \subseteq H$ or $b_l^t A \subseteq H$ we are done. So assume that $a_s^t b_l^t \subseteq F - ann(A)$. Then $a_s b_l \subseteq \sqrt{F - ann(A)} = P$ and so $a_s \subseteq P$ or $b_l \subseteq P$ since P is prime F.ideal of R. It is clear that $P = \sqrt{F - ann(A)} \subseteq \sqrt{(H:_R A)}$. Therefore $a_s \subseteq \sqrt{(H:_R A)}$ or $b_l \subseteq \sqrt{(H:_R A)}$

b) The proof is similar to that of part (a).

Proposition 3.18: Let Y be F.md. of an R-md. M and let $\{K_i\}_{i\in I}$ be a chain of T-ABSO F.second submd. of Y. Then $\bigcup_{i\in I} K_i$ is T-ABSO F.second submd. of Y.

Proof: Let a_s, b_l be F.singletons of R and H be acompletely irred.F.submd.of Yand $a_s b_l (\bigcup_{i\in I} K_i) \subseteq H$. Assum that $a_s(\bigcup_{i\in I} K_i) \nsubseteq H$ and $b_l (\bigcup_{i\in I} K_i) \nsubseteq H$. Then there are m,n ∈ I, where $a_s k_n \nsubseteq H$ and $b_l k_m \nsubseteq H$. Hence for every $k_n \subseteq k_c$ and $k_m \subseteq k_d$, c,d ∈ I, we have $a_s k_c \nsubseteq H$ and $b_l k_d \nsubseteq H$. Therefore for each F.submd. k_h such that $k_n \subseteq k_h$ and $k_m \subseteq k_h$ we have $a_s b_l k_h \subseteq 0_1$. Hence $a_s b_l(\bigcup_{i\in I} K_i) \subseteq 0_1$, so that$a_s b_l \subseteq F - ann(\bigcup_{i\in I} K_i)$.

Definition 3.19: We say that T-ABSO F.second submd.A of F.md. Y of an R-md. M. is a maximal T-ABSO F.second submd.A of submd. K of Y, if A⊆K and there does not exist T-ABSO F.second submd. H of Y such that A⊂ H ⊂ K.

Lemma 3.20: (**Fuzzy Zorn′s lemma**) let X be F.ordered set with F.order R. If everyF.chain in X has an upper bound,then X has a maximal element, [14].

Proposition 3.21: Let Y be F.md. of an R-md. M. Then every T-ABSO F.second submd. of Y is contained in a maximal T-ABSO F.second submd. of Y.

Proof: This proved easily by using F. Zorn′s lemma and proposition (3.18).

4. Strongly T-ABSO F. Second Submds.

In this section, we will define a strongly T-ABSO F.second submd., and discuss its relationship to T-ABSO F.second submd., and a quasi T-ABSO F.second submd.

Definition 4.1: Let $A \neq 0_1$ be F.submd. of F.md. Y of an R-md. M. We say that A is a strongly T-ABSO F.second submd. of Y if whenever F.singletons a_s, b_l of R, and H_1, H_2 are completely irred.F.submd. of Y and $a_s b_l A \subseteq H_1 \cap H_2$, then $a_s A \subseteq H_1 \cap H_2$ or $b_l A \subseteq H_1 \cap H_2$ or $a_s b_l \subseteq F - ann(A)$.

Remark 4.2: A is T-ABSO F.second submd. of F.md. Y of an R-md. M iff A is strongly T-ABSO F.second submd. of Y.

Proof: ⟹) Let a_s, b_l are F.singletons of R and H is completely irred.F.submd. of Y such that $a_s b_l A \subseteq H \cap H$, then $a_s A \subseteq H \cap H$ or $b_l A \subseteq H \cap H$ or $a_s b_l \subseteq F - ann(A)$ Then A is strongly T-ABSO F.second submd. of Y.

⟸) This is clear.

Theorem 4.3: Let A be F.submd.of Y F.md.of an R-md.M.The following statements are equivalent:

a. A is a strongly T-ABSO F.second submd. of Y F.md. of an R-md. M.
b. If $A \neq 0_1$, KNA⊆ C for some F.ideals K,N of R and F.submd. C of Y, Then KA⊆ C or NA⊆ C or KN⊆ F-ann(A).
c. $A \neq 0_1$ and for each F.singletons a_s, b_l of R we have $a_s b_l A = a_s A$ or $a_s b_l A = b_l A$ or $a_s b_l = 0_1$

Proof: (a)→(b) Assume that KNA⊆H for some F.ideals K,N of R, H F.submd.of Yand KN⊈F-ann(A). They by

Proposition (3.10) for all completely irred. F.submd. H of Y with C⊆H either KA⊆H or NA⊆H. If KA⊆H (resp. NA⊆H) for all completely irred.F.submds. H of Y with C⊆H, we are done

Now suppose that H_1 and H_2 are two completely irred.F.submds. of Y with C⊆ H_1, C⊆ H_2, KA⊈ H_1 and NA⊈ H_2. Then KA⊆ H_2 and NA⊆ H_1.Since KNA⊆ $H_1 \cap H_2$, We have either KA$\subseteq H_1 \cap H_2$ or NA$\subseteq H_1 \cap H_2$. As KA$\subseteq H_1 \cap H_2$, we have KA$\subseteq H_1$ which is a contradiction. Similariy from NA$\subseteq H_1 \cap H_2$ we get a contradiction.

(b) →(a) this is clear.

(a) →(c) By Part (a), $A \neq o_1$, leta_s, b_l be F.singletons of R, then $a_s\, b_l A \subseteq a_s\, b_l A$ indicates that $a_s\, A \subseteq a_s\, b_l A$ or $b_l A \subseteq a_s\, b_l A$ or $a_s\, b_l A = o_1$.Thus $a_s\, b_l A = a_s\, A$ or $a_s\, b_l A = b_l A$ or $a_s\, b_l A = o_1$

(c) →(a) This is clear.

Proposition 4.4: Let A be a strongly T-ABSO F.second submd. of Y F.md. of an R-md. M. Then we Have the following:

a. F-ann(A) is T-ABSO F.ideal of R
b. If C is F.submd. of Y F.md. of R-md. M, that $A \nsubseteq C$ then $(C:_R A)$ is T-ABSO F.ideal of R
c. If T is F.ideal of R, then $T^n A = T^{n+1} A$, $\forall n \geq 2$.
d. If $(H_1 \cap H_2 :_R A)$ is a prime F.ideal of R for all completely irred. F.submd. H_1 and H_2 of Y, such that $A \neq H_1 \cap H_2$ then F-ann(A) is a prime F.ideal of R.

Proof: a) Let a_s,b_l,c_i be F.singletons of R and $a_s b_l c_i \subseteq$ F-ann(A). Then $a_s\, b_l A \subseteq a_s\, b_l A$ implies that $a_s A \subseteq a_s b_l A$ or $b_l A \subseteq a_s\, b_l A$ or $a_s b_l A = 0_1$ by Theorem (4.3) $a_s\, b_l A = 0_1$ then we finished.If $a_s A \subseteq a_s b_l A$, then $c_i a_s A \subseteq c_i a_s b_l A = 0_1$. In other case we do the same.

b) Let Let a_s,b_l,c_i be F.singletons of R and $a_s b_l c_i \subseteq (C:_R A)$. Then $a_s c_i A \subseteq C$ or $b_l c_i A \subseteq C$ or $a_s b_l c_i A = 0_1$. If $a_s c_i A \subseteq C$ or $b_l c_i A \subseteq C$, then we are done.

If $a_s b_l c_i A = 0_1$, then the result follows from part (a).

c) It is enough to show that $T^2 A = T^3 A$. It is clear that $T^3 A \subseteq T^2 A$. Since A is strongly T-ABSO F.second submd. $T^3 A \subseteq T^3 A$ implies that $T^2 A \subseteq T^3 A$ or $TA \subseteq T^3 A$ or $T^3 A = 0_1$ by theorem (4.3). If $T^2 A \subseteq T^3 A$ or $TA \subseteq T^3 A$ then we are done.

If $T^3 A = 0_1$, then the result follows from part (a).

d)Suppose that a_s,b_l be F.singletons of R and $a_s b_l A = 0_1$. Assume contrary that $a_s A \neq 0_1$ and $b_l A \neq 0_1$.Then there exist completely irred.F. submds. H_1 and H_2 of Y, such that $a_s A \nsubseteq H_1$, and $b_l A \nsubseteq H_2$. Now since $((H_1 \cap H_2):_R A)$ is a prime F.ideal of R

$0_1 = a_s\, b_l A \subseteq H_1 \cap H_2$ implies that $a_s\, A \subseteq H_1 \cap H_2$ or $b_l A \subseteq H_1 \cap H_2$.

In any cases we have a contradiction.

Proposition 4.5: If T is T-ABSO F.ideal of R then on of the following statements must hold:

a. $\sqrt{T}$ = P is a prime F.ideal of R such that $P^2 \subseteq T$.
b. $\sqrt{T}$ = P∩Q, PQ⊆ T and $\sqrt{T^2} \subseteq T$, where P and Q are the only distinct prime F.ideals of R that are minimal over T.[6]

Theorem 4.6: If A is a strongly T-ABSO F.second submd. of F.md. Y of an R-md. M,and $A \nsubseteq N$,then either $(N:_R A)$ is a prime F.ideal of R or there exists an element a_s F.singleton of R such that $(N:_R a_s A)$ is a prime F.ideal of R.

Proof: By Proposition(4.4) and Proposition (4.5) we have one of the following two case.

a. Let $\sqrt{F - \text{ann}(A)}$ = P, where P is a Prime F. ideal of R, we show that $(N:_R A)$ is a prime F.ideal of R when $P \subseteq (N:_R A)$. Assume that a_s,b_l be F.singletons of R and $a_s b_l \in (N:_R A)$. Hence $a_s A \subseteq N$ or $b_l A \subseteq N$ or $a_s b_l \subseteq$ F-ann(A).
b. If either $a_s A \subseteq N$ or $b_l A \subseteq N$, we are done. Now assume that $a_s, b_l \subseteq$ F-ann(A). Then $a_s b_l \subseteq P$ and so $a_s \subseteq P$ or $b_l \subseteq P$. Thus $a_s \subseteq (N:_R A)$ or $b_l \subseteq (N:_R A)$ and the assertion follows. If $\nsubseteq (N:_R A)$. Then there exists $a_s \subseteq P$ such that $a_s A \nsubseteq N$ By Proposition (4.5), $P^2 \subseteq$ F-ann(A) $\subseteq (N:_R A)$, thus $P \subseteq (N:_R a_s A)$. Now a similar argument shows that $(N:_R a_s A)$ is a prime F.ideal of R.
c. Let $\sqrt{F - \text{ann}(A)}$ = P∩Q, where P and Q are distinct prime F.ideals of R. If $P \subseteq (N:_R A)$ then the result follows by a similar proof to that of part (a). Assume that $P \nsubseteq (N:_R A)$ then there exist $a_s \subseteq P$ such that $a_s A \nsubseteq N$. By Proposition (4.5) we have $PQ \subseteq$ F-ann(A) $\subseteq (N:_R A)$ thus $Q \subseteq (N:_R a_s A)$ and the result follws by a similar proof to that of part (a).

Theorem 4.7: Let A be F.submd. of F.md. Y of a comultiplication R-md. M. Then we have the following:

a. If F-ann(A) is T-ABSO F.ideal of R,then A is a strongly T-ABSO F.second submd. of Y. In particular, A is T-ABSO F.second submd. of Y.
b. If Y is a cocyclic F. md. and A is T-ABSO F.second submd. of Y,then A is a strongly T-ABSO F.second submd. of Y.

Proof:

a) let a_s,b_l be F.singletons of R,K be F.submd.of Yand $a_s b_l A \subseteq K$.Then we have F-ann(K) $a_s b_l A = 0_1$ so by assumption, F-ann(K) $a_s A = 0_1$ or F-ann(K) $b_l A = 0_1$ or $a_s b_l A = 0_1$.If $a_s b_l A = 0_1$,we are done.If F-ann(K) $a_s A = 0_1$or F-ann(K) $b_l A = 0_1$,then F-ann(K) ⊆ F -ann(a_sA) or F-ann(K) ⊆ F -ann(b_lA).

Hence $a_s A \subseteq K$ or $b_l A \subseteq K$ since M is a comultiplication R-md.

b) By proposition(2.17), F-ann(A) is T-ABSO F.ideal of R. Thus the result follows from part (a).

Lemma 4.8: Let X,Y be F.mds. of M, Ḿ an R-mds. resp. and let F: X→ Y be a F-monomorphism of R-mds. If H is a completely irred. F.submd. of F(X) then $F^{-1}(H)$ is a completely irred.F.submd. X.

Proof: This is strighat forward.

Lemma 4.9: Let F: X→ Y be F-monomorphism of R-md. If H is a completely irred. F.submd. of X F.md. of an R-md. M,then F(H)is a completely irred.F.submd.of F(X).

Proof: Let $\{\acute{A_\imath}\}_{i\in I}$ be a family of f.submds. of F(Y) such that F(H) = $\bigcap_{i\in I} \acute{A}_\imath$.

Then H = $F^{-1}F(H)$ = F^{-1} ($\bigcap_{i\in I} \acute{A}_\imath$) = $\bigcap_{i\in I} F^{-1}(\acute{A}_\imath)$. This denotes that there exists i ∈ I such that H = $F^{-1}(\acute{A}_\imath)$ since H is a completely irred. f.submd. Y. Therefore,

F(H) = $FF^{-1}(\acute{A}_\imath)$ = F(X)∩ $\acute{A}_\imath$= $\acute{A}_\imath$ as needed.

Theorem 4.10: Let F:X→ Y be F-monomorphism of R-md.Then we have the following:

a. If A is a strongly T-ABSO F.second submd. of F.md. X, then F(A) is T-ABSO F.second submd. of Y.
b. If A is T- ABSO F.second submd. of X, then F(A) is T- ABSO F.second submd. of F(X).
c. If Á is a strongly T- ABSO F.second submd. of Y and Á ⊆ F(X), then F^{-1}(Á) is T-ABSO F.second submd. of X.
d. If Á is T- ABSO F.second submd. of F(X), then F^{-1}(Á) is T-ABSO F.second submd. of X.

Proof: a) Since A≠ 0_1 and F is F-monomorphism, we have F(A)≠ 0_1. Let a_s, b_l F. singltons of R, Ḣ be a combletely irred. F. submd. of Y and $a_s b_l F(A) \subseteq$ Ḣ, then $a_s b_l A \subseteq F^{-1}$(Ḣ). As A is strongly T- ABSO F.second submd. $a_s A \subseteq F^{-1}$(Ḣ) or $b_l A \subseteq F^{-1}$(Ḣ) or $a_s b_l A = 0_1$. Therefore $a_s F(A) \subseteq F(F^{-1}$(Ḣ))= F(X)∩ Ḣ ⊆ Ḣ, $b_l F(A) \subseteq F(F^{-1}$(Ḣ))= F(X)∩ Ḣ ⊆ Ḣ or $a_s b_l A = 0_1$, as needed.

c) If F^{-1}(Á) $= 0_1$, then F(X)∩ A´= $F(F^{-1}$(Á))= F(0_1)= 0_1. Thus Á $= 0_1$, is a contradiction.Therefore F^{-1}(Á) $\neq 0_1$. Now let a_s, b_l F.singltons of R, H be a combletely irred. F. submd. of X and $a_s b_l F^{-1}$(Á) ⊆ H then $a_s b_l$Á = $a_s b_l$(F(X) ∩ Á)= $a_s b_l$ F^{-1}(Á)⊆ F(H). As Á is strongly T- ABSO F.second submd. a_sÁ ⊆ F(H) or b_l Á ⊆ F(H) or $a_s b_l$ Á $= 0_1$. Hence $a_s F^{-1}$(Á) $\subseteq F^{-1}F(H) = H$ or $b_l F^{-1}$(Á) $\subseteq F^{-1}F(H) = H$ or $a_s b_l F^{-1}$(Á) $= 0_1$, as required.

d) By using lemma (4.8), this is similar to the part (c).

Corollary 4.11: Let Y F.md. of an R-md. M and A⊆ K be two F.submds. of Y. Then we have the following:

a. If A is a strongly T- ABSO F.second submd. of K then A is T- ABSO F.second submd. of Y.
b. If A is a strongly T-ABSO F.second submd. of Y,then A is T-ABSO F.second submd.of K.

Proof: This follows from Theorem (4.10) by using the natural F-monomorphism K→ Y.

Theorem 4.12: Let A be F.submd. of Y F.md. of an R-md. M. Then the following statements are equivalent:

a. A is a strongly quasi-prime F. second submd. of Y
b. F-ann of any nonzero homomorphic image of A is Prime F.ideal.
c. A≠ 0_1 and $a_s b_l A \subseteq H$, where a_s, b_l F.singltons of R and H is a finite intersection of completely irred.F. submds.of Y, implies either $a_s A \subseteq H$ or $b_l A \subseteq H$.
d. A≠ 0_1 and for each a_s, b_l F.singltons of R either $a_s b_l A = b_l A$ or $a_s b_l A$ =a_sA.

e. F-ann(A) is a prime F.ideal of R and the set $\{(K:_R A)$: K is a proper completely irred. F.submd. of Y with $A \not\subseteq K\}$ is a chain of Prime F.ideals of R.

Proof : (a) →(b) and (a) →(c) there are clear.
(c) →(a) Assume that $a_s b_l A \subseteq Q$, where a_s, b_l F.singltons of R and Q is submd. of Y, but $a_s A \not\subseteq Q$ and $b_l A \not\subseteq Q$. There exists a collection $\{K_i\}_{i\in I}$ of completely irred.F. submds. of Y such that $Q = \bigcap_{i\in I} K_i$

Therefore $a_s A \not\subseteq K_i$ and $b_l A \not\subseteq K_j$ for some I,j∈ I. But by assumption, $a_s b_l A \subseteq Q \subseteq K_i \cap K_j$ implies either $a_s A \subseteq K_i \cap K_j$ or $b_l A \subseteq K_i \cap K_j$. Thus in any case, we have a contradiction.

(a) →(d) Let A be a strongly quasi-prime F.second submd. of Y and a_s, b_l F.singltons of R.Then $a_s b_l A \subseteq a_s b_l A$ implies that $a_s A \subseteq a_s b_l A$ or $b_l A \subseteq a_s b_l A$ as needed.

(d) →(a) Suppose that A has the stated property and $a_s b_l A \subseteq Q$, where a_s, b_l F. singltons of R and Q is F. submd. of Y. Then either $a_s A = a_s b_l A \subseteq Q$ or $b_l A = a_s b_l A \subseteq Q$.

(a) →(e) By part (b), for each proper completely irred. submd. K of Y with $A \not\subseteq K$,we have $(K:_R A)$ is a prime f.ideal of R. Let K_1 and K_2 be two proper completely irred. F. submds. of Y such that $(K_1:_R A) \not\subseteq (K_2:_R A)$ and $(K_2:_R A) \not\subseteq (K_1:_R A)$. Then there exist a_s, b_l f.singltons of R such that $a_s A \subseteq K_1$, $a_s A \not\subseteq K_2$, $b_l A \subseteq K_2$, and $b_l A \not\subseteq K_1$. Hence $a_s b_l A \subseteq K_1 \cap K_2$. Since A is strongly quasi-prime F.second submd., this implies that either $a_s A \subseteq K_2$ or $b_l A \subseteq K_1$. In any case we have a contradiction.

(e) →(a) Let a_s, b_l F. singltons of R, Q be F.submd. of Y with $a_s b_l A \subseteq Q, a_s A \not\subseteq Q$ and $b_l A \not\subseteq Q$. Then there exist completely irred. F. submds. K_1 and K_2 of Y such that $Q \subseteq K_1$, $a_s A \not\subseteq K_1$, $Q \subseteq K_2$ and $b_l A \subseteq K_2$. By assumption, we may assume that $(K_1:_R A) \subseteq (K_2:_R A)$ but $a_s b_l A \subseteq Q \subseteq K_1$ and $(K_1:_R A)$ is a prime F.ideal of R by assumption. Hence either $a_s \subseteq (K_1:_R A)$ or $b_l \subseteq (K_1:_R A) \subseteq (K_2:_R A)$ in any case we have a contradiction, and the proof is completed.

Remark 4.13: Every strongly quasi prime F.second subm. of Y F.md. of an R-md. M is strongly T-ABSO F.second submd. but the converse is not true in general, for example:

Let y: $Z_{p^\infty} \oplus Z_{q^\infty} \rightarrow L$ where $Y(y) = \begin{cases} 1 & \text{if } y \in Z_{p^\infty} \oplus Z_{q^\infty} \\ 0 & \text{o.w.} \end{cases}$
It is evident Y is F.md. of $Z_{p^\infty} \oplus Z_{q^\infty}$ as Z-md.
Let A: $Z_{p^\infty} \oplus Z_{q^\infty} \rightarrow L$ w here $A(y) = \begin{cases} u & \text{if } y \in \langle \frac{1}{p} + Z \rangle \oplus \langle \frac{1}{q} + Z \rangle \\ 0 & \text{o.w.} \end{cases}$

Where p, q are prime. It is evident A is F.submd. of Y.

Now,$A_u = \langle \frac{1}{p} + Z \rangle \oplus \langle \frac{1}{q} + Z \rangle$ is strongly T-ABSO second submd. of $Y_u = Z_{p^\infty} \oplus Z_{q^\infty}$ as Z-md. since $pqA_u = 0_{Y_u}$ and $pq \in ann(A_u)$, but A_u is not strongly quasi prime second submd. since $pA_u = 0 \oplus Z_{q^\infty} \neq 0_{Y_u}$ and $qA_u = Z_{p^\infty} \oplus 0 \neq 0_{Y_u}$. Thus A strongly T-ABSO F. second submd., but it is not strongly quasi prime F. second submd.

Proposition 4.14: Let A be a non zero F. submd. of Y F.md. of an R-md. M. Then A is a strongly quasi-prime F. second submd.of Y iff A is a strongly T-ABSO F.second submd.of Y and F-ann(A) is a prime F.ideal of R.

Proof: Distinctly if A is a strongly quasi-prime F.second submd. of Y, then A is a strongly T- ABSO F.second submd. of Y and by Theorem(4.12), F-ann(A) is a prime F.ideal of R. For the convers, let $a_s b_l A \subseteq H$ for some a_s, b_l F.singltons of R and F. submd. K of Y such that neither $a_s A \subseteq H$ nor $b_l A \subseteq H$. Then $a_s b_l \subseteq$ F-ann(A) and so either $a_s \subseteq$ F-ann(A) or $b_l \subseteq$ F-ann(A). This contradiction shows that A is strongly quasi-prime F.second submd.

Definintion 4.15: A non-zero F.submd. A of F. md. Y of an R-md. M is called a quasi T-ABSO F. second submd. if F-ann(A) is T-ABSO F. ideal of R.

Example 4.16: Every strongly T-ABSO F..second submd. is a quasi T-ABSO F.second submd., but the converse is not true in general, See Remarks and Example (3.7) part (6), where A is a quasi T-ABSO F.second submd. since F-ann(A)is T-ABSO F. ideal,but it's not T-ABSO F. second submd., then it's not strongly T-ABSO F..second submd.by Remark(4.2).

Proposition 4.17: Let Y be comultiplication F.md. of an R-md. M.Then F.submd. A of Y is strongly T-ABSO F.second submd. of Y iff it is a quasi T-ABSO F. second submd. of Y.

Proof: This follows from Proposition (4.4) and Theorem(4.7).

References

1. Zadeh L.A., *Fuzzy sets, information and control, 8*, 1965, 338-353.
2. Rosen feld, Fuzzy groups, *J.Math.Anal.Appl.,35*, 1971, 512-517.
3. Deniz S.; Gürsel Y.; Serkan O.; Bayram A. E.; Bijan D. On 2-Absorbing Primary Fuzzy Ideals of Commutative Rings, Hindawi, *Mathematical Problems in Engineering*. 2017. 2017, 1-7.
4. Rabi H.J. *Prime Fuzzy Submodules and Prime Fuzzy Modules.* M.Sc.Thesis, University of Baghdad. 2001.
5. Hatam Y.K. *Fuzzy quasi-prime modules and Fuzzy quasi-prime submodules,* M.sc. Thesis, university of Baghdad, 2001.
6. Wafaa H. Hanoon, *T-ABSO Fuzzy Submodules and T-ABSO Fuzzy Modules and Their Generalizations,* Ph.D. Dissertation University of Baghdad, 2019.
7. H. Ansari Toroghy and F. Mahboobi Abkenar, The dual notion of fuzzy prime submodules, *Journal of Mathematical Extension. 13*(2), 2019, 17-30.
8. Liu, W. J., Fuzzy Invariant Subgroups and Fuzzy Ideals, *Fuzzy Sets and Systems, 8,* 1982, 133-139.
9. Mohammed M.R. AL-Shamiri, On Fuzzy Semiprime Submodules", *International Journal of Innovation and Applied Studies, 13*(4), 2015, 929-934.
10. Zahedi, M.M, On L-Fuzzy Residual Quotiet Modules and P-Primary Submodules, *Fuzzy Sets and Systems, 51*, 1992, 333-344.
11. Mashinchi, M. and Zahedi, M. M., On L-Fuzzy Primary Submodules, *Fuzzy Sets Systems, 49,* 1992, 231-236.
12. Mukherjee, T.K., Sen, M.K. and Roy, D., On Fuzzy Submodules and Their Radicals, *J. Fuzzy Math., 4,* 1996, 549-558.
13. Mukhrjee, T. K., Prime Fuzzy Ideals in Rings, *Fuzzy Sets and Systems, 32*, 1989, 337-341.
14. Ismat Beg, On Fuzzy Zorn′s lemma, *Fuzzy Sets and Systems, 101,* 1999, 181-183.