Research Article

An Iterated Function System for A-contraction mapping

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Abstract: Fractals play main role in dynamical systems, quantum mechanics, biology, computer graphics, geophysics, astrophysics and astronomy etc. The Iterated Function Systems (IFS) is an emerging scheme provides an important implement to mathematicians for manipulation and description of the attractors applying common mathematical algorithms. The intension of the present study is to originate the novel IFS expressly "A-Iterated Function System" either "AIFS" defined on a complete metric space using a unique class of contraction maps known as A-contractions; that was studied by various mathematicians. This study proves the uniqueness and existence of the attractor for AIFS. The analysis also establishes the Collage theorem for AIFS. To obtain our outcomes we utilizing some basic ideas and speculations given in the literature. Our outcomes extend, unify and generalize numerous consequences present in the literature.

Keywords: Iterated Function System, Collage theorem, attractor, complete metric space, A-contraction mapping.
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1. Introduction

Banach Contraction Principle plays a vital role in Iterated Function System. The concept of a class of contractions studied by various mathematicians (see[1-5]). In 2008, M. Akram et al. [6] included the contractions and reported a novel class of contraction maps, called A-contraction maps and they also demonstrated fixed point theorems. Recently, M. Akramet al. [7] used A-contraction type cases and extended a few standard results offixed pointin thegeneralized metric spaces.

Fractal analysis is an exciting research area that provides many applications in computer science, modeling, biology, quantum physics, image processing and other fields of applied sciences. Firstly, B. Mandelbrot [8] introduced the term in 1975. The concept of fractal popularized by Hutchinson[9] and Barnsley[10]. Iterated Function System is one of the powerful & exciting developments for analysis and construction of fractal sets. In 1981, Hutchinson [9] introduced a formal definition of IFS. However,the idea of IFS popularized by Barnsley and Demko [11] in 1985 and many others (also see [12]-[18]). Initially mentioned, Singh et al. [19] introduced the Hutchinson Barnsley hypothesis for single and multivalued contractions on metric space in 2009 and shortly after him Sahu et al.[20] investigated the KIFS based onkannan mappings and established the collage theorem in the same setting. In 2011, S.C. Shrivastava and Padmavati [21] established the collage theorem for IFS under the contraction condition in two mappings. They reported the extension ofHutchinson's classical framework for commuting mapping. Recently, S. C. Shrivastava and Padmavati [22] introduced D-Iterated Function System they designed IFS in D-metric space and established collage theorem in D-metric space. In 2012, S. C. Shrivastava and Padmavati [23] introduced an Iterated Function System due to Reich. Recently, BhagwatiPrasad [24] obtained fractal sets for such A-Iterated Function System and Iterated Multifunction System satisfying some general contractive condition using projection onto sets.

In this paper, we introduce an extension of the "usual" IFS method namely "A-Iterated Function System" or "AIFS" which includes the IFS originally studied by Hutchinson [9] and Barnsley et al. [10]. We also derived a Collage theorem for AIFS.

2. Preliminaries

This segment define some essential concepts and hypotheses which is valuable for demonstrating our outcome.

Definition 2.1[25]: A self-map f is called contraction map on a complete metric space (X, d)if∃ acontractivity factor k with $0 \leq k < 1$ such that

$$d\big(f(x), f(y)\big) \leq kd(x, y) \qquad \forall x, y \in X$$

Definition 2.2[6]: Assume that $A \neq \emptyset$ consisting of all functions $\alpha: R_+^3 \to R_+$, if following properties are satisfied:

1. α be the continuous function on the set R_+^3 (regarding the Euclidean metric on R^3).
2. $a \leq kb$ for any $k \in [0,1)$ when $a \leq \alpha(a,b,b)$ either $\alpha \leq (b,a,b)$ for all a,b.

Definition 2.3[6]: A self map f is said to be A-contraction map on a complete metric space (X,d), if it satisfies theconsequtiveinequality,

$$d(f(x),f(y)) \leq \alpha\Big(d(x,y), d\big(x,f(x)\big), d\big(y,f(y)\big)\Big) \qquad \forall x,y \in X \text{ and some } \alpha \in A.$$

Remark 2.1 [6]: Every k-contraction and R-contraction is A-contraction. In any case, the opposite may not be possible.

Theorem 2.1[25]: A contraction mapping $f: X \to X$ with contractivity factor 'k', on a complete metric space (X,d). Then f has a exactly one fixed point x^* in X. Moreover, for every x in X, the sequence $\{f^n(x)\}_{n\in N}$ converges to x^*. i.e., $\lim_{n\to\infty} f^n(x) = x^*$, for any $x \in X$.

Definition 2.4[10]: Consider (X,d) be a complete metric space and $K_0(X)$ denotes the family of all non-empty compact subset of X. Assume $a,b \in X$ and $B,C \in K_0(X)$. Then the Hausdorff distance between two sets will be expressed as follows;

$h_d(B,C) = max\{d(B,C), d(C,B)\}$

and the distance between a point a to a set C is expressed as follows,

$d(a,C) = inf\{d(a,c): c \in C\}$,

Where;

$$d(B,C) = sup\{d(a,C): a \in B\}.$$

Then, the set $(K_0(X), h_d)$ is said to be Hausdorff metric space (or fractal space).

Definition 2.5 ([9], [10]): If a metric space (X,d) is complete, then the Hausdorff metric space $(K_0(X), h_d)$ is also complete.

Definition 2.6 ([9], [10]): A hyperbolic IFS consisting of a compete metric space (X,d) forthwith a definite collection of continuous mappings $f_n: X \to X$ with regard to contractivityratio k_n, where $n \in N$. The representation of the IFS is $\{X; f_n, n = 1,----,N\}$ and its contractivity ratios $k = max\{k_n: n = 1,---,N\}$.

"IFS"definedin a simple way a definite collection of operatorsperforming on a metric space. The successive resultshows the fundamental details for IFS.

Theorem 2.2([9],[10]): Consider an IFS $\{X; f_n, n = 1,----,N\}$ alongcontractivity ratios.Then the mapping $F: K_0(X) \to K_0(X)$ represented by $F(B) = \cup_{n=1}^{N} f_n(B)\, \forall B \in K_0(X)$, be a contraction mapping on the complete metric space $(K_0(X), h_d)$ alongcontractivity ratios k. i.e.,

$h_d\big(F(B), F(C)\big) \leq k h_d(B,C)$,

Then F has the exactly one fixed point (fractal or attractor) $A^* \in K_0(X)$, Observe that,

$$A^* = F(A^*) = \cup_{n=1}^{N} f_n(B),$$

Which is provided by, $A^* = \lim_{n\to\infty} F^{on}(B)$ for any $B \in K_0(X)$.

Where; F^{on} represents the $n-$ fold composition of F.

3. A-iterated function system

Now, we present in this section the standard concepts of A-Iterated Function System based on classical framework of Hutchinson's IFS which was provided by ([9], [10]).Initially, in this investigation we introduce and prove the following lemmas, which is helpful for finding the attractor in the framework of A-contraction mapping.

Lemma 3.1: Consider(X, d) be a complete metric space and $f: X \to X$ be an A-contraction mapping for any $x, y \in X$ and some $\alpha \in A$. Then

1. $f$maps the elements $K_0(X)$ to the element $K(X)$.
2. If for every $B \in K_0(X)$;

$f(B) = \{f(x): x \in B\}$

Then $f: K_0(X) \to K_0(X)$ is also a$A -$ contraction on $(K_0(X), h_d)$.

Proof:(i) Consider $f$be a continuous mapping.
Consequently; by Lemma 2 of Ref [10], f maps $K_0(X)$ intoitself.
Thus, image of a compact subset under $f: X \to X$ is compact, that is
$B \in K_0(X) \Rightarrow f(B) \in K_0(X)$

(ii) Assume that,$B, C \in K_0(X)$.

Now,

$$h_d\big(f(B), f(C)\big) = d\big(f(B), f(C)\big) \vee d\big(f(C), f(B)\big)$$
$$\leq \alpha\{[d(B,C), d(B, f(B)), d(C, f(C))] \vee [d(C,B), d(C, f(C)), d(B, f(B))]\}$$
$$\leq \alpha[d(B,C), d(B, f(B)), d(C, f(C))]$$
$$\leq \alpha[h_d(B,C), h_d(B, f(B)), h_d(C, f(C))]$$

Therefore;

$$h_d\big(f(B), f(C)\big) \leq \alpha[h_d(B,C), h_d(B, f(C)), h_d(C, f(C))]$$

Thus, the desired verification completes.

Lemma 3.2: Consider(X, d) be a complete metric space and $\{f_n\}_{n\in N}$is a continuous A-contraction mappings on$(K(X), h_d)$. Let$F: K_0(X) \to K_0(X)$be represented by,

$$F(B) = f_1(B) \cup f_2(B) \cup - - - - - \cup f_N(B) = \bigcup_{n=1}^{N} f_n(B)$$

for any$B \in K_0(X)$. Then F be an A-contraction mapping on $(K_0(X), h_d)$.

Proof: We have to using mathematical induction method to prove of the lemma;

The lemma is obviously accurate for $N = 1$.
Now, for $N = 2$, we see that
Let $f_1, f_2: X \to X$ are two A-contractions.

Assume that$B, C \in K_0(X)$, According to the Lemma (3.1)

$$h_d\big(F(B), F(C)\big) = h_d\left(f_1(B) \cup f_2(B), f_1(C) \cup f_2(C)\right)$$
$$\leq h_d\big(f_1(B), f_1(C)\big) \vee h_d\big(f_2(B), f_2(C)\big)$$
$$\leq \alpha_1[h_d(B,C), h_d(B, f_1(B)), h_d(C, f_1(C))] \vee \alpha_2[h_d(B,C), h_d(B, f_2(B)), h_d(C, f_2(C))]$$
$$\leq (\alpha_1 \vee \alpha_2)[h_d(B,C), h_d(B, f_1(B)) \vee h_d(B, f_2(B)), h_d(C, f_1(C)) \vee h_d(C, f_2(C))]$$
$$\leq \alpha[h_d(B,C), h_d(B, f_1(B) \cup f_2(B)), h_d(C, f_1(C) \cup f_2(C))]$$

$= \alpha[h_d(B,C), h_d(B,F(B)), h_d(C,F(C))]$

Therefore;

$$h_d(F(B),F(C)) \leq \alpha[h_d(B,C), h_d(B,F(B)), h_d(C,F(C))]$$

Lemma (3.2) is demonstrated by the method of mathematical induction.

Hence; it can be possible to describe AIFSwith supportof previous provided outcomes and definitions. Now, we introduce an interesting result for AIFS.

Theorem 3.1: An AIFS consisting of a complete metric space (X,d) forthwith a definite collection of contractions f_n along contractivityratiosk_n, where$n \in N$. It is expressed by $\{X;\ (f_0), f_1, f_2, ------, f_N\}$ where, f_0 denote the condensation mapping along contractivity ratios$k = max\{k_n : n = 1 ----, N\}$. Let$F: K_0(X) \to K_0(X)$ represented by

$$F(B) = f_1(B) \cup f_2(B) \cup ------ \cup f_N(B) = \bigcup_{n=1}^{N} f_n(B)$$

for any $B \in K_0(X)$ is a continuous $A-$ contraction mapping on the Hausdorff space $(K_0(X), h_d)$.Then F has an exactly one fixed point (attractor)$A^* \in K_0(X)$, that is

$$A^* = F(A^*) = \bigcup_{n=1}^{N} f_n(A^*)$$

Which isprovidedby; $A^* = \lim_{n\to\infty} F^{on}(A^*), \forall\, B \in K_0(X)$.Where, F^{on} represents the $n-$ fold composition of$F$.

Proof: Consider(X,d) is a complete metric space and each f_n is A-contraction.

Then $(K_0(X), h_d)$ is a complete Hausdorff metric space by definition (2.5).

Also, the HB operator F is A-contraction mapping by Lemma (3.2).

Hence, we conclude that F has a unique fixed point by theorem (2.1).

Finally, we formulate the Collage theorem by using the concept of proposition (3.3).

Theorem 3.2 Assume$V \in K_0(X)$and $\varepsilon \geq 0$ be given. Take an AIFS $\{X; (f_0), f_1, f_2, -----, f_N\}$, where $f_0$denote the condensation mapping , such that

$$h_d\left(V, \bigcup_{n=0,n=1}^{N} f_n(V)\right) \leq \varepsilon$$

Then;

$$h_d(V, A^*) \leq \varepsilon.\left(\frac{1}{1-k}\right)$$

Where; $A^* \in K_0(X)$is anattractor of the AIFS, Symmetrically, the equality

$$h_d(V, A^*) \leq \left(\frac{1}{1-k}\right).\, h_d\left(V, \bigcup_{n=0,n=1}^{N} f_n(V)\right)$$

holdsfor every$V \in K_0(X)$.

Proof: Applying triangular property of metric space, we know that

$$h_d(V, F^n(V)) \leq h_d(V, F(V)) + h_d(F(V), F^n(V))$$
$$\leq h_d(V, F(V)) + [h_d(F(V), F^2(V)) + h_d(F^2(V), F^n(V))]$$
$$\leq h_d(V, F(V)) + h_d(F(V), F^2(V)) + ------- + h_d(F^{n-1}(V), F^n(V)) \quad (1)$$

Since, F is an A-contraction mapping, we have

$$h_d\left(F^n(V),F^{n+1}(V)\right)=h_d\left(F\left(F^{n-1}(V),F\left(F^n(V)\right)\right)\right)$$
$$\leq \alpha\left(h_d\left(F^{n-1}(V),F^n(V)\right),h_d\left(F^{n-1}(V),F^n(V)\right),h_d\left(F^n(V),F^{n+1}(V)\right)\right)$$
$$\leq k\left(h_d\left(F^{n-1}(V),F^n(V)\right)\right)$$
$$\leq k^2\left(h_d\left(F^{n-2}(V),F^{n-1}(V)\right)\right)$$

$$\leq k^n\left(h_d\left(V,F(V)\right)\right)$$

From equation (1), we get

$$h_d\left(V,F^n(V)\right)\leq h_d\left(V,F(V)\right)+kh_d\left(V,F(V)\right)+k^2h_d\left(V,F(V)\right)+\;-----+k^{n-1}h_d\left(V,F(V)\right)$$
$$\leq [1+k+k^2+--------+k^{n-1}]h_d\left(V,F(V)\right)$$
$$\leq\left[\frac{1-k^n}{1-k}\right]h_d\left(V,F(V)\right);\ \text{for} k<1$$

Taking the limit $n\to\infty$; we get

$$h_d(V,A^*)\leq\left(\frac{1}{1-k}\right)h_d\left(V,F(V)\right)$$
$$h_d(V,A^*)\leq\left(\frac{1}{1-k}\right)h_d\left(V,\bigcup_{n=0,n=1}^{N}f_n(V)\right)$$

Holds for all$V\in K_0(X)$.

4. Conclusion

The study presented an AIFS and the fractal is obtained for AIFS in the new setting. Finally, in this analysis formulated theCollage theorem.

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