# Pairwise Locally Compact Space and Pairwise Locally Lindelőf Space

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**Abstract**. In this paper we define pairwise locally compact space and pairwise locally lindelöf space and study their properties and their relations with other bitopological spaces. several examples are discussed and many will known theorems are generalized concerning pairwise locally compact space and pairwise locally lindelöf space. and we shall investigate subspaces of pairwise locally compact space and pairwise locally lindelöf space and also bitopological spaces which are related to pairwise locally compact space and pairwise locally lindelöf space.

**Keywords**: pairwise locally compact space, pairwise locally lindelöf space, Pairwise regular, pairwise completely regular, Pairwise paracompact:

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## Introduction

In [3],Kelly introduced the notion of a bitopological space, i.e.

a triple $(X,\tau_1,\tau_2)$ where X is a non-empty set and $\tau_1$, $\tau_2$ are two topologies on X. He also defined pairwise regular (P−regular), pairwise normal (P−normal), and obtained generalization of several standard results such as Urysohn′s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim in [4] and Fletcher in [1] . Also Fletcher in [1] gave the definitions of $\tau_1\tau_2$−open and P−open covers in bitopological spaces as A cover Ủ of the bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$ -open if

Ủ$\subseteq \tau_1 \cup \tau_2$. If, in addition, Ủ contains at least one non-empty member of $\tau_1$ and at least one non-empty member of $\tau_2$, it is called p-open cover.

Dissanayake in[5]studied some properties of locally lindelöf space. Also, in 1972 Ivan in [6] defined a bitopological local compactness as $(X, \tau_1, \tau_2)$ is a bitopological space then $\tau_1$ said to be locally compact with respect to $\tau_2$ if for each point $x \in X$, there is a $\tau_1$ open neighborhood of x whose $\tau_2$ closure is pairwise compact and $(X, \tau_1, \tau_2)$ is pairwise locally compact if $\tau_1$ is locally compact with respect to $\tau_2$ and $\tau_2$ is locally compact with respect to $\tau_1$.

A.FORA and H.HDEIB [2] in 1983 give a definition of pairwise lindelöf bitopological spaces and derive some related results.

Let R, I, N denote the set of all real numbers, the interval [0,1], and the natural numbers respectively. Let $\tau_d$, $\tau_u$, $\tau_c$, $\tau_l$, $\tau_r$, $\tau_{ind}$ denote the discrete, Usual, cocountable, left ray, right ray, and the indiscrete topologies on R (or I). Also, The $\tau_i$ closure of a set A will be denoted by $cl_{\tau i}A$.

***Definition 1.1.*** A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise compact (p-compact) if every pairwise open cover(p-pen cover) has a finite subcover which contains at least one non empty member of $\tau_1$ and at least one non empty member of $\tau_2$.

***Definition 1.2.*** A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise locally compact if for each point x ϵ X, there is a $\tau_1$-open neighbourhood of x whose $\tau_1$-closure is pairwise compact or a
$\tau_2$-open neighbourhood of x whose $\tau_2$-closure is pairwise compact.
To illustrate the above definition of a p- locally compact.

***Example 1.3.*** Let X= $\mathbb{R}$ . $\tau_1$ ={ ϕ, $\mathbb{R}$, $\mathbb{R}$-{1}}. $\tau_2$ ={ ϕ, $\mathbb{R}$, $\mathbb{R}$-{2}}. Then $(\mathbb{R}, \tau_1, \tau_2)$ is a pairwise locally compact.

To show this, let x ϵ $\mathbb{R}$, then any $\tau_1$-open neighbourhood of x, such that case 1: if x= {1}, $U_x$= $\mathbb{R}$ is an open set containing 1, cl($\mathbb{R}$)= $\mathbb{R}$ which is pairwise compact.

Case 2: if x ϵ $\mathbb{R}$-{1}, cl($\mathbb{R}$-{1})=$\mathbb{R}$ (in $\tau_1$) which is pairwise compact. On the hand, case 1: if x= {2}, $U_x$= $\mathbb{R}$ is an open set containing 2, cl($\mathbb{R}$)= $\mathbb{R}$ which is pairwise compact.

Case 2: if x ϵ $\mathbb{R}$-{2}, cl($\mathbb{R}$-{2})=$\mathbb{R}$ (in $\tau_2$) which is pairwise compact. So $(\mathbb{R}, \tau_1, \tau_2)$ is a pairwise locally compact.

***Example 1.4.*** Let X= $\mathbb{R}$ . Then $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ is a pairwise locally compact.

To show this, let x ϵ $\mathbb{R}$, let $U_1$ be any $\tau_{dis}$ -open neighborhood containing x, any p-open cover of $U_1$ must contains $U_1$ or $V_1$ such that $U_1 \subseteq V_1$, so{$U_1$} is a finite subcover or {$V_1$:$U_1 \subseteq V_1$} is a finite subcover.

On the other hand, let x ϵ $\mathbb{R}$, let $U_2$ be any $\tau_{coc}$ -open neighborhood containing x, then cl($U_2$) is $\mathbb{R}$ (which is pairwise compact) or a finite set(which is pairwise compact).

***Example 1.5.*** Let X= $\mathbb{R}$. Then $(\mathbb{R}, \tau_l, \tau_{dis})$ is not a pairwise locally compact.

To show this, let x ϵ $\mathbb{R}$, let U be any $\tau_l$-open neighborhood containing x, then U= $\mathbb{R}$, but cl($\mathbb{R}$)=$\mathbb{R}$ which is not pairwise compact.

For any pairwise open cover say
$\mathring{U}$= {(-∞, x): x ϵ $\mathbb{R}$}∪{{x}: x ϵ $\mathbb{R}$} which has no finite subcover.

***Theorem 1.6.*** If X is finite, then $(X, \tau_1, \tau_2)$ is pairwise locally compact.

Proof. Assume that X= {$x_1$, $x_2$, $x_3$,…, $x_n$}. Let x ϵ X, and $U_x$ be a
$\tau_l$-open set or a $\tau_2$ open set such that x ϵ $U_x$. without loss of generalization assume $U_x$ ϵ $\tau_1$, then x ϵ $U_x$. cl($U_x$) is finite, so cl($U_x$) is pairwise compact.

***Theorem 1.7.*** Every pairwise compact space is a pairwise locally compact.

Proof. Assume that $(X, \tau_1, \tau_2)$ is pairwise compact. Let x ϵ X, and let $U_x$ be a $\tau_l$-open set or a $\tau_2$ open set such that x ϵ $U_x$. without loss of generalization assume $U_x$ ϵ $\tau_1$, then we show

that cl($U_x$) is a pairwise compact in $\tau_1$. Let Ủ be a pairwise open cover of cl($U_x$) subset of (cl($U_x$), $\tau_1^*$, $\tau_2^*$), Where
$\tau_1^*$={U ∩ cl($U_x$): U∈ $\tau_1$} and $\tau_2^*$={V ∩ cl($U_x$): V∈ $\tau_2$}. Then Ủ ∪{X- cl($U_x$)} is a pairwise open cover of a pairwise compact space (X, $\tau_1$, $\tau_2$), so it has a finite subcover of X. Hence Ủ for cl($U_x$). So cl($U_x$) is pairwise compact.

The converse of above theorem need not be true, as shown in the following example:

***Example 1.8.*** Let X= ℝ . Then (ℝ, $\tau_{dis}$, $\tau_{coc}$) is a pairwise locally compact, but not pairwise compact.
To see this; let Ủ={$A_x$}: x ∈ Q}∪{$Q^c$; $Q^c$ ⊆ $\tau_{coc}$} be an pairwise open cover.
If Ủ has a finite subcover, then
ℝ ⊆ $Q^c$ ∪ {{$x_1$}, {$x_2$},…, {$x_n$}: $x_i$ ∈ Q; i=1, 2, ., ., ., n}: which is impossible.

***Theorem 1.9. :*** A pairwise locally compactness is hereditary with respect to closed subspace.
***Proof.*** Let F be a closed subspace in a pairwise locally compact space X= (X, $\tau_1$, $\tau_2$). Let x∈ F, so x∈ X, since X is a pairwise locally compact, so there exists an open set containing x in $\tau_1$ or in $\tau_2$ say W such that cl(W) is pairwise compact. Thus F∩W is open in F with respect to $\tau_1$ or $\tau_2$, and x∈ F∩W, and
cl(F∩W)$^F$ = cl(F∩W)∩F ⊆ cl(F) ∩ cl(W) ∩ F =F∩ cl(W)
⊆ cl(W) which is pairwise compact. We get cl(F∩W)$^F$ is pairwise compact. Hence the result.

***Theorem 1.10. :*** A pairwise locally compactness is hereditary with respect to open subspace.
***Proof.*** Let V be an open subspace in a pairwise locally compact space X. Let x∈ V, so x∈ X, since X= (X, $\tau_1$, $\tau_2$) is a pairwise completely regular space, then X is a pairwise regular space, since V is open, there exists an open set say $U_x$ in X with respect to $\tau_1$ or $\tau_2$ such that x∈ $U_x$⊆ cl($U_x$) ⊆ V, now; since X is pairwise locally compact, there exists an open set in X containing x in $\tau_1$ or in $\tau_2$ say $W_x$ such that cl($W_x$) is pairwise compact.
So x∈ $U_x$ ∩ $W_x$= $M_x$, and $M_x$ is open in V. Since $M_x$ ⊆ $U_x$ and $U_x$⊆ V, so $M_x$ ⊆ V, cl($M_x$) ⊆ cl($W_x$), and since cl($W_x$) is pairwise compact and cl($M_x$) is closed, so cl($M_x$) is pairwise compact in V. Therefore V is pairwise locally compact.

***Corollary 1.11.*** : A pairwise locally compactness is hereditary with respect to intersection of open subspace and closed subspace.
***Proof.*** Let X= (X, $\tau_1$, $\tau_2$) be a pairwise locally compact space, and F be a closed subspace of X. and V be an open subspace of X. Now, F∩V is open in F. Because V is open subspace in X. By a previous theorem ***1.9.*** F is a pairwise locally compact, and by a previous theorem ***1.10.*** F∩V is a pairwise locally compact in F, and so in X.

Now, will define a definition of a pairwise locally lindelöf.
***Definition 1.12.*** A bitopological space (X, $\tau_1$, $\tau_2$) is said to be pairwise lindelöf (p- lindelöf) if every pairwise open cover(p-open cover) has a countable subcover which contains at least one non empty member of $\tau_1$ and at least one non empty member of $\tau_2$.

***Definition 1.13.*** A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise locally lindelöf if for each point x ϵ X, there is a $\tau_1$-open neighbourhood of x whose $\tau_1$-closure is pairwise lindelöf or a $\tau_2$-open neighborhood of x whose $\tau_2$-closure is pairwise lindelöf.

***Example 1.14.*** $(\mathbb{N}, \tau_{dis}, \tau_{ind})$ is a pairwise locally lindelöf.
To show this ; let n ϵ ℕ, then $\{n\} \epsilon \tau_{dis}$, and $cl_{\tau dis}(\{n\}) = \{n\}$ which is pairwise lindelöf.
If n ϵ ℕ, then the only $\tau_{ind}$- open set containing {n} is ℕ, so
$cl_2(\mathbb{N}) = \mathbb{N}$, which is pairwise lindelöf.

***Example 1.15.*** $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is a not a pairwise locally lindelöf.
Clearly.

***Remark 1.16.*** Every pairwise lindelöf space is a pairwise locally lindelöf.

***Theorem 1.17.*** Every pairwise locally compact space is a pairwise locally lindelöf.
Proof. Let $(X, \tau_1, \tau_2)$ be a pairwise locally compact. Let x ϵ X , then there exists $\tau_1$-open neighbourhood of x say $U_x$ whose $\tau_1$-closure is pairwise compact, or a $\tau_2$-open neighbourhood of x say $V_x$ whose $\tau_2$-closure is pairwise compact, following that $cl_{\tau 1}(U_x)$ is pairwise lindelöf or $cl_{\tau 2}(V_x)$ is pairwise lindelöf . So $(X, \tau_1, \tau_2)$ is a pairwise locally lindelöf.

***Example 1.18.*** $(\mathbb{R}, \tau_u, \tau_u)$ is a pairwise locally lindelöf . Since it is a pairwise locally compact.

***Theorem 1.19.*** If X is countable, then $(X, \tau_1, \tau_2)$ is pairwise locally lindelöf.
Proof. Assume that $X = \{x_1, x_2, x_3, \ldots\}$. Let x ϵ X, and $U_x$ be a $\tau_l$-open set or a $\tau_2$ open set such that $x \epsilon U_x$. without loss of generalization assume $U_x \epsilon \tau_1$, then $x \epsilon U_x$. $cl(U_x)$ is countable, so $cl(U_x)$ is pairwise compact.

***Theorem 1.20.*** If $(X, \tau_1, \tau_2)$ is pairwise lindelöff and *A* is a subset of X which is $\tau_1$ closed then A ia pairwise lindelöf.

Proof. Let Ů be any pairwise open cover of the subspace $(A, \tau_1^*, \tau_2^*)$. Where $\tau_1^* = \{U \cap A: U \epsilon \tau_1\}$ and $\tau_2^* = \{V \cap A: V \epsilon \tau_1\}$. Then

Ů ∪ {X - A} induces a pairwise open cover of $(X, \tau_1, \tau_2)$ which has a countable subcover for X, and hence so does for *A.*

***Example 1.21.*** Let $\tau_1 = (\mathbb{R}, \tau_{left})$ be the left ray topology and $\tau_1 = (\mathbb{R}, \tau_{right})$ be the right ray topology. Now; $(\mathbb{R}, \tau_{left})$ and $(\mathbb{R}, \tau_{right})$ are not locally lindelöf.

To show this for $(\mathbb{R}, \tau_{left})$; let x ϵ ℝ, and let $U_x \epsilon \tau_{left}$ be open neighborhood of x, then $U_x = \{(-\infty, a): x < a\}$, $cl_{left}(U_x) = \mathbb{R}$ which is not lindelöf. The same for $(\mathbb{R}, \tau_{right})$.

However, $(\mathbb{R}, \tau_{left}, \tau_{right})$ is a pairwise locally lindelöf. To show this; let x ϵ ℝ, and let $U_x \epsilon \tau_{left}$ be open neighborhood of x, then

$U_x$= {(-∞,a): x<a}, $cl_{right}(U_x)$= (-∞, a], which is a pairwise lindelöf.

***Theorem 1.22.*** If (X, $\tau_1, \tau_2$) is pairwise lindelöf and A is a subset of X which is $\tau_i$ (i=1,2) closed set, then A ia pairwise lindelöf. Proof. Let Ủ be any pairwise open cover of the subspace (A, $\tau_1^*, \tau_2^*$). Where $\tau_1^*$={U ∩ A: U ϵ $\tau_1$} and $\tau_2^*$={V ∩ A: Vϵ $\tau_1$}. Then

Ủ ∪ {X - A} induces a pairwise open cover of (X, $\tau_1, \tau_2$) which has a countable subcover for X, and hence so does for *A.*

***Corollary 1.23.*** If (X, $\tau_1, \tau_2$) is pairwise locally lindelöf and *A* is a subset of X which is $\tau_i$ (i=1,2) closed set, then A is pairwise locally lindelöf.

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