A nonlocal Cauchy problem for abstract Hilfer equation with fractional integrated semi groups

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Abstract: Nonlinear fractional Hiller differential equations with fractional integrated semi groups are studied in Banach space. Nonlocal Cauchy problem is considered. Existence and uniqueness theorems are proved. The stability of the considered Cauchy problem is studied

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1. Introduction

Let us consider the following equation.

$$u(t) = u_0 + \varphi(t)\ \ h_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Au(s) + g(s)]ds + \frac{1}{\Gamma(\gamma_1)}\int_0^t (t-s)^{\gamma_1-1}\,[\,F(s, B(s)u(s) + b(s)h_2\,]\,ds\ , \qquad (1)$$

where $0 < \alpha \le 1$, $0 < \gamma_1 \le 1$, u_0 and h_1 are given elements in Banach space E, A is a linear closed operator defined on a dense set $S1 \subset E$ and with values in E, $g(.)$ is a map defined on the closed interval $[0, T]$ and with values in E , $T > 0$, b is a real function defined on J,

$$\varphi(t) = \frac{t^{(1-\alpha)(1-\gamma)}}{\Gamma(\gamma(1-\alpha)+\alpha)}, \qquad 0 < \gamma \le 1$$

$$h_2 = \sum_{i=1}^{p} c_i u(t_i), \qquad 0 < t_1 < t_2 < \cdots < t_p < T,$$

$c_1;\ \ldots\ ;\ c_p$ are real numbers, $B(t)$, $t \in J$ is a family of linear closed operators defined on $S2 \supset S1$, F is map defined on $J \times E$ and with values in E . It is assumed that F satisfies the following Lipchitz condition

$$\|F(t_2, V_2) - F(t_1, V_1)\| \le M[|t_2 - t_1| + \|V_2 - V_1\|] \qquad , \qquad (2)$$

for all t_2 , $t_1 \in J$, $V_1, V_2 \in E$, where M is a positive constant and $\|\,.\,\|$ is the norm in E .It is supposed that the operator A generates $\beta -$ times integrated semi groups $Q(t)\colon t \ge 0$, where $Q(t)\colon t \in [0,1)$ is a family of linear bounded operators on E to E , with the following properties:

(I) $Q(t)$ is strongly continuous on $[0; 1)$,

(II) There exist positive constants, M_1 and M_2 such that

$$\|\|\ Q(t)\ \| \le M_1 e^{M_2 t}\| , \qquad t \ge 0$$

The interval $[M_2, \infty)$ is contained in the resolvent of A and $(I\lambda - A)^{-1} = \lambda^{\beta} \int_0^{\infty} for\ all\ \lambda > M_2$ where I is the identity operator defined on E, $0 < \beta < 1$,

(III) $\|AQ(t)h\| \le \frac{k}{t}\|h\|$, for all $t > 0$, $h \in E$,

(IV) $\|B(t_2)Q(t_1)h\| \le \frac{k}{t_1^{\delta}}\|\,h\|$, $t_2 \in J$, $t_1 > 0$, $0 < \delta < 1$, $h \in E$,

(V) $\beta(t)h \in \boldsymbol{C}_E(J)$, for every $h \in S_2$, where $\boldsymbol{C}_E(J)$ is the set of all continuous functions f on J , with respect to the norm in E , such that $f(t) \in S$ for every $t \in J$. Notice that $Q(t)h$ satisfies the following representation:

$$Q(t)h = \frac{t^{\beta}h}{\Gamma(1+\beta)} + \int_0^t Q(s)Ah\,ds\ \ , h \in S_1, \qquad (3)$$

(See [1],[2],[3],[4],[5])

In section 2 , we shall study a special case, when $\left(t, B(t)u(t)\right) = 0$, (the zero element in E) and $b(t) \equiv 0$. In section 3 , we shall solve the general nonlinear case. Some properties are also studied under suitable conditions. The results in this paper can be considered as a generalization of our previous results in ([4],[5]). There are many important applications of the nonlocal Cauchy problems for Hilfer fractional differential equations with integrated semi groups, (see [6],[7],[8],[9],[10],[11],[12],[13],[14]).

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2. Strong solutions

Consider the following linear fractional Hilfer abstract differential equation

$$u(t) = u_o + \varphi(t)h_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[Au(s) + g(s)]ds. \tag{4}$$

Equation (4) can be written in the form

$$u(t) = u_o + \varphi(t)h_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[Au(s) + g_1(s)]ds + \frac{t^\alpha}{\Gamma(\alpha+1)}g(0), \tag{5}$$

where $g_1(t) = g(t) - g(0)$. Consequently,

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-\alpha\beta}\, g_3(s)ds, \tag{6}$$

Where

$$g_2(t) = \frac{1}{\Gamma(1-\alpha\beta)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha\beta}\, g_1(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha\beta)}\int_0^t (t-s)^{-\alpha\beta}\, g_1(s)ds$$

$$g_3(t) = g_2(t) + \phi_1(t)u_1 + \phi_2(t)g(0)$$

$$\phi_1(t) = \frac{\Gamma(1+\gamma_2)t^{\gamma_2-\gamma_1}}{\Gamma(\gamma_3)\Gamma(1+\gamma_2-\gamma_1)}$$

$$\phi_2(t) = \frac{t^{-\alpha\beta}}{\Gamma(1-\alpha\beta)}$$

$\gamma_1 = \alpha\beta + \alpha$,

$\gamma_2 = (1-\alpha)(1-\gamma)$,

$\gamma_3 = \gamma(1-\alpha) + \alpha$,

It is supposed that $\alpha\beta < \gamma_2$. Thus $\gamma_2 - \gamma_1 + \alpha > 0$, $\alpha\beta + \alpha < 1$.

Theorem 2.1 Suppose that $\frac{dg_1(t)}{dt} \in \boldsymbol{C}_{S_1}(J)$, $g \in \boldsymbol{C}_{S_1}(E)$ if $u_0, Au_0, h_1 \in S_1$, then there exists a unique function $u \in \boldsymbol{C}_{S_1}(J)$ such that u satisfies equation (4)

Proof: It easy to get from (6):

$$u^*(s) = s^{\alpha-1}(s^\alpha I - A)^{-1}u_0 + s^{-\alpha\beta}(s^\alpha I - A)^{-1}g^*(s), \tag{7}$$

Where $u^*(s)$ and $g^*(s)$ are the Laplace transform of $u(t)$ and $g_3(t)$ respectively. From (7) and property (3), one gets:

$$u^*(s) = s^{\gamma_1-1}\int_0^\infty e^{-ts^\alpha}Q(t)u_0 dt + \int_0^\infty e^{-ts^\beta}Q(t)g^*(s)dt \tag{8}$$

From (3):

$Q(0)h = 0$, for every $h \in S_1$. (9)

Using the results in [4], we get from (8), (9) and the simple facts about the Laplace transform of fraction of derivatives, the following representation

$$u(t)=\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty \xi_\alpha(\theta)Q(t^\alpha o)u_0 d\theta+\int_0^t\int_0^\infty \alpha\theta(t-\tau)^{\alpha-1}\xi_\alpha(\theta)Q((t-\tau)^\alpha\theta)g_3(t)d\theta d\tau \qquad (10)$$

Where $\xi_\alpha(t)$ is a probability function defined on $(0,\infty)$ and satisfies the following identity:

$$\int_0^\infty e^{\theta s}\xi_\alpha(\theta)d\theta=E_\alpha(s), \qquad (11)$$

Where $E_\alpha(s)$ is the Mittag-Loffler function defined by

$$E_\alpha(s)=\sum_{j=0}^{\infty}\frac{s^j}{\Gamma(1+\alpha j)}$$

Using (3) and (10), we get

$$u(t)=\Psi_1(t)u_o+\int_0^t\Psi_2(t_i-\tau)g_3(\tau)d\tau\ , \qquad (12)$$

where

$$\Psi_1(t)=I+\frac{1}{\Gamma(1-\alpha\beta)}\int_0^t\tau^{-\alpha\beta}\Psi_2(t-\tau)Ad\tau,$$

$$\Psi_2(t)=\int_0^\infty\alpha\theta t^{\alpha-1}\xi_\alpha(\theta)Q(t\theta)d\theta.$$

From property II and (11), we can find a constant $M>0$, such that,

$$\|u(t)-u_0\|\le Mt^{\alpha(1-\beta)}[\|g(0)\|+\|Au_0\|]+Mt^{\gamma_2-\gamma_1+\alpha}\|u_1\|$$

(See [[15]-[22]]). It is clear from (12) that $u\in \boldsymbol{C}_{S_1}(J)$.

3. Nonlinear equations

Consider the following equation:

$$u(t)=u_0+\varphi(t)h_1+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[Au(s)+g(s)]ds$$

$$+\frac{1}{\Gamma(\gamma_1)}\int_0^t(t-s)^{\gamma_1-1}\big[F\big(s,B(s)u(s)\big)+b(s)h_2\big]ds \qquad (13)$$

We can write (13) in the form

$$u(t)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds+\frac{1}{\Gamma(\gamma_1)}\int_0^t(t-s)^{\gamma_1-1}\big[F\big(s,B(s)u(s)\big)+b(s)h_2+g_3(s)\big]ds \qquad (14)$$

Set:

$$V(t)=F\big(t,B(t)u(t_1)\big)+b(t)h_2 \qquad (15)$$

Thus we can write formally:

$$u(t)=\Psi_1(t)u_0+\int_0^t\Psi_2(t-\tau)[g_3(\tau)+V(\tau)]d\tau \qquad (16)$$

If equation (15) has a solution $V\in C_E(J)$, we call formula (16) a mild solution of equation (13).

Notice that

$$h_2=\sum_{i=1}^{p}c_1\Psi_1(t_i)u_0+\sum_{i=1}^{p}c_i\int_0^{t_i}\Psi_2(t_i-\tau)V(\tau)d\tau+\sum_{i=1}^{p}c_i\int_0^{t_i}\Psi_2(t_i-\tau)g_3(\tau)d\tau\ . \qquad (17)$$

Theorem 3.1 Equation (13) has a unique mild solution

Proof: Let us prove the uniqueness of the mild solution. Let u_1 and u_2 be two mild solutions of equation (13) and

$$V_j(t)=F\Big(t,B(t)u_j(t)\Big)+\sum_{i=1}^{p}c_iu_j(t_i)\,,\qquad j=1,2 \qquad (18)$$

From properties (2), (5) and (2), (16), (17), (18) one can find a constant $M>0$ such that

$$\|V_1(t) - V_2(t)\| \leq M\sum_{i=1}^{p}|c_i|\int_0^{t_i}(t_i-\tau)^{\alpha-1}\|V_1(\tau) - V_2(\tau)\|d\tau + M\int_0^{t}(t-\tau)^{\gamma_4-1}\|V_1(\tau) - V_2(\tau)\|d\tau \quad (19)$$

Where $\gamma_4 = \alpha(1 - \delta)$.
Let

$$\boldsymbol{H} = \max_{t\in J}\ [e^{-\lambda t}\|V_1(t) - V_2(t)\|]$$

where $\lambda > 0$

$$\int_0^t (t-\tau)^{\gamma_4-1}\|V_1(\tau) - V_2(\tau)\|d\tau \leq \lambda^{1-\gamma_4}H\int_0^{t-\frac{1}{\lambda}}e^{\lambda\tau}d\tau + H\int_{t-\frac{1}{\lambda}}^{t}e^{\lambda\tau}(t-\tau)^{\gamma_4-1}d\tau \leq (\frac{1}{\lambda})^{\gamma_4}[1+\frac{1}{\gamma_4}]He^{\lambda\tau}$$

(20)

$$\sum_{i=1}^{p}\int_0^{t_i}|c_i|\int_0^{t_i}(t-\tau)^{\alpha-1}\|V_1(t) - V_2(t)\|dt \leq \sum_{i=1}^{p}|c_i|(\frac{1}{\lambda})^{\alpha}[1+\frac{1}{\alpha}]He^{\lambda\tau} \quad (21)$$

From (19), (20), (21), we get

$$e^{-\lambda\tau}\|V_1(t) - V_2(t)\| \leq (\frac{1}{\lambda})^{\gamma_4}[1+\frac{1}{\gamma_4}][1+\sum_{i=1}^{p}|c_i|\, e^{\lambda T-\lambda\delta}]$$

For sufficiently large λ , one gets

$$(\frac{1}{\lambda})^{\gamma_4}[1+\frac{1}{\gamma_4}] < \frac{1}{2}$$

Now if $\sum_{i=1}^{p}|c_i| \leq e^{\lambda T}$. we get

$$H \leq cH$$

Where c ϵ $\left(0,\frac{1}{2}\right)$.

Thus

$$\boldsymbol{H} = \max_{t\in J}\ [e^{-\lambda t}\|V_1(t) - V_2(t)\|] = 0$$

(See [23]-[30]).

To prove the existence, we define a sequence $\{V_k(t)\}$ where

$$V_{k+1}(\boldsymbol{t}) = \mathbf{F}(\mathbf{t}, \mathbf{B}(\mathbf{t})\mathbf{u_k}(\mathbf{t})) + \sum_{i=1}^{p}c_i u_k(t_i), \qquad k = 1, 2, \ \ldots$$

So

$$\|V_{k+1}(t) - V_k(t)\| \leq M\sum_{i=1}^{p}|c_i|\int_0^{t_i}(t_i-t)^{\alpha-1}\|V_k(t) - V_{k-1}(t)\|dt$$

$$+M\int_0^{t}(t_i-t)^{\gamma-1}\|V_k(t) - V_{k-1}(t)\|dt$$

where $M > 0$ is a constant. Thus

$\max_{t\in J}\ [e^{-\lambda t}\|V_{k+1}(t) - V_k(t)\|] \leq c\max_{t\in J}\ [e^{-\lambda t}\|V_k(t) - V_{k-1}(t)\|]$.

By induction, one gets

$$\max_{t\in J}\ [e^{-\lambda t}\|V_{k+1}(t) - V_k(t)\|] \leq c^k\max_{t\in J}\ [e^{-\lambda t}\|V_1(t) - V_0(t)\|]$$

Where $V_0(t)$ is the zero approximation, which can be taken the zero element in E.

Thus $\sum_{k=0}^{\infty} \|V_{k+1}(t) - V_k(t)\|$ uniformly converges on J. Thus the sequence $V_{n+1}(t) = \sum_{k=0}^{\infty}(V_{k+1}(t) - V_k(t))$ is uniformly convergent to an element $C_E(J)$.

Consequently $u \in C_E(J)$. Where

$$u(t) = \Psi_1(t)u_0 + \int_0^t \Psi_2(t-\tau)V(t)d\tau + \int_0^t \Psi_2(t-\tau)g_3(t)d\tau$$

Hence the required result.

Let us prove now the stability of solutions. Consider the following equations

$$\vartheta_n(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A\vartheta_n(s)ds + \frac{1}{\Gamma(\gamma_1)}\int_0^t (t-s)^{\gamma_1-1}\left[F\big(s,B(s)u_n(s)\big) + b(s)h_{2n}\right]ds$$

$$+\frac{1}{\Gamma(\gamma_1)}\int_0^t (t-s)^{\gamma_1-1}g_{3n}(s)ds \qquad (22)$$

Where

$\vartheta_n(t) = u_n(t) - u_{on}$

$h_{2n} = \sum_{i=1}^{n} c_i\vartheta_n(t_i)$,

$$g_{3n}(t) = g_{2n}(t) + \phi_2(t)g_n(0) + \phi_2(t)Au_{on} + \phi_1(t)h_{1n} + \left(b(t)\sum_{i=1}^{p} c_i\right)u_{on}$$

$g_{2n}(t) = \frac{1}{\Gamma(1-\alpha\beta)}\int_0^t (t-s)^{-\alpha\beta}\frac{dg_{1n}(s)}{ds}$,

$g_{1n}(t) = g_n(t) - g(0)$,

$\vartheta_n(t) = \int_0^t \Psi_2(t-\tau)V_n(\tau)d\tau + \int_0^t \Psi_2(t-\tau)g_{3n}(\tau)d\tau \qquad (23)$

Where

$V_n(t) = F(t, B(t)u_n(t)) + b(t)h_{2n} \qquad (24)$

Theorem 3.2 Suppose that the sequence $\frac{dg_n}{dt} \in C_E(J)$ uniformly converges on J to $\frac{dg}{dt} \in C_E(J)$. Suppose also that the sequences $\{u_{on} \in E\}$, $\{Bu_{on} \in E\}$, $\{Au_{on} \in E\}$ and $\{h_{in} \in E\}$ are convergent such that

$$\lim_{n\to\infty} u_{on} = u_o \in S_1$$

$$\lim_{n\to\infty} Bu_{on} = Bu_o \in E$$

$$\lim_{n\to\infty} Au_{on} = Au_o \in E$$

$$\lim_{n\to\infty} h_{1n} = h_1 \in E$$

Then the sequence $u_n \in C_E(J)$ of mild solutions of equation (22) uniformly converges on J to the mild solution $\{u \in C_E(J)\}$ of equation (13)

Proof: From (4), (23) and (24), one gets

$$\|V_n(t) - V_m(t)\| \le M\int_0^t (t-\tau)^{\gamma_4-1}\|V_n(\tau) - V_m(\tau)\|d\tau + M\,\|Bu_{on} - Bu_{om}\|$$
$$+ M\sum_{i=1}^{p}|c_i|\int_0^{t_i}(t_i-\tau)^{\alpha-1}\|V_n(\tau) - V_m(\tau)\|d\tau$$

Where $M > 0$ is a constant. Since E is a complete space, it follows that for every $\varepsilon > 0$, we can find a positive integer N such that $n > N$, $m > N$ implies

$$\|V_n(t) - V_m(t)\| \le (1 - c)\varepsilon$$

Since E is a complete space, it follows that the sequence $\{ V_n(t) \}$ is uniformly convergent on J . From (23), we find that the sequence $\vartheta_n(t) \in C_E(J)$ is uniformly convergent . Hence the required result, comp [[31]-[34]].

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