

Pairwise Locally Compact Space And Pairwise Locally Lindelöf Space

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Abstract. In this paper we define pairwise locally compact space and pairwise locally lindelöf space and study their properties and their relations with other bitopological spaces. several examples are discussed and many will known theorems are generalized concerning pairwise locally compact space and pairwise locally lindelöf space. and we shall investigate subspaces of pairwise locally compact space and pairwise locally lindelöf space and also bitopological spaces which are related to pairwise locally compact space and pairwise locally lindelöf space.

Keywords: pairwise locally compact space, pairwise locally lindelöf space, Pairwise regular, pairwise completely regular, Pairwise paracompact:

Introduction

In [3], Kelly introduced the notion of a bitopological space, i.e.

a triple (X, τ_1, τ_2) where X is a non-empty set and τ_1, τ_2 are two topologies on X . He also defined pairwise regular (P -regular), pairwise normal (P -normal), and obtained generalization of several standard results such as Urysohn's lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim in [4] and Fletcher in [1]. Also Fletcher in [1] gave the definitions of $\tau_1\tau_2$ -open and P -open covers in bitopological spaces as A cover \dot{U} of the bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ -open if

$\dot{U} \subseteq \tau_1 \cup \tau_2$. If, in addition, \dot{U} contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2 , it is called p -open cover.

Dissanayake in [5] studied some properties of locally lindelöf space. Also, in 1972 Ivan in [6] defined a bitopological local compactness as (X, τ_1, τ_2) is a bitopological space then τ_1 said to be locally compact with respect to τ_2 if for each point $x \in X$, there is a τ_1 open neighborhood of x whose τ_2 closure is pairwise compact and (X, τ_1, τ_2) is pairwise locally compact if τ_1 is locally compact with respect to τ_2 and τ_2 is locally compact with respect to τ_1 .

A.FORA and H.HDEIB [2] in 1983 give a definition of pairwise lindelöf bitopological spaces and derive some related results.

Let $\mathbb{R}, I, \mathbb{N}$ denote the set of all real numbers, the interval $[0,1]$, and the natural numbers respectively. Let $\tau_d, \tau_u, \tau_c, \tau_l, \tau_r, \tau_{ind}$ denote the discrete, Usual, cocountable, left ray, right ray, and the indiscrete topologies on \mathbb{R} (or I). Also, The τ_i closure of a set A will be denoted by $cl_{\tau_i}A$.

Definition 1.1. A bitopological space (X, τ_1, τ_2) is said to be pairwise compact (p -compact) if every pairwise open cover (p -pen cover) has a finite subcover which contains at least one non empty member of τ_1 and at least one non empty member of τ_2 .

Definition 1.2. A bitopological space (X, τ_1, τ_2) is said to be pairwise locally compact if for each point $x \in X$, there is a τ_1 -open neighbourhood of x whose τ_1 -closure is pairwise compact or a τ_2 -open neighbourhood of x whose τ_2 -closure is pairwise compact.

To illustrate the above definition of a p -locally compact.

Example 1.3. Let $X = \mathbb{R}$. $\tau_1 = \{ \phi, \mathbb{R}, \mathbb{R} - \{1\} \}$. $\tau_2 = \{ \phi, \mathbb{R}, \mathbb{R} - \{2\} \}$. Then

$(\mathbb{R}, \tau_1, \tau_2)$ is a pairwise locally compact.

To show this, let $x \in \mathbb{R}$, then any τ_1 -open neighbourhood of x , such that case 1: if $x = \{1\}$, $U_x = \mathbb{R}$ is an open set containing 1, $cl(\mathbb{R}) = \mathbb{R}$ which is pairwise compact.

Case 2: if $x \in \mathbb{R} - \{1\}$, $cl(\mathbb{R} - \{1\}) = \mathbb{R}$ (in τ_1) which is pairwise compact. On the hand, case 1: if $x = \{2\}$, $U_x = \mathbb{R}$ is an open set containing 2, $cl(\mathbb{R}) = \mathbb{R}$ which is pairwise compact.

Case 2: if $x \in \mathbb{R} - \{2\}$, $cl(\mathbb{R} - \{2\}) = \mathbb{R}$ (in τ_2) which is pairwise compact. So $(\mathbb{R}, \tau_1, \tau_2)$ is a pairwise locally

compact.

Example 1.4. Let $X = \mathbb{R}$. Then $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ is a pairwise locally compact.

To show this, let $x \in \mathbb{R}$, let U_1 be any τ_{dis} -open neighborhood containing x , any p -open cover of U_1 must contains U_1 or V_1 such that $U_1 \subseteq V_1$, so $\{U_1\}$ is a finite subcover or $\{V_1: U_1 \subseteq V_1\}$ is a finite subcover.

On the other hand, let $x \in \mathbb{R}$, let U_2 be any τ_{coc} -open neighborhood containing x , then $cl(U_2)$ is \mathbb{R} (which is pairwise compact) or a finite set (which is pairwise compact).

Example 1.5. Let $X = \mathbb{R}$. Then $(\mathbb{R}, \tau_1, \tau_{dis})$ is not a pairwise locally compact.

To show this, let $x \in \mathbb{R}$, let U be any τ_1 -open neighborhood containing x , then $U = \mathbb{R}$, but $cl(\mathbb{R}) = \mathbb{R}$ which is not pairwise compact.

For any pairwise open cover say

$\dot{U} = \{(-\infty, x): x \in \mathbb{R}\} \cup \{x\}: x \in \mathbb{R}$ which has no finite subcover.

Theorem 1.6. If X is finite, then (X, τ_1, τ_2) is pairwise locally compact.

Proof. Assume that $X = \{x_1, x_2, x_3, \dots, x_n\}$. Let $x \in X$, and U_x be a

τ_1 -open set or a τ_2 open set such that $x \in U_x$. without loss of generalization assume $U_x \in \tau_1$, then $x \in U_x$. $cl(U_x)$ is finite, so $cl(U_x)$ is pairwise compact.

Theorem 1.7. Every pairwise compact space is a pairwise locally compact.

Proof. Assume that (X, τ_1, τ_2) is pairwise compact. Let $x \in X$, and let U_x be a τ_1 -open set or a τ_2 open set such that $x \in U_x$. without loss of generalization assume $U_x \in \tau_1$, then we show that $cl(U_x)$ is a pairwise compact in τ_1 . Let \dot{U} be a pairwise open cover of $cl(U_x)$ subset of $(cl(U_x), \tau_1^*, \tau_2^*)$, Where $\tau_1^* = \{U \cap cl(U_x): U \in \tau_1\}$ and $\tau_2^* = \{V \cap cl(U_x): V \in \tau_2\}$. Then $\dot{U} \cup \{X - cl(U_x)\}$ is a pairwise open cover of a pairwise compact space (X, τ_1, τ_2) , so it has a finite subcover of X . Hence \dot{U} for $cl(U_x)$. So $cl(U_x)$ is pairwise compact.

The converse of above theorem need not be true, as shown in the following example:

Example 1.8. Let $X = \mathbb{R}$. Then $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ is a pairwise locally compact, but not pairwise compact.

To see this; let $\dot{U} = \{A_x\}: x \in Q\} \cup \{Q^c\}: Q^c \subseteq \tau_{coc}$ be an pairwise open cover.

If \dot{U} has a finite subcover, then

$\mathbb{R} \subseteq Q^c \cup \{\{x_1\}, \{x_2\}, \dots, \{x_n\}\}: x_i \in Q; i=1, 2, \dots, n\}$: which is impossible.

Theorem 1.9. : A pairwise locally compactness is hereditary with respect to closed subspace.

Proof. Let F be a closed subspace in a pairwise locally compact space $X = (X, \tau_1, \tau_2)$. Let $x \in F$, so $x \in X$, since X is a pairwise locally compact, so there exists an open set containing x in τ_1 or in τ_2 say W such that $cl(W)$ is pairwise compact. Thus $F \cap W$ is open in F with respect to τ_1 or τ_2 , and $x \in F \cap W$, and $cl(F \cap W)^F = cl(F \cap W) \cap F \subseteq cl(F) \cap cl(W) \cap F = F \cap cl(W) \subseteq cl(W)$ which is pairwise compact. We get $cl(F \cap W)^F$ is pairwise compact. Hence the result.

Theorem 1.10. : A pairwise locally compactness is hereditary with respect to open subspace.

Proof. Let V be an open subspace in a pairwise locally compact space X . Let $x \in V$, so $x \in X$, since $X = (X, \tau_1, \tau_2)$ is a pairwise completely regular space, then X is a pairwise regular space, since V is open, there exists an open set say U_x in X with respect to τ_1 or τ_2 such that $x \in U_x \subseteq cl(U_x) \subseteq V$, now; since X is pairwise locally compact, there exists an open set in X containing x in τ_1 or in τ_2 say W_x such that $cl(W_x)$ is pairwise compact.

So $x \in U_x \cap W_x = M_x$, and M_x is open in V . Since $M_x \subseteq U_x$ and $U_x \subseteq V$, so $M_x \subseteq V$, $cl(M_x) \subseteq cl(W_x)$, and since $cl(W_x)$ is pairwise compact and $cl(M_x)$ is closed, so $cl(M_x)$ is pairwise compact in V . Therefore V is pairwise locally compact.

Corollary 1.11. : A pairwise locally compactness is hereditary with respect to intersection of open subspace and closed subspace.

Proof. Let $X=(X, \tau_1, \tau_2)$ be a pairwise locally compact space, and F be a closed subspace of X . and V be an open subspace of X . Now, $F \cap V$ is open in F . Because V is open subspace in X . By a previous theorem **1.9**. F is a pairwise locally compact, and by a previous theorem **1.10**. $F \cap V$ is a pairwise locally compact in F , and so in X .

Now, will define a definition of a pairwise locally lindelöf.

Definition 1.12. A bitopological space (X, τ_1, τ_2) is said to be pairwise lindelöf (p- lindelöf) if every pairwise open cover(p-open cover) has a countable subcover which contains at least one non empty member of τ_1 and at least one non empty member of τ_2 .

Definition 1.13. A bitopological space (X, τ_1, τ_2) is said to be pairwise locally lindelöf if for each point $x \in X$, there is a τ_1 -open neighbourhood of x whose τ_1 -closure is pairwise lindelöf or a τ_2 -open neighborhood of x whose τ_2 -closure is pairwise lindelöf.

Example 1.14. $(\mathbb{N}, \tau_{dis}, \tau_{ind})$ is a pairwise locally lindelöf.

To show this ; let $n \in \mathbb{N}$, then $\{n\} \in \tau_{dis}$, and $cl_{\tau_{dis}}(\{n\}) = \{n\}$ which is pairwise lindelöf.

If $n \in \mathbb{N}$, then the only τ_{ind} - open set containing $\{n\}$ is \mathbb{N} , so $cl_2(\mathbb{N}) = \mathbb{N}$, which is pairwise lindelöf.

Example 1.15. $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is a not a pairwise locally lindelöf.

Clearly.

Remark 1.16. Every pairwise lindelöf space is a pairwise locally lindelöf.

Theorem 1.17. Every pairwise locally compact space is a pairwise locally lindelöf.

Proof. Let (X, τ_1, τ_2) be a pairwise locally compact. Let $x \in X$, then there exists τ_1 -open neighbourhood of x say U_x whose

τ_1 -closure is pairwise compact, or a τ_2 -open neighbourhood of x say V_x whose τ_2 -closure is pairwise compact, following that $cl_{\tau_1}(U_x)$ is pairwise lindelöf or $cl_{\tau_2}(V_x)$ is pairwise lindelöf . So (X, τ_1, τ_2) is a pairwise locally lindelöf.

Example 1.18. $(\mathbb{R}, \tau_u, \tau_v)$ is a pairwise locally lindelöf . Since it is a pairwise locally compact.

Theorem 1.19. If X is countable, then (X, τ_1, τ_2) is pairwise locally lindelöf.

Proof. Assume that $X = \{x_1, x_2, x_3, \dots\}$. Let $x \in X$, and U_x be a

τ_1 -open set or a τ_2 open set such that $x \in U_x$. without loss of generalization assume $U_x \in \tau_1$, then $x \in U_x$. $cl(U_x)$ is countable, so $cl(U_x)$ is pairwise compact.

Theorem 1.20. If (X, τ_1, τ_2) is pairwise lindelöf and A is a subset of X which is τ_1 closed then A ia pairwise lindelöf.

Proof. Let \dot{U} be any pairwise open cover of the subspace (A, τ_1^*, τ_2^*) . Where $\tau_1^* = \{U \cap A : U \in \tau_1\}$ and $\tau_2^* = \{V \cap A : V \in \tau_2\}$. Then

$\dot{U} \cup \{X - A\}$ induces a pairwise open cover of (X, τ_1, τ_2) which has a countable subcover for X , and hence so does for A .

Example 1.21. Let $\tau_1 = (\mathbb{R}, \tau_{left})$ be the left ray topology and

$\tau_2 = (\mathbb{R}, \tau_{right})$ be the right ray topology. Now; $(\mathbb{R}, \tau_{left})$ and $(\mathbb{R}, \tau_{right})$ are not locally lindelöf.

To show this for $(\mathbb{R}, \tau_{left})$; let $x \in \mathbb{R}$, and let $U_x \in \tau_{left}$ be open neighborhood of x , then $U_x = \{(-\infty, a) : x < a\}$, $cl_{left}(U_x) = \mathbb{R}$ which is not lindelöf. The same for $(\mathbb{R}, \tau_{right})$.

However, $(\mathbb{R}, \tau_{left}, \tau_{right})$ is a pairwise locally lindelöf. To show this; let $x \in \mathbb{R}$, and let $U_x \in \tau_{left}$ be open neighborhood of x , then

$U_x = \{(-\infty, a) : x < a\}$, $cl_{right}(U_x) = (-\infty, a]$, which is a pairwise lindelöf.

Theorem 1.22. If (X, τ_1, τ_2) is pairwise lindelöf and A is a subset of X which is τ_i ($i=1,2$) closed set, then A is pairwise lindelöf. Proof. Let \dot{U} be any pairwise open cover of the subspace (A, τ_1^*, τ_2^*) . Where $\tau_1^* = \{U \cap A : U \in \tau_1\}$ and $\tau_2^* = \{V \cap A : V \in \tau_2\}$. Then

$\dot{U} \cup \{X - A\}$ induces a pairwise open cover of (X, τ_1, τ_2) which has a countable subcover for X , and hence so does for A .

Corollary 1.23. If (X, τ_1, τ_2) is pairwise locally lindelöf and A is a subset of X which is τ_i ($i=1,2$) closed set, then A is pairwise locally lindelöf.

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