

From Euclidean Geometry To Non-Euclidean Geometry, In Particular Hyperbolic Geometry, In Secondary Education Through Dynamic Geometry.

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Abstract

The current school requires teaching and learning about Euclidean and non-Euclidean geometry from the first educational levels. The research is carried out with the purpose of making a transition from Euclidean to non-Euclidean geometry, through constructions with ruler and compass, with their subsequent comparison in the computer by means of GeoGebra software, contributing to the characterization of the geometric thinking involved, in seventh grade students of Rodrigo Lara Bonilla school. In the research, 10 activities are designed based on the development of constructions with ruler-compass and their subsequent comparison with traditional instruments or dynamic geometry, as well as problem solving based on mathematical visualization and motivation towards the study of mathematics. Through the development of the activities, students begin to build their own concepts about these new geometries, using their geometric thinking on notions of plane geometry. The implementation of these activities and the results obtained, allows improving their learning process, as well as favoring their interest in learning geometry. From personal interviews with the working groups, it is possible to verify the elements that characterize the geometric thinking used and involved by the students.

Key words: Euclidean geometry; GeoGebra software; mathematics; secondary education.

1. Introduction

The teaching and learning of geometry in elementary and middle school students has been a problem studied and exposed by researchers in different congresses and meetings. Geometry is one of the branches of mathematics that has greater intra- and extra-mathematical applications, however, its teaching and learning at different educational levels shows limitations. These difficulties are evidenced in the results achieved by students in international tests (PISA) and national tests (Saber Tests).

Research on the teaching and learning of geometry at different levels seeks solutions that bring about a change in response to the school's problems. In this sense, it is important to highlight the different changes that the teaching of geometry has undergone throughout history. Meanwhile, other researches have been conducted, whose objective is to explain the reasons why in school geometry topics are not studied in depth and topics such as non-Euclidean geometries are not treated (Jones, 2002; Keith, 2001; Christi, 2011; Ribeiro, 2013; Henderson, 2000; Rosenfeld, 1988).

The use of technology in the classroom enhances the teaching and learning of this branch of mathematics in educational institutions. In this sense, it takes a significant role the teaching and learning of Euclidean and non-Euclidean geometry from the first educational levels in school (Ribeiro, 2013; Marcondes, 2014; Lénárt, 2010; among others). This process is done with the help of dynamic geometry software (SGD), particularly GeoGebra.

The teaching and learning of geometry supported by didactic materials and GeoGebra favors a robust construction of geometric content in students. This process contributes to the understanding of other

geometries, since it is necessary, "... to know Euclidean axiomatic geometry, since it was from the questions raised about Euclid's axiomatic model, that it was possible to build other geometries".

Focusing on the development of geometric thinking, without specifying or characterizing such thinking, has led different national and international researches to present their results in congresses and meetings. Among these, the series of studies of the International Commission on Mathematical Instruction (ICMI), the International Congress on Mathematics Education (ICME), the Congress of the European Society for Research in Mathematics Education (CERME), the Inter-American Conference on Mathematics Education (CIAEM), the Conference of the International Group for the Psychology of Mathematics Education (PME), the Latin American Meetings on Educational Mathematics (RELME), and the Latin American Meetings on Educational Mathematics (RELME) stand out.

At the national level, the Colombian Congresses of Mathematics (CCM), the Meeting of Geometry and its Applications (EGA), and the Colombian Meeting of Educational Mathematics (ECME), among others, stand out. In the proceedings of these congresses, models, methodologies, didactic sequences and activities proposed to contribute to the teaching and learning of geometry at school can be specified.

Mammana and Villani (1998), in the discussion paper for an ICMI study, focus their study on "Perspectives on Teaching Geometry for the 21st Century". This paper recognizes the most relevant challenges and emerging trends for the future, highlighting the use of both manipulative and visual materials to improve the teaching and learning of geometry in schools.

In an ICMI study (2001), it was suggested that geometry is conducive to understanding and improving the intuitive part of mathematics that relates to its context. On the other hand, the teaching and learning of geometry using dynamic geometry software (SGD) was discussed at the ICME 2008. In this congress, it was proposed that manipulating geometric figures and having students learn from these experiences are potentialities of using DGS in the teaching and learning of geometry.

In ICME 13 (2016) the thematic study group (TSG 13) addresses the teaching of geometry at the secondary level. In TSG 9, discussions will be held on this topic, but also on various geometry curricula at the secondary level. At ICME 14 to be held in 2021, in TSG 9, the Teaching and Learning of Geometry at the secondary level is discussed. In these TSGs of ICME 13 and 14, argumentation, proof, visualization, figuration and instrumentation are discussed, aspects that are determinant in the achievement of a robust teaching and learning process of geometry, specifically for the development of geometric thinking.

On the other hand, in CERME (2013), spatial thinking applied to the teaching of geometry is addressed, and in this same congress different strategies are discussed based on different didactic materials that allow enhancing spatial perception.

The teaching and learning of geometry at school from non-Euclidean geometry has been studied by Olive, (2000); Jones, (2002); González and Vélchez, (2002); Komarnicki (2013); Marcondes (2014), among others. These investigations have as a point of relation to establish that most students do not know axiomatic Euclidean geometry, therefore, they do not know the existence of non-Euclidean geometries. One of the causes is the inadequate training of teachers in geometry, which leads to a low interest in teaching geometric concepts in the classroom (Rojas 2009).

On the other hand, Jones (2002) states that the study of geometry contributes to the development of critical thinking skills, visualization, intuition, perspective, problem solving, conjecture, synthetic thinking, deductive reasoning and demonstration in students. For Pavanello (1989), sometimes geometry classes at school are not taught in depth, they are limited to formulas and calculations, ignoring the development of geometric thinking and deductive argumentation as a fundamental part in the construction of their knowledge.

Likewise, the inclusion of non-Euclidean geometries in high school makes it possible to improve the teaching and learning of geometry, and it is also important to use dynamic geometries, since this is of interest to students (Ribeiro 2013).

On the other hand, teachers feel insecure with the treatment of geometric content and thus arithmetic or algebraic concepts are favored. In this regard, Gravina (2001) states that the low academic training in geometry of high school students is demonstrated in their performance when they reach the university and take the Geometry I course. According to Gravina (2001) "... these students arrive at university devoid of the intellectual skills necessary for the construction of geometric knowledge. Abstracting, generalizing, establishing relationships, making mistakes, making conjectures, demonstrating... actions that characterize the process of creation in mathematics".

In order to verify the geometric knowledge of students, it is necessary to know the curricular path through the different school grades in secondary education. On the other hand, school curricula in Colombia do not contemplate the study of non-Euclidean geometry, perhaps because of the little importance given to geometry or because of the way it is taught.

This research emphasizes the teaching and learning of geometry in secondary education from the study of non-Euclidean geometries, which leads to reflect on their own pedagogical practices, and also on historical aspects related to the teaching and learning of this branch of mathematics. In this process, geometric constructions are made with ruler and compass where students use geometric concepts of plane geometry with the objective of making them feel identified, motivated and interested.

In recent years, the advancement of technology has been reflected in the classroom, exponentially growing the rise of information and communication technologies (ICT) as a teaching and learning mechanism, which leads to its use as an educational tool. The Colombian Ministry of National Education (MEN) develops various strategies to integrate these technologies in the educational environment; it is here where an opportunity is found to incorporate the teaching and learning of non-Euclidean geometry in the educational institution, through the GeoGebra program, by means of Poincaré's model (Poincaré's disk).

2. General Objective

To characterize the geometric thinking involved in the process of making constructions in which the way in which they relate Euclidean geometry and the hyperbolic model of non-Euclidean geometry is contrasted, through a system of activities, in seventh grade students of the Rodrigo Lara Bonilla School.

Research methodology

About the teaching-learning process of plane geometry concepts (Euclidean) in transition to non-Euclidean geometry in basic secondary education, it is necessary to make an analysis of the disciplinary needs, students and the teaching context. For this reason, the type of research, the scope of the study and the methods used are presented below.

3.1 Methodological design

This research is based on a qualitative approach, with an action research design. In this design, the conceptual level of the seventh-grade students is determined, evaluating the changes and improvements in their learning in each of the stages proposed through the activities. In order to clearly determine and fulfill this objective, the work is also developed through a thematic analysis, interpreting the real level of the students through their representations, solution strategies and attitudes.

The student's development process is established for each stage and the processes of their geometric thinking to build solid meaning of the concept to be studied and to categorize them. Didactic activities are carried out to be solved individually by the students and in some cases work groups will be established to analyze their development and answers and, therefore, actively seek to elucidate what they are grasping.

Motivation is fundamental in the development of the activities, since it allows the teacher to transform, improve and enrich his/her teaching work. In addition, by allowing independent, free and autonomous reasoning, students check their ideas and share them with others, enriching their learning.

3.2 Population and sample

For this research, the population is taken as the students of the Rodrigo Lara Bonilla Educational Institution and the sample is a group of seventh grade students of the morning session of the academic year 2018, 2019, 2020.

3.3 Methods, Techniques and Instruments Used

The thesis combines scientific research methods and techniques, on a theoretical and empirical level. The following theoretical methods are used:

Historical-logical: it is used with the objective of estimating the progress of the teaching-learning process of geometry in basic education.

Analysis-synthesis: there is an interrelation between these processes present in the research, both in its theoretical structure and in the study of the diagnostic results related to the teaching of geometry in basic education, which allows establishing the meaning of the facts in order to interpret, synthesize the results and elaborate conclusions and generalizations.

From the **empirical** level, the following are used

Participant observation: for the observation of classes and other teaching activities, to obtain information on the teaching and learning of geometry in basic education.

Survey: to students (see Annex 1) to obtain information on the teaching and learning of geometry in basic education. A student satisfaction survey is also applied.

Statistical mathematical methods (percentage calculation procedure) are used to process the information obtained through the methods and techniques of the empirical level, at different moments of the research.

3.4 Phases of the investigation

High school. Participant observation is designed. In this phase, an exhaustive search of the state of the art is made, which allows confirming the research problem, restructuring the general objective and determining the current trends on the teaching-learning of Euclidean and non-Euclidean geometry at school. In addition, the theoretical framework of the thesis is specified.

Methodology design and analysis of results. This phase includes the design and elaboration of ten activities, which are shown in Chapter 4. The phase considers the field work, the application of instruments, the application of the activities, the application of the satisfaction survey, the collection of information, the information process and the analysis of the results.

Informative phase. In this phase, the preparation of the written document, the publication or socialization of the results and the presentation of the thesis are carried out.

4. Design of activities

The following chapter describes the elaboration of the activities used for the development of this research. Initially, a diagnostic test is elaborated and applied in order to know the level of knowledge that the students possess and thus have a clearer idea of the geometric thinking that they have previously developed.

The activities are developed in three stages. The first is based on carrying out different constructions with ruler and compass, seeking to motivate students to learn. The second stage is based on carrying out a process of construction of the propositions of Euclid's Book I, a stage that is complemented with the development

of non-routine problems based on this geometry. Finally, in the third stage, new geometries are investigated with the help of the computer.

4.1 Design and implementation of activities

Next, ten activities are designed with the aim of generating in students the motivation to learn, an important key in the teaching and learning of mathematics and in particular of geometry, in order to characterize the geometric thinking involved, the objective of this research. To this end, the activities are supported by constructions with ruler and compass, the use of GeoGebra software, and the development of non-routine problems that go beyond the daily routine in the classroom. The aim is not to develop the ability only to answer a test, but to see if the student constructs his own concepts, and in this way to analyze which are the thought processes that the student develops.

4.1.1. Activity 1. Circles with straight line strokes.

Objective: to promote geometric thinking through the manipulation and appreciation of flat geometric figures (circles) from different positions and perspectives.

Methodological suggestions: as a first activity, a playful action is proposed to begin to introduce geometric concepts, where the student begins to feel interested, motivated and curious about this new subject. The activity is developed individually by the students, its objective is to get out of the everyday, generating interest in learning.

Purpose of the activity: the instructions for the development of the activity seek that the students themselves discover the circles as they progress in their solution and, in addition, that they find a series of questions that require analysis and reflection on their part.

Materials to be used: ruler, colored pencils and graph paper.

Development of the activity: each student must carry out the construction of the Cartesian planes as indicated in the following figure (see Figure 1); the sheet is worked horizontally.

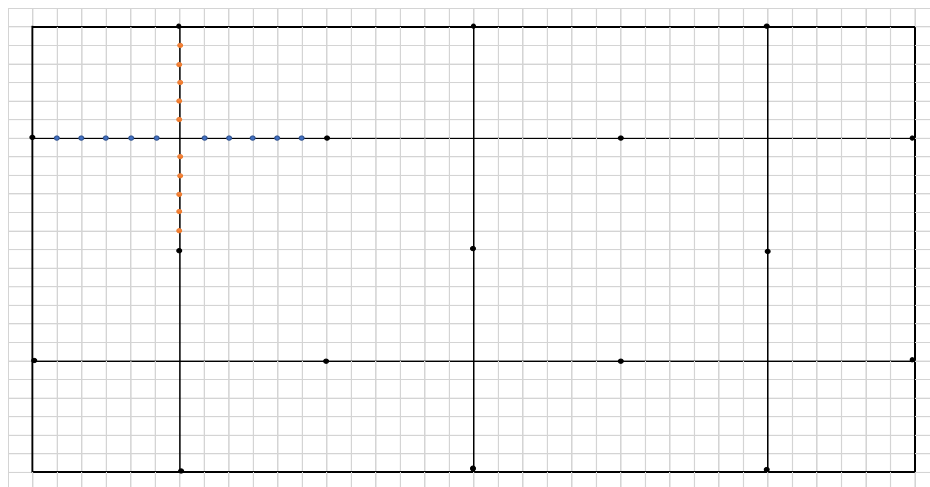


Figure 1. Horizontal leaf.

Once the previous construction has been elaborated, in the first Cartesian plane (upper left part), follow the indications as shown in the figure (see Figure 2).

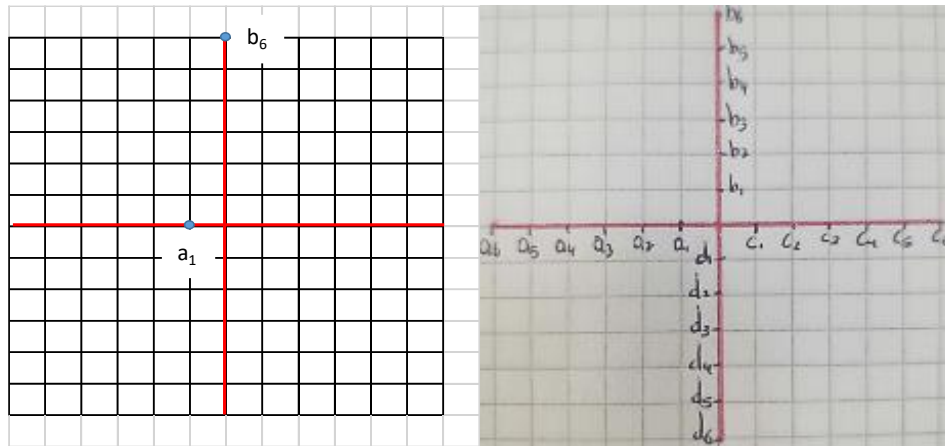


Figure 2. Cartesian Plane

Then follow the instructions below:

1. Join point a_1 with point b_6 .
2. Next, join point a_2 with point b_5 , continue in quadrant II until joining point a_6 with b_1 .
3. Now, join point c_1 with point b_6 .
4. Next, join point c_2 with point b_5 , continue in quadrant I until joining point c_6 with b_1 .
5. Now, join point a_1 with point d_6 .
6. Next, join point a_2 with point d_5 , continue in quadrant III until joining point a_6 with d_1 .
7. Now, join point c_1 with point d_6 .
8. Next, join point c_2 with point d_5 , continue in quadrant IV until joining point c_6 with d_1 .

Continue the above process in the next Cartesian plane, as shown in the figure (see Figure 3) and repeat the instructions given. Finally repeat these steps in each Cartesian plane.

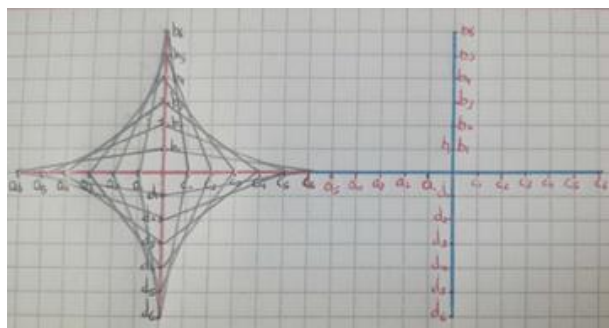


Figure 3. Continued Activity.

Answer the following questions:

1. If the size of each square of the sheet is 0.5cm x 0.5cm, calculate the radius of the circle obtained.
2. Calculate the diameter of the circle found.
3. Is the radius of the circles the same, why?

4.1.2. Activity 2. Construction of figures with ruler and compass: triangle, square, pentagon, hexagon, octagon, octagon, enneagon.

Objective: to promote geometric thinking through the construction with ruler and compass of the following figures: triangle, square, pentagon, hexagon, octagon.

Methodological suggestions: for the beginning of the activity a playful action is taken to begin to introduce the subject of plane geometry, where the student begins to feel interested, motivated and curious about this topic. The development of this activity consists of 6 different classes. The activity must be developed individually, its objective is to get out of the everyday, generating an interest in learning. To do so, it is proposed to build with ruler and compass the following figures: triangle, square, pentagon, hexagon, octagon.

Purpose of the activity: the instructions of the activity seek that students discover and assimilate geometric concepts through the progress of the constructions with ruler and compass. In addition, there are a series of questions that require students to analyze, verify and reflect on the properties of geometric figures. With the help of these constructions, they must correctly answer the questions posed.

Each student is asked to have the geometric instruments at hand and follow the instructions below.

1. Draw a 5cm long segment in the middle of the sheet and at each end locate points A and B respectively.
 - a. Locate the compass at point A and open it to point B; trace the circle, repeat this at point B. Next, locate point C, the point of intersection of the circles at their bottom, join the points from point A to point C and from point B to point C.

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are the sides of this figure equal? If the answer is yes, explain why.
2. On a new sheet of paper, draw a 4 cm long segment in the middle of the sheet and at each of its ends locate points A and B respectively.
 - a. Locate the compass at point A and open it beyond point B; trace the circle, repeat this at point B. Next, locate point C which is the point of intersection of the circles at the bottom, join the points from point A to point C and from point B to point C.

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are all sides of this figure equal? If the answer is yes, explain why, otherwise explain which ones are and why.
3. Construction with three given segments. On a new sheet, draw three segments with the following dimensions: the first segment ℓ_1 of 10cm long red color, the second segment ℓ_2 of 7cm long green color and finally the segment ℓ_3 of 5cm long blue color, (see Figure 4).

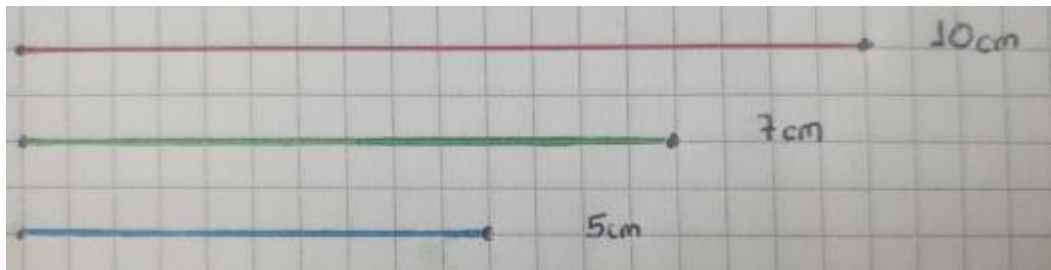


Figure 4. Given segments.

- a. Next, in the middle of the sheet draw again the segment ℓ_1 , locate at the ends of this segment the points A and B respectively.
- b. Take the distance of segment ℓ_2 , now, place the compass at point A and draw circle c_1 .
- c. Take the distance of segment ℓ_3 , now, place the compass at point B and draw circle c_2 .
- d. Select the upper intersection point of circles c_1 and c_2 respectively, which corresponds to point C. Then draw a line from point A to point C and from point B to point C.

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are all sides of this figure equal? If the answer is yes, explain why, otherwise explain which ones are not and why.
4. On a new sheet, draw an 8cm segment at the bottom of the sheet and locate at its ends points A and B respectively; this segment will be ℓ_1 .
- a. Note point A and draw a circle c_1 (approximately 2cm radius), locate point A_1 , which is the point of intersection of segment ℓ_1 with circle c_1 .
 - b. Now place yourself at point A_1 and draw a new circle c_2 , with the same compass opening.
 - c. Next, locate point A_2 , which is the intersection point of the top of circles c_1 and c_2 respectively.
 - d. Proceed to point A_2 and trace circle c_3 with the same previous compass opening.
 - e. Now, locate point A_3 , which is the intersection between circles c_3 and c_1 , respectively. This point must be different from point A_1 .
 - f. Next, look at point A_3 and with the same previous compass opening, trace circle c_4 , then locate point A_4 , which is the intersection of circles c_4 and c_3 . This point must be different from point A.
 - g. Now, look at point A and extend the compass to point B and trace semicircle c_5 , at the top of segment ℓ_1 .
 - h. Next, I plot the segment ℓ_2 , which passes through point A and point A_4 , so that it intersects semicircle c_5 , then proceed to locate point D, which corresponds to the point of intersection of segment ℓ_2 with semicircle c_5 .
 - i. Now proceed to fix at point B and extend the compass to point A; then trace semicircle c_6 , on top of segment ℓ_1 .
 - j. Then, look at point D and extend the compass to point A; next, trace semicircle c_7 , then proceed to locate point C, which corresponds to the point of intersection of semicircles c_6 and c_7 , respectively, which is different from point A.
 - k. Finally, trace the segments from point A to point B, from point B to point C and from point C to point D.

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are all sides of this figure equal? If yes, explain why. If not, explain which ones are and why.
 - Are the sides of this figure parallel and why?
5. Draw two-line segments ℓ_1 and ℓ_2 10 cm long that are perpendicular at their respective midpoints. Let X be the point of intersection and with center at point X draw circle c_1 of approximately 6cm radius.
- a. Now proceed to locate point A_1 , which is located to the right of the intersection between segment ℓ_1 and circle c_1 respectively (see Figure 5).

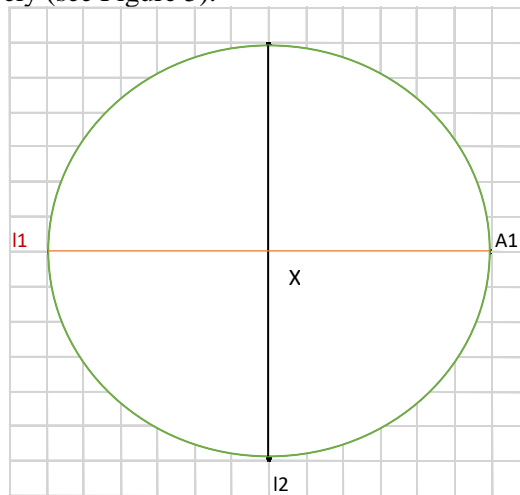


Figure 5. Location of point A_1 .

- b. Next, look at point A_1 and extend the compass to point X, then draw the circle c_2 .
- c. Now, through the intersection points of circles c_1 and c_2 , draw the segment s_1 .
- d. Next, proceed to locate point A_2 , which is the point of intersection between line ℓ_1 and segment s_1 .
- e. Proceed by locating point A_3 , which is the intersection between line ℓ_2 and circle c_1 at its top.
- f. Now, look at point A_2 and extend the compass to point A_3 and draw the semicircle c_3 .
- g. Next, locate point A_4 , which is the point of intersection between semicircle c_3 and line ℓ_1 .
- h. Now, look at point A_3 and extend the compass to point A_4 and draw semicircle c_4 , so that it intersects circle c_1 .
- i. Next, proceed to locate points A and D, which are the intersection points between circle c_1 and semicircle c_4 .
- j. Now, look at point A and extend the compass to point A_3 (this is the same distance as above). Then draw the semicircle c_5 , so that it intersects the circle c_1 . Next, locate the intersection points between circle c_1 and semicircle c_5 , one of them is point A_3 and the other will be point B.
- k. Next, position yourself at point D and extend the compass to point A_3 (this is the same distance as above). Now, draw the semicircle c_6 , proceed to locate the intersection points between the circle c_1 and the semicircle c_6 . One of them is point A_3 and the other will be point C.
- l. Finally, join the points from point A_3 to point A, from point A to point B, from point B to point C, from point C to point D and finally from point D to point A_3 .

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are all sides of this figure equal? If the answer is yes, explain why, otherwise explain which ones are not and why.
 - How long is its apothem?
 - Identify the central angle, what is its amplitude?
6. Draw two-line segments ℓ_1 and ℓ_2 10 cm long that are perpendicular at their respective midpoints. Let X be the point of intersection and with center at point X draw circle c_1 of approximately 6cm radius.
 - a. I placed the intersection points between the line ℓ_2 and the circle c_1 , name them A_1 and A_2 respectively.
 - b. Next, look at point A_1 and extend the compass to point X, then draw circle c_2 .
 - c. Now, determine the two intersection points between circles c_1 and c_2 , name these points A and D respectively.
 - d. Next, look at point A_2 and extend the compass to point X, proceed to draw circle c_3 .
 - e. Now, proceed to determine the two intersection points between circles c_1 and c_3 , these points are designated as B and C respectively.
 - f. Finally, one from point A_1 to point A, now from point A to point B, from point B to point A_2 , from point A_2 to point C, from point C to point D and finally from point D to point A_1 .

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
 - Are all sides of this figure equal? If the answer is yes, explain why, otherwise explain which ones are not and why.
 - How long is its apothem?
 - How long is the central angle?
7. Draw two-line segments ℓ_1 and ℓ_2 10 cm long that are perpendicular at their respective midpoints. Let X be the point of intersection and with center at point X draw circle c_1 of approximately 6cm radius.
 - a. Next, locate the points of intersection between the line ℓ_1 and the circle c_1 and name them points C and G, respectively.

- b. Now, determine the intersection points between the line l_2 and the circle c_1 , call them points A and E, respectively.
- c. Then, proceed to trace the segment s_1 , which passes through points A and C, respectively.
- d. Now, find the perpendicular bisector of the segment s_1 , sketch the perpendicular bisector so that it intersects the circle c_1 at two points, the points of intersection between the perpendicular bisector and the circle c_1 correspond to points B and F, respectively.
- e. Next, trace the segment s_2 , which passes through the points between C and E, respectively.
- f. Now, find the perpendicular bisector of the segment s_2 , sketch the perpendicular bisector so that it intersects the circle c_1 at two points, the points of intersection between the perpendicular bisector and the circle c_1 correspond to the points D and H, respectively.
- g. Finally, one from point A to point B, from point B to point C, from point C to point D, from point D to point E, from point E to point F, from point F to point G, from point G to point H, and finally from point H to point A.

With the help of the above construction answer the following questions:

- What is the name of the figure obtained?
- Are all sides of this figure equal? If the answer is yes, explain why, otherwise explain which ones are not and why.
- How long is its apothem?
- How long is the central angle?

4.1.3 Activity 3. Mediatrix, bisector and problems.

Objective: to promote geometric thinking by means of constructions with ruler and compass, applying acquired knowledge to problem solving.

Methodological suggestion: this activity is proposed to be developed in two different phases, in the first one the students work individually, while in the second one they meet in groups of a maximum of three students. Three different classes are assigned for the development of the activity, one class for phase 1 and two classes for phase 2. In the first class, the circumcenter and incenter are constructed with ruler and compass; in the second and third classes, solutions are given to the problems posed.

Purpose of the activity: through the construction with ruler and compass of the circumcenter and incenter, students understand concepts such as perpendicular bisector, bisector, among others, thus improving the internalization of this knowledge. Once the first part of the activity is completed, analysis, verification and reflection of the properties of the triangle's perpendicular bisectors and perpendicular bisectors are required. In addition, the circumcenter and incenter of the triangle are analyzed, so that students can develop the second and third part of the activity that corresponds to the resolution of the problems.

Each student is asked to have the geometric instruments at hand and follow the instructions below.

Instructions for part one

1. Draw a segment of 9cm and locate on these two points A and B respectively.
 - a. Next, look at point A and open the compass to point B and draw the circle c_1 .
 - b. Now, place the compass at point B and open it to point A and draw the circle c_2 .
 - c. Next, locate points A_1 and A_2 , which are the intersections of circles c_1 and c_2 .
 - d. Finally, draw the line ℓ_1 passing through the points A_1 and A_2 , then ℓ_1 is called the bisector of the segment AB.
2. Construct a scalene triangle, and denote the vertices by A, B and C.
 - a. Then, with the previous indications, draw the perpendicular bisectors of the sides \overline{AB} , \overline{AC} y \overline{BC} respectively.
 - b. Now, locate the point of intersection (concurrence) of the three medians Or (circumcenter).
 - c. Finally, look at point O_r and open the compass to point A, then draw the circumscribed circle c_1 .

3. Construct a scalene triangle whose sides all have lengths greater than 2 cm; denote the vertices by A, B and C.
 - a. Now, look at point A and draw the circle c_1 of radius 2cm.
 - b. Locate points A_1 and A_2 , which are the intersection points of circle c_1 and $\triangle ABC$.
 - c. Next, embed yourself at point A_1 with the same previous compass opening and draw circle c_2 .
 - d. Now, look at point A_2 and with the same previous compass opening draw the circle c_3 .
 - e. Next, locate point X, which is the intersection point of circles c_2 and c_3 , inside the $\triangle ABC$.
 - f. Now draw the line through ℓ_1 which is the one that passes through the points A and X respectively, therefore, the line ℓ_1 is the first bisector of $\triangle ABC$.
 - g. Then, look at point B and draw the circle c_4 of radius 2cm.
 - h. Now, locate points B_1 and B_2 , which are the intersection points of circle c_4 and $\triangle ABC$.
 - i. Next, embed at point B_1 with the same previous compass opening and trace circle c_5 .
 - j. Now, look at point B_2 and with the same previous compass opening draw the circle c_6 .
 - k. Next, locate point Y which is the point of the intersection of circles c_5 and c_6 , inside the $\triangle ABC$.
 - l. Now draw the line ℓ_2 , which is the one that passes through the points B and Y, respectively. The line ℓ_2 is the second bisector of $\triangle ABC$.
 - m. Then, look at point C and draw the circle c_7 of radius 2cm.
 - n. Now, locate the points Q_1 and Q_2 , they are the intersection points of the circle c_7 and the $\triangle ABC$.
 - o. Next, insert yourself in point Q_1 and with the same previous compass opening draw the circle c_8 .
 - p. Now look at the point Q_2 and with the same previous compass opening draw the circle c_9 .
 - q. Next, locate point Z, which is the point of the intersection of circles c_8 and c_9 , inside the $\triangle ABC$.
 - r. Now draw the line ℓ_3 , it is the one that passes through the points C and Z respectively. Therefore, the line ℓ_3 is the third bisector of $\triangle ABC$.
 - s. Next, locate the point I_r , which is the point resulting from the intersection of the lines ℓ_1 , ℓ_2 and ℓ_3 , respectively; this point is called the incenter (I_r).
 - t. Construct the perpendicular segment to side AB from point I_r and draw a circle with center at I_r , whose radius is equal to the length of that perpendicular.

Instructions for part two

A farmer has a piece of land with triangular dimensions. He buys a cattle lot of 20 cattle. In order to take advantage of and improve the yield of the pasture, the land must be divided into equal plots and the cattle remain in each plot for 15 days. After this time, the cattle must move to the next plot, and so on until they reach the first plot again.



- 1.) The terrain has the following proportions, which are given by the coordinates A (1, -2), B (8, -2) and C (7,4) by locating it in the Cartesian plane.
 - a. Suppose that each unit on the plan corresponds to 200 m. What is the area of the land?
 - b. To solve and improve grass production, divide the land into 6 parts of equal area. What is the individual area of each part of the land?
 - c. Given the importance of keeping the cattle lot irrigated, the farmer needs to buy watering cans. When he arrives at the hardware store, he is shown five types of watering cans, which are differentiated by their range. What should be the range of each watering can, where should the farmer place, each watering can according to the plot where the cattle are located?

- d. If the water intake is located at vertex C, how many meters of hose should the farmer use to meet the demand of the watering cans?
- e. During a visit from aqueduct officials, they inform the farmer that when placing the sprinklers, it is necessary to place a central drainage point and establish gutters where the water moves to the outside of the land. Locate the point where the drainage should be placed inside the field. Draw the gutters to the outside of the lot.

2.) Construct the triangle $\triangle ABC$ and on each side construct an equilateral triangle. Locate the notable points of these three equilateral triangles. What type of triangle is obtained by joining these notable points?

The activities have been designed with the objective of improving the teaching and learning of geometry at school, since this branch of mathematics is of great interest to students. The implementation of constructions with ruler and compass in the classroom results in the transformation of traditional and monotonous classes to motivating and interesting classes for the students, which leads to an improved understanding of new geometric concepts.

The comparison of different geometric constructions is proposed, both with GeoGebra software and geometric instruments. This is done in both geometries, Euclidean and hyperbolic, with the aim of encouraging students to analyze, argue and justify, in the discovery of different properties inherent to geometric figures.

4.1.4 Activity 4 Non-routine problems about circumcenter and incenter.

Objective: to promote geometric thinking through construction with ruler and compass and the resolution of non-routine problems.

Methodological suggestions: The development of the activity is proposed for two different classes. In the first class, students work individually, make constructions with ruler and compass on the perpendicular bisectors of the triangle, locating the circumcenter point, and trace the circle that passes through the three vertices. Then, the process of construction of the bisectors of the triangle and the location of the incenter point is performed, and the inner circle that is tangent to the three sides is traced. The second class is designed to be developed in group; non-routine problems are posed for which the student must apply what was previously seen in the first part of the activity. For the problem-solving process, the phases or strategies proposed by Polya (1965) are considered.

Purpose of the activity: students observe, once the first class is finished, the construction made with ruler and compass, where the analysis, verification and reflection of the properties of the bisectors and bisectors of the angles of the triangle are required. In addition, they analyze the circumcenter and incenter of the triangle, with the objective of developing the second class that corresponds to the resolution of the challenging problems for the students.

Next, students are presented with challenging problems that go beyond the traditional and every day, allowing them to analyze possible strategies for solving the problem.

1. Locate the coordinates A (-6, -7); B (7, -6); C (-2, 6), and form the triangle $\triangle ABC$ with the above coordinates in the Cartesian plane.
 - a. From the necessary constructions in the triangle $\triangle ABC$ place the circumcenter and name it O_r .
 - b. Now, draw the cevian line, which passes through vertex C and contains the circumcenter O_r .
 - c. Let T be the point of intersection of the cevian line with the side \overline{AC} . If the angle $\angle ABT = 40^\circ$ and angle $\angle TBC = 20^\circ$. What is the value of angle $\angle CTB$?
2. Let $\triangle ABC$ be a triangle (see Figure 5). Construct in the triangle $\triangle ABC$ the circumcenter O_r and incenter I_r .

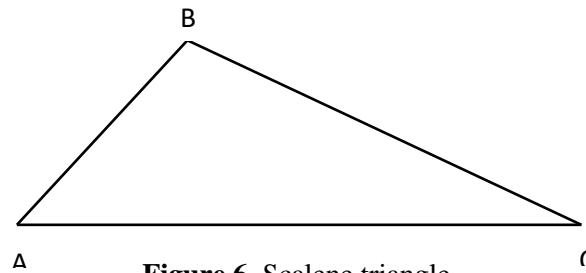


Figure 6. Scalene triangle.

- a. If the measure of angle $\angle ACB$ is equal to 120° , determine the sum of the measures of the angles $\angle AIB$ + $\angle AOB$.
- Note the triangle $\triangle ABC$ (see Figure 6).

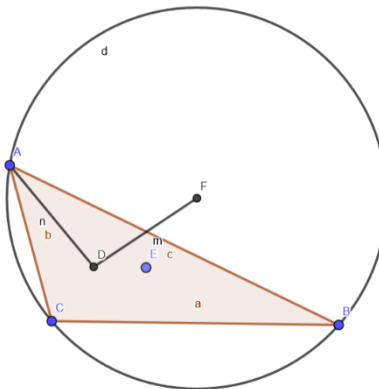


Figure 6. Problem.

In triangle $\triangle ABC$ the points D, F and E correspond to the incenter, circumcenter and barycenter (point of intersection of the medians of the triangle) respectively, also the segment \overline{DE} is parallel to the segment \overline{CB} . Show that the angle $\angle ADF = 90^\circ$.

4.1.5 Activity 5: Challenging Problems

Objective: to promote geometric thinking through the resolution of challenging problems.

Methodological suggestions: for the development of this activity, students should work in groups of four students. The activity consists of four problems, based on the concepts previously seen about demonstrations with ruler and compass; these problems should be developed through the phases or strategies proposed by Polya (1965).

Purpose of the activity: students must develop the activity by applying the concepts seen previously and the notions used in the constructions with ruler and compass to explain each step used in the solution of the problems they are presented with.

Inscribe in three concentric circles of center A, an equilateral triangle (see Figure 7).

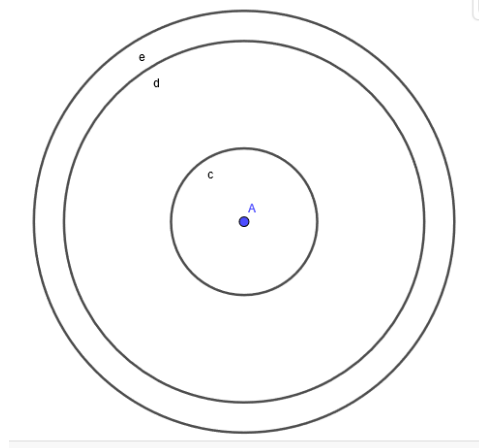


Figure 7. Concentric circles.

2. Construct a scalene triangle, find the circumcenter O_r , the incenter I_r , such that r is the radius of the inscribed circle. If the bisector between the segment of the circumcenter O_r and the incenter I_r cuts the circumscribed circle at L , and the segment between the point L and the point of the incenter I_r cuts again at M the circumscribed circle. Prove that $I_r M = 2r$.

4. Let $\triangle ABC$ be a triangle. The circle inscribed in $\triangle ABC$ is tangent to the segments \overline{AB} y \overline{AC} at points D and E , respectively. Let O be the circumcenter of the triangle $\triangle BCE$, show that the angle $\angle ODB = \angle OEC$.

5. Let $\triangle ABC$ be a triangle and P be any point on the circle circumscribed to the triangle $\triangle ABC$. Let X , Y , Z be the points of the perpendiculars from P on the segments \overline{BC} , \overline{CA} y \overline{AB} . Show that the points X , Y , Z are collinear (Simson's line).

6. Let $\triangle ABC$ be an equilateral triangle and a , b , c the lengths of its sides. The tangent lines to the inscribed circumference of the triangle, which are parallel to the sides, divide the triangle $\triangle ABC$ into three small triangles. In each small triangle consider the inscribed circumference. Calculate the sum of the areas of the four inscribed circles (problem, Yugoslavia).

4.1.6. Activity 6. Introduction to non-Euclidean geometry.

Objective: to enhance geometric thinking through the introduction of new geometries by using the history of geometry as a didactic resource.

Methodological suggestions: for the development of this activity it is proposed that students work in small groups of four and a brief historical tour is made through the development of three readings where the history of geometry is used as a didactic resource. Next, the students are asked for materials for the construction of figures, which propose the existence of non-Euclidean geometries. In this activity, students must complete each reading and answer the questions proposed at the end of each one.

Purpose of the activity: through the reading, students can observe the historical path of the emergence of non-Euclidean geometries. In addition, it is proposed that with the construction of the figures they can verify their existence. At the end there is a series of questions to verify the fulfillment of the objective of the activity.

Construction 1. To improve and facilitate understanding and to be able to imagine geometry without parallel lines, observe the following graphic.

Take the sphere of styrofoam, you can imagine the lines as big circles.

1. Cut 3 strips 1.5 cm wide and the length that covers the surface of the sphere over the equator, color each black, red and green respectively.

2. Next, glue the black strip on the equator, as shown in Figure 1, the red strip glue it so that it is perpendicular to the black line, forming a right angle, and finally, glue the green strip so that it also makes a right angle with the equator (see Figure 8).

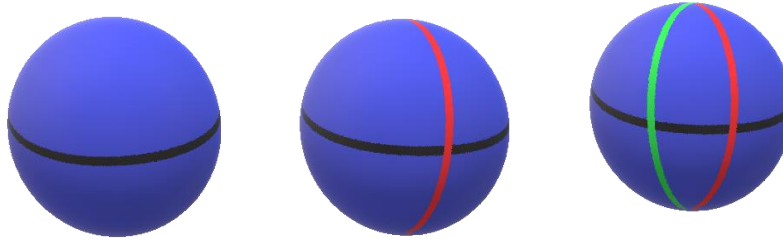


Figure 8. Spherical geometry.

- Rotate the sphere, through that point on the equator.
- The more they move to the top of the sphere, the closer the lines come together, meeting at the pole.

Considering the work done, answer the following questions:

1. Do the angles of intersection of the green line with the black line and the red line with the black line add up to 180 degrees? Explain your answer.
2. The more they move to the top of the sphere the closer the lines come together, meeting at the pole. Are the red and green lines parallel? If your answer is yes or no, explain why.
3. How can this geometry be called on this surface? Argue.
4. Take equal-sized pieces of construction paper and form an equilateral triangle. Can the triangle be constructed? Explain your answer.
5. What is the shape of the sides? Justify your answer.

Construction 2. Given the materials supplied, perform the following orientations:

1. Take the cardboard and draw a circle of radius 15cm.
2. Place the circle on the table.
3. On the circle, place the waltz stick vertically, this will represent the line a (Initial Line).
4. Take the curved stick, place it 5cm from the base of the previous waltz stick, this will represent the curved line.
5. The circle on the table will now be the plane, as shown in Figure 9.

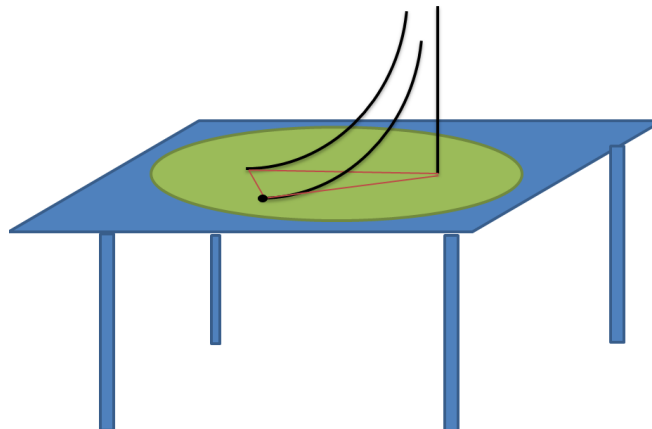


Figure 9. Construction.

6. The curved line is now an asymptotic parallel to the initial line.
7. Choose another point on the table base.
8. A non-Euclidean triangle is obtained, the new point forms a right angle.
9. At this point place the other curved valso stick, it is another asymptotic parallel, perpendicular to the first one.

Taking into account the above instructions, answer the following questions and justify your answer:

- a) The initial line is?
 - Parallel to the plane.
 - Perpendicular to the plane.
- b) Does any trigonometric formula describe the spherical triangle in our peculiar non-Euclidean world?
- c) Knowing the shape of the spherical triangle is it possible to determine the shape of the non-Euclidean triangle? Explain.
- d) Does spherical geometry depend on Euclid's postulates?

Construction three

1. Choose the bowl and position it so that the starting line is in the center of the bowl, see Figure 3.

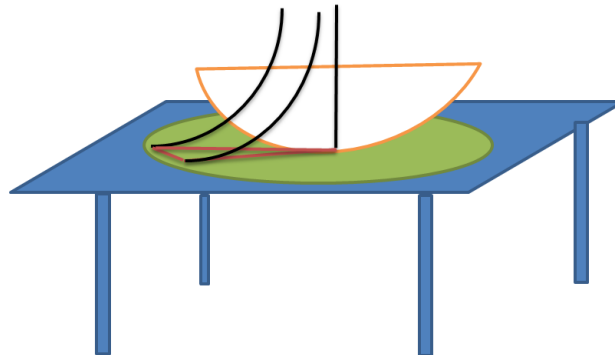


Figure 10. Construction three.

2. With the 1.5cm wide pieces of cardboard join the intersection points of the walnut sticks with the bowl.
3. Is every parallel asymptotic to the original perpendicular, are its angles right angles, explain your answer.
4. What is the sum of the internal angles of the triangle inside the bowl? Give the reason.

Construction Four

Given the materials provided, construct the given figure (see figure 11):

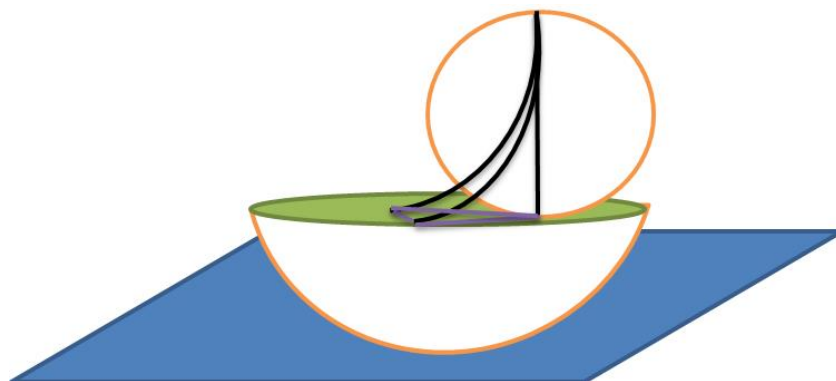


Figure 11. Construction four.

Given the constructed figure answer:

1. What are the asymptotic parallels to the original line? Explain why the ones you point out are.
2. Is the point that appears to be the point of contact between the asymptotic parallels and the original line infinitely far away in non-Euclidean space? Justify your answer.
3. Can it be stated that this atlas confirms the non-Euclidean three-dimensional model? Explain your answer.

Finally, returning to this model, there are two equivalent forms of configuration, viewed from the outside it is a three-dimensional map of a non-Euclidean space discovered by Bolyai and Lobachevsky.

While inside it is a non-Euclidean three-dimensional space, in which one has a precise figure of another geometry, perhaps strange to students, but logically possible. These ideas show the remarkable work of two-dimensional Euclidean geometry.

4.1.7. Activity 7. GeoGebra activities: hyperbolic geometry.

Objective: to promote geometric thinking through the manipulation of lines and points in GeoGebra software with the theme of hyperbolic geometry.

Methodological suggestions: work is proposed in small groups of two in the systems room. Through the guide, students must make certain constructions in the Poincaré disk. For each series of constructions, questions are posed with the objective of encouraging students to make comparisons between both geometries, the Euclidean and the hyperbolic.

Purpose of the activity: students solve the activity on hyperbolic geometry, using constructions elaborated through the computer by means of GeoGebra software. For the development of this activity 2 different classes are assigned; it is proposed to make constructions with GeoGebra simulating the constructions with ruler and compass elaborated in class on Euclidean geometry. Students must make a comparison between Euclidean and hyperbolic geometry; each construction has a series of questions that they must answer to improve the internalization of the concepts. Students are then familiarized with the program.

Enter the GeoGebra software, observe the following screen (see Figure 12).

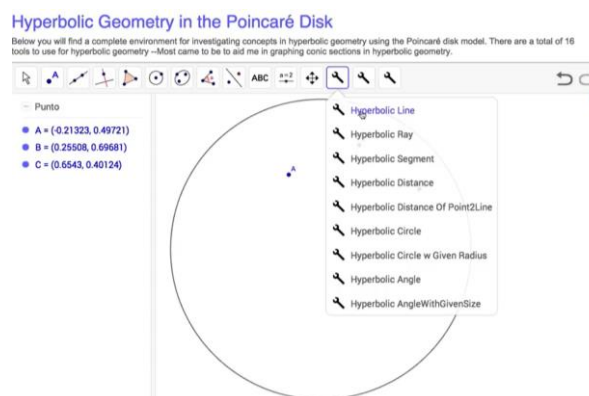


Figure 12. GeoGebra Poincaré disk.

1. Then follow the instructions given and answer the questions.
 - b. Select the point button, now place three points in different parts of the Poincaré disk.
 - c. Choose the hyperbolic line button, select point A and point B and draw the hyperbolic line.
- 1.1. With the help of the instructions above, answer the following questions.
 - Are these hyperbolic lines perpendicular arcs of circumference at the two points of the Poincaré disk? Explain.

- By choosing a point A and moving the pointer the perpendicularity is maintained? Does the same thing happen when choosing point B? Argument.
2. Then follow the instructions given and answer the questions.
 - a. Choose another point on the Poincaré disk such that another hyperbolic line is drawn and this one intersects the previous line.
 - b. Choose two other points on the middle of the Poincaré disk such that it does not intersect the two previous hyperbolic lines.
 - 2.1. With the help of the above instructions, answer the following question:
 - Will these three hyperbolic lines be parallel? Justify your answer.
 3. Then follow the instruction and answer the question.

Proceed to place three points on the Poincaré disk, select the "Hyperbolic Segment" button.

 - What figure is formed? Explain its difference with respect to Euclidean geometry.
 - 4.2. With the "Hyperbolic Angle" button proceed to calculate the angles of the triangle, and answer the following questions:
 - Do the angles of the triangles in this hyperbolic model add up to more or less than 180 degrees, how much do their three angles add up to?
 - 4.3. Select the points of the triangle and scroll the Poincaré disk:
 - What happens to the angles of the triangle if you bring the three vertices close to the ends of the Poincaré disk?
 - What happens to the angles of the triangle if you bring the three vertices closer to the center of the Poincaré disk?
 - 4.4. Construct a perpendicular bisector on the Poincaré disk (hyperbolic geometry). Follow the given instructions and answer the question.
 - a. Locate two points A and B inside the Poincaré disk.
 - b. Select the "Hyperbolic Segment" button.
 - c. Now, place yourself at point B and select the "Hyperbolic Circle" button, draw the hyperbolic circle passing through point A, and proceed in the same way at point A. At the intersection points of the two circles draw the hyperbolic line passing through these two points. Now, select point B and start moving it.
 - What happens with the hyperbolic mediatrix? Argument.
 - 4.5. Check with the "Hyperbolic Angle" button that the intersection point of the segment between points A and B and the perpendicular bisector is 90 degrees. \overline{AB} measure the same with the "Hyperbolic Distance" button.
 - 4.6. Construct a bisector on the Poincaré disk (hyperbolic geometry). Follow the given instructions. At the end answer the question.

- Construct with the "Hyperbolic Ray" button two hyperbolic ray lines \overrightarrow{AB} y \overrightarrow{AD} respectively (see Figure 13),

Figure 13. Bisector construction.

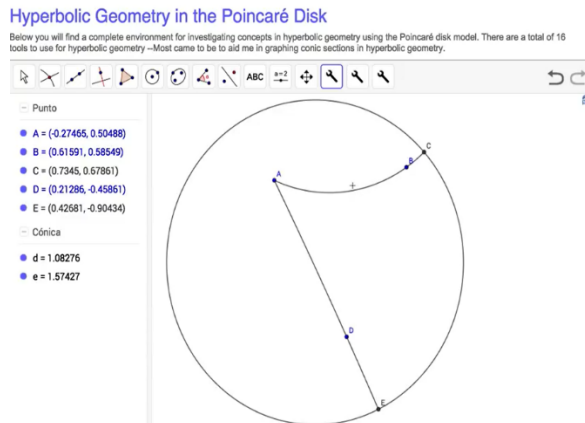
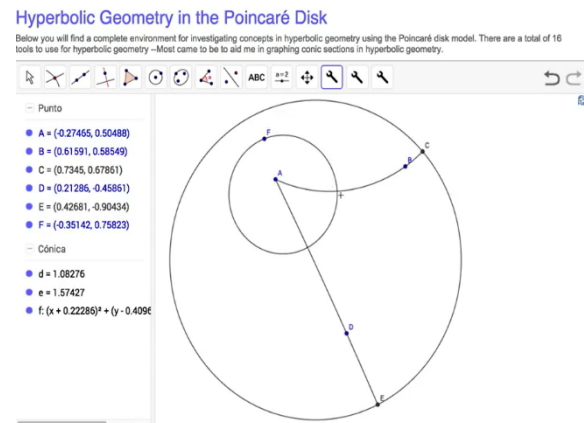


Figure 14. Bisector construction.



- Then choose the Hyperbolic Circle button and, looking at point A, draw a circle c1 (see Figure 10).
- Locate the intersection points G and H of the circle c1 with the two hyperbolic ray lines.
- Now, with the tool "Hyperbolic Circle" look at point G and as it passes through point H draw the circle c2.
- Repeat this at point H passing through point G and trace circle c3.
- Two intersection points are generated between circles c2 and c3 respectively; choose a point of the intersection and denote it by I.
- Finally, with the "Hyperbolic Line" tool draw a hyperbolic line passing through points A and I respectively.

Check that the hyperbolic line is the bisector by calculating the measure of the angles.

4.1.8. Median, perpendicular bisector and bisector in a hyperbolic triangle

Objective: to enhance geometric thinking through the realization of constructions about the median, perpendicular bisector and bisector in a hyperbolic triangle with GeoGebra software.

Methodological suggestions: each student must construct the medians, medians and bisectors of a hyperbolic triangle with GeoGebra software. A series of questions are posed comparing the constructions with ruler and compass elaborated in Euclidean geometry class with the objective that students compare Euclidean and hyperbolic geometries.

Purpose of the activity: students solve the activity on hyperbolic geometry using constructions elaborated through the computer by means of GeoGebra software. In this process, students can compare the work done in the classroom on the topic based on Euclidean geometry with that done on the computer. Each construction has a series of questions that students must answer to improve their internalization of geometric concepts and procedures.

Construction of remarkable lines of a hyperbolic triangle.

1. Construct the perpendicular bisectors of a hyperbolic triangle $\triangle ABC$ inside the Poincaré disk. Follow the instructions given.

- a. Using the "Hyperbolic Midpoint" button, find the points D, F and E, which are the midpoints of the hyperbolic segments \overline{AB} , \overline{AC} and \overline{BC} , respectively, in the hyperbolic triangle $\triangle ABC$.
- b. Now, look at the point D and with the tool "Hyperbolic Perpendicular at Point" and clicking on the segment \overline{AB} the segment, draw the perpendicular bisector of the mentioned segment. Repeat the process at points F and E, respectively.
- 1.1. With the help of the above construction, answer the following questions.
 1. Do the three perpendicular bisectors intersect at the same point? Justify your answer.
 2. Is it the same for any triangle? Explain.
- 1.2. Construct the circumscribed circle and then answer the questions.
 1. Does the circumscribed circumference hold for different types of triangles? Justify your answer.
 2. What is the point of intersection of the three hyperbolic lines (medians) called?
2. Construct the medians of a hyperbolic triangle inside the Poincaré disk following the instructions.
 - a) Using the "Hyperbolic Midpoint" button, find the midpoint of each hyperbolic segment (side) of the triangle.
 - b) Now, choose the "Hyperbolic Line" button and draw a hyperbolic line from each vertex to the midpoint of the opposite side of the hyperbolic triangle.
- 2.1. With the help of the above construction, answer the following questions.
 1. Will the three medians cut at the same point? Explain.
 2. Is it the same for any hyperbolic triangle? Justify your answer.
 3. What is the point of intersection of the three hyperbolic lines (medians) called?
3. Construct the bisectors of a hyperbolic triangle $\triangle ABC$ inside the Poincaré disk. Follow the instructions given.
 - a) Look at point A and with the "Hyperbolic Angle Bisector" button, find the midpoint of the hyperbolic segment. \overline{AB} Repeat the process at point B and the hyperbolic segment, repeat the process at point C and the hyperbolic segment. \overline{AC} Repeat the process at point C and the hyperbolic segment. \overline{BC} .
 - b) With this tool select each vertex, draw the bisector through the selected vertex, repeat the process until the three bisectors are constructed.
- 3.1. With the help of the above construction, answer the following questions.
 1. Do the three bisectors intersect at the same point? Justify your answer.
 2. Does it happen for any hyperbolic triangle? Explain.
 3. What is the point of intersection of the hyperbolic bisectors called?
 5. Is it possible to construct a circle inscribed in the triangle from the point of intersection of the three hyperbolic bisectors? Justify.

4.1.9. Construction of the equilateral triangle and square in the Poincaré disk using GeoGebra

Objective: to enhance geometric thinking through constructions of the equilateral triangle and square in the Poincaré disk with GeoGebra software.

Methodological suggestions: each student must carry out the construction with GeoGebra of an equilateral triangle in the Poincaré disk and compare the result with the construction in Euclidean geometry. This process is repeated with the construction of the square. After this, another construction model is made inside the inversion disk.

Purpose of the activity: by performing the activity the student compares the results obtained in Activity 3 with those obtained here and then answers a series of questions that help him/her to consolidate the concepts in Euclidean and hyperbolic geometry.

1. Construct an equilateral triangle in hyperbolic geometry from the following instructions and answer the given questions.

- a) With the Hyperbolic Segment button draw a segment inside the Poincaré disk to obtain the A and B vertices.
- b) Look at point A and with the Hyperbolic Circle tool draw the circle with center at point A and passing through point B, repeat the process with center at point B and passing through point A.
- c) Locate the point of intersection of the two hyperbolic circles; this will be called point C.
- d) With the Hyperbolic Segment button draw the hyperbolic segments from point A to point C and from point B to point C.

1.1. With the help of the above construction, answer the following questions.

1. How long is each side of your triangle? Are they equal?
2. What is the measure of each angle of your triangle? Are they equal?
3. If points A and B are placed in different regions of the Poincaré disk and this same construction is performed, do the previous results hold?
4. What is the sum of the internal angles of the constructed triangle?

2. Construct a square in hyperbolic geometry on a given side from the following instructions and answer the given questions.

- a) With the Hyperbolic Segment button draw a segment inside the Poincaré disk to obtain the A and B vertices.
- b) Now, with the Hyperbolic Perpendicular at Point button look at point B and click on the segment AB. This line is called ℓ_1 .
- c) Repeat the process at point A. This line is called ℓ_2 . Now you have two hyperbolic perpendicular lines ℓ_1 and ℓ_2 at the two vertices A and B, respectively.
- d) With the Hyperbolic Circle tool position, yourself at point B and draw the circle c_1 passing through point A. Repeat the process at point A and draw the circle c_2 passing through point B.
- e) Then locate the intersection point between circle c_2 and line ℓ_2 , this will be point E. Continue with the same process locating the intersection point between circle c_1 and line ℓ_1 .
- f) With the Hyperbolic Segment button draw the segments from point A to point E, from point E to point F and from point F to point B.

2.1. With the help of the above construction, answer the following question:

1. In the above construction, the same procedure used in Euclidean geometry is used. Are the same results obtained in the Poincaré disk by applying the same procedures?

3. Construct a quadrilateral with equal sides from the following instructions and answer the questions posed.

- a) Locate two points A and B on the Poincaré disk.
- b) With the Hyperbolic Line tool draw the hyperbolic line ℓ_1 .
- c) Through the Hyperbolic Perpendicular at Point tool, draw a line ℓ_2 perpendicular at point A. On this perpendicular line with the Hyperbolic Circle button, draw a circle c_1 .
- d) Finally, locate the intersection points between the hyperbolic line ℓ_1 and circle c_1 . Perform the same procedure, locating the points of intersection between the hyperbolic line ℓ_2 and circle c_1 .
- e) With the Hyperbolic Segment tool draw the segments joining these four intersection points.

3.1. With the help of the above construction, answer the following questions.

1. Do these segments have the same length? Justify your answer.
2. Do the angles of this figure have the same measure? Justify your answer.
3. What happens if the points are shifted? Explain your answer.

Activity 10. Constructions with ruler and compass in the inversion disk or Poincaré disk.

Objective: to promote geometric thinking through the contrast of construction in GeoGebra and with ruler and compass.

Methodological suggestion: each student develops with ruler and compass orthogonal circles, medians, perpendicular bisectors, circumcenters and tessellations within the inversion disk or Poincaré disk, comparing the result of their constructions made in GeoGebra with those implemented with ruler, square and compass in hyperbolic geometry.

Purpose of the activity: by performing the activity through constructions with ruler and compass within the inversion disk, the student reinforces his learning processes, compares the results obtained in the computer through GeoGebra software and answers a series of questions that help him to reinforce concepts in Euclidean and hyperbolic geometry.

Next, you must make the following constructions using a ruler, square and compass.

1. Draw a circle c_1 of radius 10cm (this will represent the Poincaré disk).
 - a. Construct two orthogonal circles with ruler, square and compass. Are the radii of the three circles orthogonal to each other?
2. Construct the inversion of a point from the interior of the Poincaré disk to the exterior and vice versa.
3. Locate any two points on the inversion disk.
 - a. Draw a circle orthogonal to c_1 passing through the two points.
4. Calculate the perpendicular bisector of the two previous points.
5. Construct an equilateral triangle inside the inversion disk.
6. Elaborate a square inside the Poincaré disk.
7. Construct on the Poincaré disk the tessellation given in Figure 11 below.

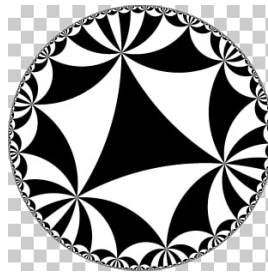


Figure 15. Tessellation.

6. Conclusions

The activities have been designed with the objective of improving the teaching and learning of geometry at school, since this branch of mathematics is of great interest to students. The implementation of constructions with ruler and compass in the classroom results in the transformation of traditional and monotonous classes to motivating and interesting classes for the students, which leads to an improved understanding of new geometric concepts.

The comparison of different geometric constructions is proposed, both with GeoGebra software and geometric instruments. This is done in both geometries, Euclidean and hyperbolic, with the aim of encouraging students to analyze, argue and justify the discovery of different properties inherent to geometric figures.

7. Recommendations

- To continue researching on the inclusion of hyperbolic geometry in school and to propose new challenging constructions for classroom work with seventh grade students in Basic Education.
- Motivate students for the study of geometry, specifically towards the process of making constructions with ruler and compass, due to the potential that this work has for the development of geometric thinking.

- Encourage the work of geometric constructions with the use of GeoGebra software as a didactic tool as a way to achieve robust knowledge of the contents.
- Include in the school curriculum the study of non-Euclidean geometry, particularly hyperbolic geometry, in order to favor the teaching and learning process of Euclidean geometry and thus contribute to the development of geometric thinking.
- Investigate the inclusion in the school curriculum of other non-Euclidean geometries, to strengthen the process of geometric thinking in students.

References

1. Arboleda, L. C. & Anaconda, M. P. (1994). *Non-Euclidean Geometries in Colombia, Professor Julio Garavito (1865-1920)*.
2. Arcavi, A. (2003). The role of visual representations in the teaching and learning of mathematics. *Educational Studies in Mathematics*, 52(3).
3. Atanasova-Pacemska, T., Gunova, V., Lazarova, L., & Pacemska, S. (2016). Visualization of the Geometry Problems in Primary Math Education-Needs and Challenges. *IMO-Istrazivanje matematickog obrazovanja*, 8(15), 33-37.
4. Ballester, S. et al. (1992). *Metodología de la Enseñanza de la Matemática*. Havana: Pueblo y Educación.
5. Barker, S. (1969). *Filosofia da Matemática*. Rio de Janeiro: Zahar Editores.
6. Boyer, C. B. (1996). *Historia da Matemática*. Sao Paulo: Editora Edgard Blucher.
7. Cabariti, E. (2004). Geometria Hiperbólica: uma proposta didactica em ambiente informatizado. *Dissertacao de Mestrado em Educacao Matemática*, Pontifícia Universidade Católica de Sao Paulo.
8. Cantoral, R., & Montiel, G. (2003). Visualization and mathematical thinking. *Acta Latinoamericana de Matemática Educativa*, 16(2), 694-701.
9. Chiappini, G., & Bottino, R. M. (1999). *Visualisation in teaching-learning mathematics: the role of the computer*. Computer Graphics And Visualization Education.
10. Christi, D. (2011). *Hyperbolic geometry in the high school geometry classroom* (Senior thesis). Iowa State University, Iowa, USA. Retrieved from http://www.math.iastate.edu/thesisarchive/MSM/Donald_C_MSM_F05.pdf.
11. Cowley, C. S. (2009). *Spherical and hyperbolic geometry in the high school curriculum* (Doctoral dissertation, University of Texas).
12. David, H. (1971). *Foundations of geometry (Grundlagen der Geometrie)*, translated by Leo Unger, Open Court, La Salle, IL.
13. De Guzmán, M. (1996). The Blackboard Corner. Chap. 0. *The role of visualization*. Madrid: Pirámide.
14. Ding, L., Fujita, T. and Jones, K. (2005). CERME 4. *Developing geometrical reasoning in the classroom: Learning from highly experienced teachers from China and Japan*. University of Southampton, University of Glasgow, University of Southampton. Retrievable December 11, 2014 from URL: http://www.mathematik.uni-dortmund.de/~erme/CERME4/CERME4_WG7.pdf.
15. Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for research in mathematics education*, 110-136.
16. Duval, R. (2005), Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de Didactique et Sciences Cognitives*, 10, 5-53.
17. Dreyfus T. (1995), Imagery for Diagrams in R. Sutherland and J. Mason (eds) *Exploiting mental imagery with computer in Mathematics education*, NATO ASI Series, Springer - Verlag, Berlin, Series F, Vol 138, 3-19

18. Escalona, M. (2007). *The use of computer resources to favor the integration of contents in the area of exact sciences in pre-university*. 2007. Unpublished doctoral thesis. Instituto Superior Pedagógico "José de la Luz y Caballero", Holguín. p. 65.
19. Euclid (2007). *Elements of Euclid. Books I - VII*. Biblioteca Gredos.
20. Eves, H. (2002). *Introdução à História da Matemática*. Translated by Hygino H. Domingues. São Paulo: Unicamp.
21. Fish, W. (1996). *Non-Euclidean geometry and its possible role in the secondary school mathematics syllabus* (Doctoral dissertation).
22. Fitzpatrick, M. O. M. (1964). *Saccheri, forerunner of non-Euclidean geometry*. The Mathematics Teacher, P. 323-332.
23. García, J. (1983). *Motivation and self-learning key elements in student learning and study*, retrievable on 14 06 06 2014 from URL: http://www.uclm.es/ab/educacion/ensayos/pdf/revista17/17_11.pdf, p.6
24. Geometry, *Uniciencia*, 27(1), 74-94. p. 75. p. 75
25. González, Á. and Vílchez N. (2002). *Teaching Geometry Using Multimedia Resources*. Application to the First Stage of Basic Education. San Jacinto Sector. Trujillo. Venezuela. Recoverable on October 03, 2014 from URL: http://www.saber.ula.ve/bitstream/123456789/14318/1/vilchez_nieves2.pdf
26. González, L. (2004). *Motivation to study. Fundamentals and methodology for its evaluation in elementary school*. Thesis presented in option to the scientific degree of doctor in pedagogical sciences. University of Pinar del Río "Hermanos Saíz Montes de Oca". Havana.
27. Gravina, M. (1996). *Geometria dinamica uma nova abordagem para o aprendizado da geometria*.IN: Anais do VII Simpósio Brasileiro de Informática na Educação, p.1-13, Belo Horizonte.
28. Gutiérrez, A., & Jaime, A. (1998). *On the assessment of the Van Hiele levels of reasoning. Focus on learning problems in mathematics*, 20, 27-46.
29. Guzman, M. (2000). *The role of visualization In the Teaching and Learning of Mathematical Analysis*.
30. Henderson, D. (2000). *Experiencing Geometry in Euclidean, Spherical, and Hyperbolic Spaces*, Upper Saddle River, NJ: Prentice Hall.
31. Henderson, D. (2001). *Mathematical Intelligencer*, Vol. 23, No. 2, pp. 17-28.
32. Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, 74(1), 11-18.
33. Hohenwarter, Markus & Preiner, Judith. (2007). *Dynamic Mathematics with GeoGebra*. The Journal of Online Mathematics and Its Applications RECOVERABLE, https://www.maa.org/external_archive/joma/Volume7/Hohenwarter/index.html
34. John, P. (1839). *The Elements of Geometry: Containing the First Six Books of Euclid*, from the last London Edition, W.E. Dean, New York.
35. Jones, K. (2002). Issues in the teaching and learning of geometry In, Haggarty, Linda (eds.) *Aspects of teaching secondary mathematics: Perspectives on practice*. London, GB, Routledge pp. 121-139.
36. Keith Jones, July (2001). *Teaching and learning geometry*, Recoverable <https://pdfs.semanticscholar.org/9205/1fc5874bd732a3a1ecbecae5f9f221f2c040.pdf>
37. Keller, J. (1983,1984,1987). Motivational model. Retrievable, 06 7, 2014, from URL: <http://files.eric.ed.gov/fulltext/ED409895.pdf>
38. Kline, M. (1986). El fracaso de la Matemática Moderna - *Por qué Juanito no sabe sumar Siglo Veintiuno Editores* - México, p.p. 189 y 190.
39. Komarnicki, N. et al. (2013). *100 Geometric Constructions with Manual and Computer Tools*. Buenos Aires: Duken.
40. Krulik, S., & Rudnik, J. (1980). *Problem solving: a handbook for teachers*. Boston: Allyn and Bacon . p. 4.

41. Labarrere, A. (1988). *How to teach elementary school students to solve problems*. Havana: Editorial Pueblo y Educación. p. 6.
42. Lénárt, I. (2010). *Gauss, Bolyai, Lobachevsky-in General Education?* (Hyperbolic Geometry as Part of the Mathematics Curriculum). In *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture* (pp. 223-230). Tessellations Publishing.
43. Leoni, M. & Reinaldo, F. (2013) *O Ensino das Geometrias Não-Euclidianas: uma prática metodológica investigativa e reflexiva no Ensino Médio*. book Brasil: PDF, pp.15. Available at: <http://www.diaadiaeducacao.pr.gov.br/portals/pde/arquivos/698-4>
44. Liljedahl, P., Santos-Trigo, M. (2019). *Mathematical Problem-Solving Current Themes, Trends, and Research*. ICME-13 Topical Study Groups. Springer Nature Switzerland AG 2019
45. M. Becheanu, International Mathematical Olympiads. 1959-2000, The Academic Distribution Center, 2001.
46. Mammana, C. & Villani, V. (1998). *Perspectives on the Teaching of Geometry for the 21st Century*. Kluwer Academic Publishers.
47. Marcondes, T. (2014). *Geometrias hiperbólicas com uso do GeoGebra*. Dissertação (Mestrado Profissional em Matemática em Rede Nacional) - Universidade Tecnológica Federal do Paraná, Curitiba.
48. María, M. (2004). *A pedagogical conception of motivational stimulation in the teaching-learning process*. PhD Thesis. Retrievable on 10/17/2014, from URL: <http://www.bibliociencias.cu/gsd/collect/tesis/index/assoc/HASHe802.dir/doc.pdf>, p. 3.
49. Martin, G. E. (1975). Saccheri's Three Hypotheses. In *The Foundations of Geometry and the Non-Euclidean Plane*. Springer, New York, NY.
50. Mason, J., Burton, L. and Stacey, K. (1988). *Thinking mathematically*. Madrid: Editorial Labor.
51. Mathematization (2009). retrievable on 5 01 01 2015, from URL: <http://ciberdocenciagobpe.blogspot.com/2009/11/matematizacion.html>.
52. Miller, J. (1998). *The psychology mathematical*. Princenton University Press, Princenton.
53. Minerva, F. (2006). *The scientific research process*. Zulia, Venezuela: Universidad del Zulia.
54. Ministry of Education of Peru. Problem Solving. (2015) from URL: http://www2.minedu.gob.pe/digesutp/formacioninicial/wp-descargas/educacionprimaria/didactica_mat/04_resolucion_de_problemas.pdf.
55. Ministry of National Education (2006). *Estándares Básicos de Competencias en Lenguaje, Matemáticas, Ciencias y Ciudadanas* (Basic Competency Standards in Language, Mathematics, Science and Citizenship). Bogotá: Imprenta Nacional, p. 62.
56. Moreno, M. (2004). *A pedagogical conception of motivational stimulation in the teaching-learning process*. Thesis in option to the degree of doctor in pedagogical sciences. Higher Pedagogical Institute "Enrique José Varona" Faculty of Educational Sciences. Havana, Cuba.
57. National Council of Teachers of Mathematics. (2003), *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA.
58. Ensino das Geometrias Não-Euclidianas (2013): *uma prática metodológica investigativa e reflexiva no Ensino Médio*. 1st ed. [ebook] Brazil. Available at: <http://www.diaadiaeducacao.pr.gov.br/portals/pde/arquivos/698-4>.
59. Olive, J. (2000). *Implications of using dynamic geometry technology for teaching and learning*. Ensino e Aprendizagem de Geometria. Lisboa: SPCE, 7.
60. Pavanello, M. (1996). *O abandono do ensino da Geometria: Uma visão histórica*. GRAVINA, M. A. Geometria dinâmica uma nova abordagem para o aprendizado da geometria. IN: Anais do VII Simpósio Brasileiro de Informática na Educação. Belo Horizonte.
61. Petakos, K.& Sgreccia, N. (n.d). *Euclid's fifth postulate: A classroom proposal for senior high school*. Journal of Mathematics Education, 25(2).

-
62. Poincaré, H. and Einstein, A. (1948). *Fundamentals of Geometry*. Buenos Aires: Iberoamericana. pp. 92 and 93.
63. Polya, G. (1957). *How to Solve It*, second edition, retrievable 2 Aug 2014, from URL: https://notendur.hi.is/hei2/teaching/Polya_HowToSolveIt.pdf, p.6.
64. Polya, G. (1965). *Cómo plantear y resolver problemas*. Mexico City: Editorial Trillas. p.117
65. Presmeg, N. (2006). *Research on visualization in learning and teaching mathematics: emergence from psychology*. In: Gutiérrez, A. and Boero, P. (Eds), *Handbook of research on the psychology of mathematics education*. Rotterdam: Sense.
66. Troubleshooting. (2014), compilation, from URL: http://www2.minedu.gob.pe/digesutp/formacioninicial/wp-descargas/educacionprimaria/didactica_mat/04_resolucion_de_problemas.pdf
67. Ribeiro, R. (2013). *Geometrias não-euclidianas na escola: uma proposta de ensino através da geometria dinâmica*.
68. Richey, Fields, & Foxon, (2001). *Instructional design*. Retrievable 12 09 2015 from URL: <http://www.ugr.es/~sevimeco/revistaeticanet/numero12/Articulos/Formato/articulo4.pdf>.
69. Rocha, Marília Valério. (2009). *Uma proposta de ensino para o estudo da geometria hiperbólica em ambiente de geometria dinâmica*. Dissertation (Master in Education) - Pontifícia Universidade Católica de São Paulo, São Paulo.
70. Rojas, O. (2009). *Didactic model to favor the teaching - learning of geometry with a developmental approach*. (Unpublished doctoral thesis). University of Pedagogical Sciences José de la Luz y Caballero. Holguín, Cuba. P48.
71. Rosenfeld, B. (1988), *History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*. With the editorial assistance of Hardy Grant. New York (Springer-Verlag). 1988. Xi, 471 pp.
72. Royer, J. M. (1979). Theories of the transfer of learning. *Educational Psychologist*, 14(1), 53-69.
73. Saccheri, G. (1903). *Revista filosófica*.
74. Sampieri, R. H., Collado, C. F., Lucio, P. B., & Pérez, M. D. L. L. C. (1998). *Research methodology* (Vol. 1). Mexico: Mcgraw-hill.
75. Sandín Esteban, M. P. (2003). *Qualitative research in education. Fundamentals and traditions*. Madrid: Mc Graw Hill.
76. Santos, T. A (2009). *Inclusão das Geometrias Não-Euclidianas no Currículo da Educação Básica*. Mestrado em Educação para a Ciência e o Ensino de Matemática - PCM - Maringá.
77. Schumann, H. and Green, D. (1994). *Learning geometry through interactive construction*. In: *Discovering Geometry with a computer*. Using CabriGéomètre. Ed. Chartwell-Bratt.
78. Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.
79. Sierra, S. and Alicia, R. (2002). *Modeling and strategy: Some considerations from a pedagogical perspective*. Havana: Pueblo y Educación. p. 317.
80. Tejada, D. Non-Euclidean geometry: Friedrich Gauss, 1777-1855, John Bolyai, 1802-1860; Nikolai Lobachevsky, 1793-1856; Bernhard Riemann, 1826-1856.
81. Taken from Ribeiro, R. S. (2013). *Geometrias não-euclidianas na escola: uma proposta de ensino através da geometria dinâmica*, p. 30.
82. Retrieved Henderson D, W. (2001), *Mathematical Intelligencer*, Vol. 23, No. 2, pp. 17-28, retrievable <http://www.math.cornell.edu/~dwh/papers/crochet/crochet.html>.
83. Valdeni Franco, & Luana P. Goulart de Menezes, (2012) *Utilizando O Geogebra Para Construção De Modelo Plano Para A Geometria Elíptica*, Recoverable <http://www.geogebra.org.uy/2012/actas/11.pdf>.
84. Vargas, G., and Gamboa, R. (2013). *Van Hiele's model and the teaching of geometry*.
85. Wolfe, H. E. (1945). *Introduction to non-euclidean geometry*. New York: The Dryden Press, 1945.
-

86. Zimmermann, W. and S. Cunningham (1991). Editor's introduction: *What is mathematical visualization?* In W. Zimmerman and S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics*. Washington, DC: Mathematical Association of America.