Research Article

Bifurcation of Periodic Solution in Singular Perturbed Parameterized Ordinary Differential Equation (ODEs)

Khalid Frhan Fazea [a], Kamal Hamid Yasir Al-Yassery[b]

[a] Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Nasiriyah, Iraq.
[b] Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Nasiriyah, Iraq.
[a]Email:kf97355@gmail.com ; k.frhan@utq.edu.iq
[b]Email:Kamalyasser2016@gmail.com ; Kamalyasser @utq.edu.iq

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Abstract:
In this paper, we study the bifurcation of the periodic solutions of the singularly perturbed parameterized differential equation(ODEs) of the form

$$\dot{x} = f(x, y, \beta)$$
$$\epsilon\dot{y} = g(x, y, \beta)$$

where β is a bifurcation parameter and ϵ is a perturbed parameter, $0 < \epsilon \ll 1$. Our study focuses on Poincare map as a periodic solution of such ODEs. We have discussed and study the basic types of bifurcation in ODE that is saddle node, transcritical, pitchfork, and Hopf bifurcation.

Keywords: Bifurcation, Periodic solution, Singular parameterized (ODEs).

1. Introduction

Many of the nonlinear issues that arise in mathematics and physics are understood to be written in the form of an operator equation

$$\mathrm{F(x,y,\beta) = 0}, \qquad \mathrm{x \in R^n, y \in R^m, \beta \in R} \tag{1.1}$$

In such kind of equatisons, we reduce them to an equation of finite dimension which is given as:

$$\begin{aligned}\dot{x} &= f(x, y, \beta)\\ \epsilon\dot{y} &= g(x, y, \beta)\end{aligned} \tag{1.2}$$

where $f : R^n \times R^m \times R \rightarrow R^n$, $g: R^n \times R^m \times R \rightarrow R^m$, and β is bifurcation parameter, ϵ is perturbation parameter, $0 < \epsilon \ll 1$. Equation(1.2) is the reduced equation of (1.1) that has the same properties and solutions. We can rewrite (1.2) as:

$$A(\epsilon)\,\dot{z} = F(z, \beta) \tag{1.3}$$

where $\dot{z} = (\dot{x}, \dot{y})^T$ denotes the derivative of z, $z = (x, y)^T$, $z \in R^{n+m}$, and

$$A(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix},$$

Systems of the form (1.3) are ubiquitous in mathematical models in physics, engineering, chemistry, economics, finance, etc [4]. When $\epsilon \rightarrow 0$ in (1.3) we get:

$$\begin{aligned}\dot{x} &= f(x, y, \beta)\\ 0 &= g(x, y, \beta)\end{aligned} \tag{1.4}$$

The last equation (1.4) is the differential algebraic equation (DAE). Due to the generality of this type to deal with the resulting simulation, its study increases with the growth of modern modeling that focuses on dynamic systems. However, many of the problems that researchers face in this field, are challenging to study DAE. The existence of constraint $\mathbf{0 = g(x, y, \beta)}$ makes DAE not easy to be studied and understand.

Perturbation theories emerged in 1886 at the hands of Poincare and Stielties and then standard further in the nineteenth century, and has been a lot in the application of this field from 1905[10]. Wazewska-Czyzewska and Lasota (1976) study of evolutionary biology [14].Chow and Mallet-Paret(1983) studied equation (1.3) and demonstrated that for certain parametric values, Hopf bifurcation can occur[3].Mallet-Paret(1983) analyzed equation (1.3) and gave the product of Hopf bifurcation and the continuity of periodic solutions globally [9].Weishi Liu(1999)study exchange lemmas for singular perturbation problems with certain turning Points [8].Abdulah.Jamil Tamraz (1988) discussed the bifurcation of solution occurs near the non-hyperbolic fixed and periodic orbit of singularly perturbed delay differential equation [13]. Mohan K. Kadalbajoo (2002) studied a survey of numerical techniques for solving singularly perturbed ordinary differential equations [6].

For equation (1.3) periodic solutions depend on small perturbations. In other words equation(1.3) has T-periodic solution if and only if $z(t + T) = z(t)$, based on ϵ. The study of the solution of equation (1.3) in one dimension is easier than if it has two dimensions. That is because step space can become overcrowded when plotting the solutions to any nonlinear issues and the underlying structure can become blurred [6]. At the end of the nineteenth century, a simple instrument was proposed by Henri Poincare is called the Poincare map, and it is a useful tool when it comes to understanding the behavior of periodic orbits of two-dimensional systems. It reduces the problem in one dimension and provides a deeper understanding [7]. In this paper, we study the bifurcations of the periodic solution of the equation of type (1.3) by studying the conditions on its Poincare map, by assuming $P(z,\beta)$ is the Poincare map of the singularity perturbed for the equation (1.3), and the periodic solution is non-hyperbolic. As a result, this enables us to study the bifurcation of the periodic solution of two-dimensional systems.

2. Related definitions and theorems

In this section, we compose the basic definitions and theorems we need and build upon for our subsequent study.

Definition 2.1.[11] Assume that $\mathrm{z(t)} = \Phi_{\mathrm{t}}(z_0,\beta_0) = \Phi(\mathrm{t}, \mathrm{z}_0, \beta_0)$ is a periodic solution of equation (1.3) and $P(z,\beta)$ is the Poincare map then z(t) is a non-hyperbolic periodic solution iff $DP(z_0,\beta_0)$ has eigenvalues with unit modulus, and it is a hyperbolic periodic solution iff none of the eigenvalues of $DP(z_0,\beta_0)$ has unit modulus where $DP(z_0,\beta_0)$ is Jacobian matrix at (z_0,β_0).

The following lemma is related to this topic.

Lemma 2.1. [5]. If $\boldsymbol{\Phi(t, z_0, \beta_0)}$ is the periodic solution of the equation (1.3) with initial condition $\boldsymbol{\Phi(t, z_0, \beta_0) = z_0, t \in [0, T]}$, then $\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{z_0}}\boldsymbol{(t, z_0, \beta_0)}$ is a solution for the equation :

$$\boldsymbol{\dot{H}(t) = A^{-1}(\epsilon)\frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\Phi(t, z_0, \beta_0))H \ , H(0) = 1}, \qquad (\mathbf{2.1})$$

where,

$$H(t) = \frac{\partial \Phi}{\partial z_0}(t, z_0, \beta_0) = e^{\boldsymbol{A^{-1}}(\epsilon)\int_0^T \frac{\partial \mathrm{F}}{\partial z}(t, \Phi(t, z_0, \beta_0))dt} \qquad (2.2)$$

Proof: See reference [5].

Consider singularly perturbed ODEs in the form:

$$A(\epsilon)\dot{z} = F(z, \beta) \qquad (2.3)$$

where $z \in R^{n+m}$, $\beta \in R$; $0 < \epsilon \ll 1$,$F = (f, g)^T \in R^{n+m}$. Let $\Phi(t, z_0, \beta_0)$ is a periodic solution to the system (2.3). Assume that for $\beta = \beta_0$, the system has a periodic orbit Γ_0 given by $z(t) = \Phi_{\mathrm{t}}(z_0,\beta_0)$. Let Σ hyperplane perpendicular to periodic orbit at $(z_0,\beta_0) \in \Sigma$, then $\forall\ (z,\beta) \in \Sigma$ near (z_0,β_0), $\Phi(t,z,\beta)$ will cross hyperplane perpendicular again at a point $\mathrm{P}(z,\beta)$ near (z_0,β_0)[11] . The mapping $P : (z,\beta) \to \mathrm{P}(z,\beta)$ is called the Poincare map[2], and can be defined by :

$$z_{n+1} = P(z_n) \qquad (2.4)$$

where z_k maps to z_{k+1} by the map P [8]. Because z is a fixed point in the map we can write the Poincare map $P : \Sigma \to \Sigma$ by [5, 7] :

$$P: z \to P(z,\beta) = \Phi(t,z,\beta) \qquad (2.5)$$

This means that

$$P(z,\beta) = \Phi(t,z,\beta) \qquad (2.6)$$

If we differentiate both sides for z, we have:

$$P'(z,\beta) = \frac{\partial \Phi}{\partial z}(t,z,\beta) \tag{2.7}$$

About equation (2.2), it is clear that:

$$P'(z,\beta) = \frac{\partial \Phi}{\partial z}(t,z,\beta) = e^{A^{-1}(\epsilon)\int_0^T \frac{\partial F}{\partial z}\Phi(t,z,\beta)dt} \tag{2.8}$$

Let

$$\Lambda = A^{-1}(\epsilon) \int_0^T \frac{\partial F}{\partial z} \Phi(t,z,\beta)dt \,, \tag{2.9}$$

then we get :

$$P'(z,\beta) = e^{\Lambda} \tag{2.10}$$

Equation (2.10) is useful to study the stability of the periodic solution according to the following theorem :

Theorem 2.2.[4]. Given $z(t) = \Phi_t(z_0, \beta_0)$ is a T- periodic solution of (2.3) satisfying (2.10). Then :
(i) If $\Lambda < 0$, then $z(t)$ is asymptotically stable.

(i) If $\Lambda > 0$, then $z(t)$ is unstable.
Proof: see reference [4].

The following theorem shows the existence and continuity of the Poincare map.
Theorem 2.3.[11] Let the equation (2.3) has a periodic solution given by $z(t) = \Phi_t(z_0, \beta_0)$ with period T and that periodic orbits are given by: $\Gamma = \{ z \in R^n \times R^m : z(t) = \Phi_t(z_0, \beta_0) \ , 0 \leq T \leq t\}$ such that $\Sigma \perp \Gamma$. Then there exists a $\delta > 0$ and $\tau(z,\beta)$, defined and differentiable, continuous for $(z,\beta) \in N_\delta(z_0,\beta_0)$, such that $\tau(z_0,\beta_0) = T$, and

$$\phi_{z,\beta}(z,\beta) \in \Sigma, \forall (z,\beta) \in N_\delta(z_0,\beta_0)$$

Proof: see reference [11].

Definition 2.2.[11] Assume Γ, Σ, δ, and $\tau(z,\beta)$ are defined in theorem (2.3). So, $\forall (z,\beta) \in N_\delta(z_0,\beta_0) \cap \Sigma$. The function $P(z,\beta) = \phi_{\tau(z,\beta)}(z,\beta) \in \Sigma$ is called the Poincare map for Γ at (z_0,β_0).

3. Bifurcations at non-hyperbolic periodic solution

The periodic solution bifurcates at a non-hyperbolic periodic orbit. Types of bifurcations which can be occurred are (saddle-node, transcritical, pitchfork) bifurcations .These bifurcations occur when $DP(z_0,\beta_0)$ has one eigenvalue with modulus equal to 1, $(z_0,\beta_0) \in \Gamma$ [11] . Rewrite equation (2.3) in the form :

$$\dot{z} = A^{-1}(\epsilon)\, F(z,\beta) \tag{3.1}$$

Now we are going to show the basic kinds of bifurcations of periodic solutions that can be occurred in the singularly perturbed parameterized ODE.

3.1. Saddle-node bifurcation

When $DP(z_0,\beta_0)$ has one eigenvalue equal to one, then the periodic orbit can bifurcate into several branches. In the saddle-node bifurcation diagram, there is a single continuous curve of fixed points near the point $(z,\beta) = (0,0)$.
Theorem 3.1. Assume that singular perturbed ODEs (3.1) has Γ_0 (periodic orbit) with the Poincare map $P(z,\beta)$

who is defined in a neighborhood $N_\delta(0,\beta_0)$. If $P(0,\beta_0) = 0$, $DP(0,\beta_0) = 1$, and P satisfying the conditions :

1. $D^2P(0,\beta_0) \neq 0$
2. $P_\beta(0,\beta_0) \neq 0$

Then equation (3.1) expresses a saddle-node bifurcation at a non-hyperbolic periodic orbit Γ_0.

Proof Define the function

$$G(z,\beta) = P(z,\beta) - z$$

and differentiate the equation for β, then by condition 1 we have :

$$G_\beta(0,\beta) = P_\beta(0,\beta) \neq 0$$

and by implicit function theorem, there is a function $\beta : R^{n+m} \to R$, such that

$$G(z,\beta(z)) = 0, \beta(0) = 0 \tag{3.2}$$

So we have :

$$G(0,\beta_0) = P(0,\beta_0) = 0 \quad \& \quad DG(0,\beta_0) = DP(0,\beta_0) - 1 = 0.$$

By condition 1 and 2 we get :

$$D^2G(0,\beta 0) = D^2P(0,\beta 0) \neq 0 \quad \& \quad G_\beta(0,\beta_0) = P_\beta(0,\beta_0) \neq 0.$$

We want to prove properties :

$$\frac{\partial\beta}{\partial z}(0) = 0, \quad \frac{\partial^2\beta}{\partial z^2}(0) \neq 0$$

Now, differentiate the equation (3.2) for z we get :

$$\frac{\partial G}{\partial z}(z,\beta(z)) + \frac{\partial G}{\partial\beta}(z,\beta(z))\frac{\partial\beta}{\partial z}(z) = 0. \tag{3.3}$$

Setting $(z,\beta) = (0,\beta_0)$ in the above equation :

$$\frac{\partial G}{\partial z}(0,\beta_0) + \frac{\partial G}{\partial\beta}(0,\beta_0)\frac{\partial\beta}{\partial z}(0) = 0. \tag{3.4}$$

By condition $DG(0,\beta_0) = 0$, and $G_\beta(0,\beta_0) \neq 0$, then we get : $\frac{\partial\beta}{\partial z}(0) = 0$.

Now to prove that $\frac{\partial^2\beta}{\partial z^2}(0) \neq 0$, differentiate equation (3.4) twice to z, then we get:

$$\frac{\partial^2 G}{\partial z^2}(z,\beta(z)) + \frac{\partial G}{\partial\beta}(z,\beta(z))\frac{\partial^2\beta}{\partial z^2}(z) + 2\frac{\partial^2 G}{\partial z\partial\beta}(z,\beta(z))\frac{\partial\beta}{\partial z}(z) + \frac{\partial^2 G}{\partial\beta^2}(z,\beta(z))\left(\frac{\partial\beta}{\partial z}(z)\right)^2 = 0.$$

Setting $(z,\beta) = (0,\beta_0)$, then will be:

$$\frac{\partial^2 G}{\partial z^2}(0,\beta_0) + \frac{\partial G}{\partial\beta}(0,\beta_0)\frac{\partial^2\beta}{\partial z^2}(0) + 2\frac{\partial^2 G}{\partial z\partial\beta}(0,\beta_0)\frac{\partial\beta}{\partial z}(0) + \frac{\partial^2 G}{\partial\beta^2}(0,\beta_0)\left(\frac{\partial\beta}{\partial z}(0)\right)^2 = 0.$$

By condition $\frac{\partial^2\beta}{\partial z^2}(0) \neq 0$, and from $\frac{\partial\beta}{\partial z}(0) = 0$, we have :

$$\frac{\partial^2 G}{\partial z^2}(0,\beta_0) + \frac{\partial G}{\partial\beta}(0,\beta_0)\frac{\partial^2\beta}{\partial z^2}(0) = 0$$

And the condition $G_\beta(0,\beta 0) \neq 0$, yields :

$$\frac{\partial^2\beta}{\partial z^2}(0) = -\frac{\partial^2 G}{\partial z^2}(0,\beta_0)\left(\frac{\partial G}{\partial\beta}(0,\beta_0)\right)^{-1} \neq 0$$

Then equation (3.1) expresses a saddle-node bifurcation at non-hyperbolic periodic orbit Γ_0.

Example 3.1. Consider the Poincare map for ODE is given by :

$$P(z,\beta) = \beta + z - 2z^2$$

The Jacobian matrix at (0,0)given by:

$$DP(0,0) = 1 - 4z = 1.$$

Hence , (0,0) is non-hyperbolic, and there is a bifurcation at $\beta = 0$ .We see that

$$P(0,0) = 0, DP(0,0) = 1$$

then the conditions :

$$D^2P(0,0) = -4 \neq 0$$
$$P\beta(0,0) = 1 \neq 0.$$

Then the ODE has a saddle node bifurcation at $P(0,0)$.

3.2. Transcritical bifurcation

In this kind of bifurcation, we have two fixed points, where they exchange stability one for the other. And there are two curves of fixed points near the point (0,0) and they meet at the origin.

Theorem 3.2. Consider the singular perturbed ODEs (3.1) has Γ_0 (periodic orbit) with the Poincare map $P(z,\beta)$
who is defined in a neighborhood $N_\delta(0,\beta_0)$. If $P(0,\beta_0) = 0$, $DP(0,\beta_0) = 1$, and P satisfying the conditions :

1. $P_\beta(0,\beta_0) \neq 0$
2. $DP_\beta(0,\beta_0) \neq 0$
3. $D^2P(0,\beta_0) \neq 0$

Then equation (3.1) expresses a transtritical bifurcation at non-hyperbolic periodic orbit at Γ_0.

Proof Define the function

$$G(z,\beta) = P(z,\beta) - z$$

Since $z = 0$ is a curve of fixed points so the function G can be written as follows :

$$G(z,\beta) = A^{-1}(\epsilon)z\,V(z,\beta). \tag{3.5}$$

Where

$$V(z,\beta) = \begin{cases} \frac{G(z,\beta)A(\epsilon)}{z}, & if \;\; z \neq 0\,; \\ \frac{\partial G}{\partial z}(0,\beta), & if \; z = 0. \end{cases}$$

and $V(z,\beta)$ satisfies the following conditions :

$$V(0,\beta_0) = \frac{\partial G}{\partial z}(0,\beta_0) = DP(0,\beta_0) - 1 = 0$$
$$\frac{\partial V}{\partial z}(0,\beta_0) = \frac{\partial^2 G}{\partial z^2}(0,\beta_0) = D^2P(0,\beta_0) \neq 0$$
$$\frac{\partial V}{\partial \beta}(0,\beta_0) = \frac{\partial^2 G}{\partial z \partial \beta}(0,\beta_0) = DP_\beta(0,\beta_0) \neq 0$$

We want to prove that $\frac{\partial \beta}{\partial z}(0) \neq 0$
then, by implicit function theorem, there is a function $\beta : R^{(n+m)} \rightarrow R$ such that :

$$V\big(z,\beta(z)\big) = 0, \beta(0) = 0 \tag{3.6}$$

Differentiate equation (3.6) for z, we get :

$$\frac{\partial V}{\partial z}\big(z,\beta(z)\big) + \frac{\partial V}{\partial \beta}\big(z,\beta(z)\big)\frac{\partial \beta}{\partial z}(z) = 0.$$

By conditions :

$$\frac{\partial V}{\partial z}(0,\beta_0) \neq 0 \quad \& \quad \frac{\partial V}{\partial \beta}(0,\beta_0) \neq 0$$

we have :

$$\frac{\partial \beta}{\partial z}(0) = -\frac{\partial V}{\partial z}(0,\beta_0)\left(\frac{\partial V}{\partial \beta}(0,\beta_0)\right)^{-1} \neq 0.$$

Then equation (3.1) undergoes trans critical bifurcation from non- hyperbolic periodic orbit Γ_0 at $(0,\beta)$.

3.3. Pitchfork bifurcation

In this kind of bifurcation, there are two curves of fixed points passing through the point(0,0), one of which $z = 0$ on either side of $\beta = 0$ and the second $z^2 = \beta$ on one side of the curve $\beta = 0$ [12].

Theorem3.3. Consider the singular perturbed ODEs(3.1). Assume Γ_0 is the periodic orbit, and $P(0,\beta_0)$ is Poincare map who defined in a neighbourhood $N_\delta(0,\beta_0)$. Suppose that

$$P(0,\beta_0) = 0, DP(0,\beta_0) = 1$$

and the following conditions hold:

1. $P_\beta(0,\beta_0) = 0$
2. $D^2P(0,\beta_0) = 0$
3. $D^3P(0,\beta_0) \neq 0$
4. $DP_\beta(0,\beta) \neq 0$

Then the ODE (3.1) expresses a pitchfork bifurcation at a non-hyperbolic periodic orbit Γ_0.

Proof Define the function

$$G(z,\beta) = P(z,\beta) - z$$

Since $z = 0$ is a curve of fixed points so the function G can be written as follows :

$$G(z,\beta) = A^{-1}(\epsilon) z\, V(z,\beta). \tag{3.7}$$

Where

$$V(z,\beta) = \begin{cases} \frac{A(\epsilon)G(z,\beta)}{z}, if \ z \neq 0 ; \\ \frac{\partial G}{\partial z}(0,\beta), \quad if \ z = 0. \end{cases}$$

and $V(z,\beta)$ satisfies the following conditions :

$$V(0,\beta 0) = \frac{\partial G}{\partial z}(0,\beta_0) = DP(0,\beta_0) - 1 = 0$$
$$\frac{\partial V}{\partial z}(0,\beta_0) = \frac{\partial^2 G}{\partial z^2}(0,\beta_0) = D^2 P(0,\beta_0) \neq 0$$
$$\frac{\partial V}{\partial \beta}(0,\beta_0) = \frac{\partial^2 G}{\partial z \partial \beta}(0,\beta_0) = DP_\beta(0,\beta_0) \neq 0$$
$$\frac{\partial^2 V}{\partial z^2}(0,\beta_0) = \frac{\partial^3 G}{\partial z^3}(0,\beta_0) = D^3 P(0,\beta_0) \neq 0$$

We want to prove that $\frac{\partial \beta}{\partial z}(\beta(0)) = 0$.

Since $\frac{\partial V}{\partial \beta}(0,\beta) \neq 0$, then by implicit function theorem, there is a function $\beta : R^{(n+m)} \to R$ such that :

$$V(z,\beta(z)) = 0, \beta(0) = 0 \tag{3.8}$$

Differentiate the equation (3.8) w.r.t z, we get :

$$\frac{\partial V}{\partial z}(z,\beta(z)) + \frac{\partial V}{\partial \beta}(z,\beta(z))\frac{\partial \beta}{\partial z}(z) = 0.$$

By conditions : $\frac{\partial V}{\partial z}(0,\beta_0) = 0$, then we get:

$$\frac{\partial \beta}{\partial z}(0) = -\frac{\partial V}{\partial z}(0,\beta_0)\left(\frac{\partial V}{\partial \beta}(0,\beta_0)\right)^{-1} = 0.$$

Now we want to prove the property $\frac{\partial^2 \beta}{\partial z^2}(0) \neq 0$ of the pitchfork bifurcation. Differentiate (3.8) twice for z and evaluate at $(0,\beta_0)$:

$$\frac{\partial^2 V}{\partial z^2}(0,\beta_0) + \frac{\partial V}{\partial \beta}(0,\beta_0)\frac{\partial^2 \beta}{\partial z^2}(0) + 2\frac{\partial^2 V}{\partial z \partial \beta}(0,\beta_0)\frac{\partial \beta}{\partial z}(0) + \frac{\partial^2 V}{\partial \beta^2}(0,\beta_0)\left(\frac{\partial \beta}{\partial z}(0)\right)^2 = 0$$

And apply the property $\frac{\partial^2 V}{\partial z^2}(0) \neq 0$, we get : $\frac{\partial^2 \beta}{\partial z^2}(0) = -\left(\frac{\partial V}{\partial \beta}(0,\beta_0)\right)^{-1}\frac{\partial^2 V}{\partial z^2}(0,\beta_0) \neq 0.$

Then, we conclude that (3.1) undergoes a pitchfork bifurcation at a non-hyperbolic periodic orbit Γ_0.

3.4. Hopf bifurcation in singular perturbed parameterized ODE

In this section, we study the Hopf bifurcation of solution of singular perturbation parameterized differential equation. Consider the system of singular perturbed parameterized ODEs given by :

$$\begin{aligned} \dot{x} &= \beta x - y + S(x,y), \\ \epsilon \dot{y} &= x + \beta y + Q(x,y) \end{aligned} \tag{3.9}$$

where $x \in R^n$, $y \in R^m, 0 < \epsilon \ll 1$ is perturbed parameter and β is bifurcation parameter, and the analytic functions $S(x,y), O(x,y)$, in the form :

$$S(x,y) = \sum_{i+j \geq 2} a_{ij} x^i y^j$$
$$= (a_{20}x^2 + a_{11}xy + a_{02}y^2) + (a_{30}x^3 + a_{21}x^2 y + a_{12}xy^2 + a_{03}y^3) + \cdots$$
$$O(x,y) = \sum_{i+j \geq 2} b_{ij} x^i y^j$$
$$= (b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2 y + b_{12}xy^2 + b_{03}y^3) + \cdots$$

The fixed point of the system (3.9) is (0,0), and the linearization of (3.9) at (0,0) is given by:

$$DF(0,0,\beta)=\begin{pmatrix}\beta & -1\\ \frac{1}{\epsilon} & \frac{\beta}{\epsilon}\end{pmatrix},\ F(x,y)=\begin{pmatrix}f\\ \epsilon g\end{pmatrix}$$

For $\beta = 0$, the Jacobian matrix at (0,0) for equation (3.9) has a pair of purely complex eigenvalues, $\lambda,\bar{\lambda}=\pm\frac{i}{\sqrt{\epsilon}}$, and (0,0) is called a weak focus.

The Liapunov number $(\upsilon)$for (3.9) given by [1]:

$$\upsilon=\frac{3\pi}{2}\left[3\left(a_{30}+\frac{1}{\epsilon}\,b_{03}\right)+\left(a_{12}+\frac{1}{\epsilon}\,b_{21}-\frac{2}{\epsilon}(a_{20}b_{20})-a_{02}b_{02}\right)-\frac{1}{\epsilon}\,b_{11}\big(b_{0,2}b_{20}\big)\right] \qquad (3.10)$$

In particular, if $\upsilon > 0$ then the origin is a stable weak focus, and if $\upsilon < 0$ then the origin is unstable a weak focus, and a Hopf bifurcation occurs at β = 0.

For $\beta \neq 0$ then $DF(0,0,\beta)$ has a pair of complex eigenvalues $\left(\lambda,\bar{\lambda}\simeq\frac{1}{\epsilon}(\beta\pm i)\right)$. So (0,0) is unstable focus when $\beta > 0$ and (0,0) is stable when $\beta < 0$.

Theorem 3.4. (The Hopf Bifurcation)[11]. If $\upsilon \neq 0$, then Hopf bifurcation it arises in the origin of the system(3.9) at $\beta = 0$, if $\upsilon < 0$ there is a unique stable periodic orbit when $\beta > 0$, and no periodic orbit when $\beta \leq 0$, and if $\upsilon > 0$ there is a unique unstable periodic orbit when $\beta < 0$, and no periodic orbit when $\beta \geq 0$.

Example 3.4. Consider the singular perturbed ODEs given by :

$$\dot{x}=\beta x-y+x(-x^2-y^2),$$
$$\epsilon\dot{y}=x+\beta y+y(-x^2-y^2)$$

the only fixed point in this system is the origin and

$$DF(0,0,\beta)=\begin{pmatrix}\beta & -1\\ \frac{1}{\epsilon} & \frac{\beta}{\epsilon}\end{pmatrix}$$

For $\beta \neq 0$ then $DF(0,0,\beta)$ has a pair of complex eigenvalues $(\lambda,\bar{\lambda}\simeq\frac{1}{\epsilon}(\beta\pm i))$. So (0,0) is unstable focus if $\beta > 0$ and (0,0) is stable if $\beta < 0$. For $\beta = 0$, Jacobin at (0,0) has a pair of purely complex eigenvalues are $\lambda,\lambda = \pm i\frac{1}{\sqrt{\epsilon}}$, then (0,0) is called the center. We write this system in polar coordinates :

$$\dot{r}=\frac{r}{\epsilon}(\beta-r^2),\quad \dot{\theta}=\frac{1}{\epsilon}.$$

When $\beta = 0$, then (0,0) becomes a stable focus, while if $\beta > 0$ that means there is a stable limit cycle (periodic orbit Γ_β) in the form :

$$\Gamma_\beta:\ \gamma_\beta=\sqrt{\beta}(cost,sint)^T.$$

Now we recognize the type of Hopf bifurcation at the bifurcation point β = 0 by finding the Liaponov number ν in (3.10) we get $\upsilon = -6\pi\frac{\epsilon+1}{\epsilon}<0$. This means the critical point generates a periodic orbit(limit cycle) through the bifurcation value $\beta = 0$ and the type of Hopf bifurcation here is a supercritical Hopf bifurcation.

5. Conclusion

In this paper, we have investigate the bifurcation such as (saddle-node, transcritical, pitchfork) that occurred in the periodic solution of singularly perturbed parameterized ordinary differential equation. The conditions on the Poincare map have been given. These conditions are necessary for bifurcation to have occurred in such ODE. Also, the Hopf bifurcation has been studied.

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