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*Research Article*

# Well-Posedness For Generalized Equilibrium Problems With B$h$-Monotone Trifunction

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**Abstract.**
The aim of this paper is to establish the existence and uniqueness of solutions of well-posedness for $Ep(\beta_1, \beta_2)$in connection with β$h$-monotone trifunction. To this aim, we deal with a subset A of Banach space V. A useful characterization of metric projection and sufficient conditions for these types of well-posedness are derived. Finally, the discussion provided by this paper can be used to prove that the concept of well-posedness for $Ep(\beta_1, \beta_2)$is equivalent to the existence and uniqueness of its solution.

**Key Words:** Equilibrium Problem; Well-Posedness; Optimization Problem; Monotonicity; Metric Characterization.

## 1. INTRODUCTION

The existence and uniqueness of the solution of well-posed problem have undoubtedly shown to be very useful in optimization theory and nonlinear operator equations which constitute relevant particular cases of (EP). It is well known that the mathematical term well-posedness was first coined and popularized by the 20th century French mathematician Jacques Hadamard (for further details, see [1] and the references therein). The basic idea of well-posedness is meant to satisfy, according to Hadamard, three fundamental conditions, and the applications of these conditions inspired much of the work for the next decades. These conditions are the existence, the uniqueness and the stability of the solution. A problem within the boundary of the preceding definition is said to be well-posed.
In the mathematical literature, several methods have been proposed to discuss different well-posedness in connection with optimization problems and variational inequalities (see, for instance, [2], [3], [4], [5]). Mathematicians like Levitin and Polyak [6] as well as Tykhonov [7] and others have worked to develop and study well-posed problems. They were able to give a precise mathematical definition of approximate solutions of well-posedness for optimization problems and variational inequalities. Today, these problems provide a very rich framework for research. Several cases of well-posedness coming from problems (i.e., optimization and variational inequality), have later been extended to scalar problems, saddle point problems, fixed-point problems, Nash equilibrium problems and others. For more details on dealing with the family of problems that surround well-posedness (for further details, see [8, 9, 10, 11, 12, 13, 14]).
One kind of well-posedness has originated in Bianchi et al. [15] who introduced and formulated two classes of well-posedness for vector equilibrium problems. Another case of well-posedness has been given in terms of a parametric form of equilibrium problem with a new well-posedness concept. This approach, proposed by Salamon has been already exploited by many authors (see, for instance, [16], [17]. Yet another concept of well-posedness has been extended to vector equilibrium problems of which Kimura et al.[18] provided us with an entering wedge into a new problem of well-posedness. Furthermore, Peng et al. [19] took the case further into several types of Levitin-Polyak well-posedness in generalized vector equilibrium problems. There are also cases for Levitin-Polyak well-posedness for explicit constrained EPs and generic well-posedness for EPs, provided by Long et al. [20] and Zaslavski [21], respectively.
Inspired and motivated by the comprehensive literature in the field where different kinds of well-posedness for equilibrium problems have been introduced and the relationship between them has been explored, this paper is an attempt to make a contribution to this glut of literature. By such an attempt, we introduce a newconcept of relaxed monotone trifunctions. The results on the existence of solutions

forgeneralized equilibrium problems withtrifunction in such a class are obtained. Ourresults in this paper extend and improve the solutions of many survey of well-posedness in the current literature.

Aiming to obtain these results, the current study is made up of the following sections. Section 2 starts with recalling the necessary definitions and refers to some results in the field. Section 3 establishes and extends the concept of well-posedness to generalized equilibrium problems $Ep(\beta_1,\beta_2)$.

. Additionally, we derive some metric characterizations of well-posedness. In Section 4, we present a new concept of well-posedness for optimization problems with constraints defined by a parametric form of generalized equilibrium problem and a relevant well-posedness. We also give a sufficient condition that enables us to prove that the well-posedness of generalized equilibrium problems is equivalent to the existence and uniqueness of its solution.

## 2. PRELIMINARIES

Unless otherwise specified, we assume that V and W are two Banach spaces and A is a nonempty, convex and closed subset of Banach space V and K subset of W.

For the convenience of the reader, we recall some definitions and results that need to to be imposed in order to prove our main results.

We recall the following generalized equilibrium problem $Ep(\beta_1,\beta_2)$ [22]. In fact to find $b \in A$ such that

$$\beta_1(b,e) + \beta_2(a,b,e) \geq 0 \qquad \forall\, a,e \in A$$

In which $\beta_1: A \times A \to \mathbb{R}$ , $\beta_2: A \times A \times A \to \mathbb{R}$ with $\beta_2(a,b,b) = 0\ \forall\, a,b \in A$.

In what follows, we introduce the formulation of optimization problems with equilibrium constraint.

Let $H: K \times A \longrightarrow \mathbb{R}$ and $\beta_1: K \times A \times A \to \mathbb{R}$ be two function and the optimization problem with generalized equilibrium constraint (denoted by (OPGETC)) is formulated as follows:

$\min H(s,b) \in K \times A$ such that $b \in I(s)$

Where $I(s)$ is the solution set of the parametric generalized equilibrium problem $Ep_S(\beta_1,\beta_2)$ defined by $b \in I(s)$ if

$$\beta_1(s,b,e) + \beta_2(a,b,e) \geq 0 \ \ \forall a,e \in A \tag{1}$$

Instead of writing $\{Ep_S(\beta_1,\beta_2), s \in k\}$ family of generalized equilibrium problems (parametric Problem).

In particular cases, one can obtain the following:

i. If $\beta_2 \equiv 0$ and $\beta_1(s,b,e) = -h(s,b,b-e)$ then the problem is reduced to parametric quasivariational inequality QVI(s) [24].

ii. If $\beta_2 \equiv 0$, then the problem is reduces to the Parametric equilibrium problem EP(s) (See [29,30]).

iii. If $\beta_2(a,b,e) = \beta(b) - \beta(e)$ and $\beta_1(s,b,e) = \langle \beta(b), e-b \rangle$, then the problem is reduced to the mixed variational inequalities (MVI) see [25].

iv. If $\beta_2(a,b,e) = \beta(b,e)$ then the problem is reduced to the optimization problem with generalized equilibrium constraint (OPGPEC) (see [23]).

In recent years, some of authors have proposed many essential generalizations of monotonicity. We will use a kind of generalized monotonicity, So called βh-monotone trifunction.

**Definition 2.1.[22]** Let $h: A \times A \to \mathbb{R}$ be a real-valued function then a trifunction $\beta: A \times A \times A \to \mathbb{R}$ is called βh-monotone if :

$$\beta(a,b,e) + \beta(a,e,b) + h(b,e) \leqslant 0 \qquad \forall a,b,e \in A \tag{2}$$

**Definition 2.2.[26]** A real-valued functionβ, defined on a convex subset A of V, is soid to be hemicontiauous if :

$$\lim_{r\to 0}\beta[ra+(1-r)b]=\beta(b)\ \forall a,b\in A$$

**Definition 2.3.[27]** Let V be a Banach space and A be anon-empty convex and subset of V. $h,\beta: A\times A\to\mathbb{R}\cup\{+\infty\}$ two real bifunction and $a_r=rb+(1-r)a$ and $a,b\in A, r\in[0,1]$ . Then

$$\lim_{r\to 0}\frac{h(a,a_r)}{r}=0;$$
$$h(a,b)\le\lim_{r\to 0}\frac{r-1}{r}[\beta(a,a)+h(a,a)].$$

**Definition 2.4. [28]** Let $A, B$ be a nonempty subset of V. The Hausdorff metric $L(.,.)$ between A and B is defined by

$$L(A,B)=\max\{e(A,B),e(B,A)\}$$

where $e(A,B)=\sup_{a\in A}d(a,B)$ with $d(a,B)=\inf_{b\in B}\|a-b\|$

**Definition 2.5. [28]** Assume that A is a nonempty subset of V. Then the measure of noncompactness$\mu$ of the Set A is defined by

$$\mu(A)=\inf\left\{\delta>0: A\subset\bigcup_{i=1}^{n}A_i, \mathrm{diam}\ A_i<\delta i=1,2\ldots n\right\}$$

Where diam means the diameter of a set.

**Theorem 2.6.[22]** Suppose that $\beta_2: A\times A\times A\to\mathbb{R}$ is $\beta h-$ monotone trifunction, hemicontinuouse in the second argument, convex in third argument and $h,\beta_1: A\times A\longrightarrow\mathbb{R}$ be convex in the second argument, then the problem $Ep(\beta_1,\beta_2)$ is equivalent to the following problem find $b\in A$ such that:

$$\beta_2(a,e,b)+h(b,e)\le\beta_1(b,e)\qquad\forall a,e\in A\tag{3}$$

### 3. WELL - POSEDNESS OF $Ep(\beta_1,\beta_2)$ WITH METRIC CHARACTERIZATIONS

In this section we establish some concepts of well-posed for generalized equilibrium problem $Ep(\beta_1,\beta_2)$. To start our analysis.through the results of the section, we will give some conditions under which the equilibrium problem is strongly well-posed in the generalized sense.

**Definition 3.1.** A sequence $\{(s_n,b_n)\}\subset K\times A$ is said to be approximating sequence for $Ep(\beta_1,\beta_2)$ if there exists a nonnegative sequence $\{c_n\}$ with $c_n\to 0$ as $n\longrightarrow\infty$ such that

$$\beta_1(s,b_n,e)+\beta_2(a,b_n,e)+c_n\|b_n-e\|\ \|a\|\ge 0\ \forall n\in N,a,e\in A\tag{4}$$

**Definition 3.2.** The problem $Ep(\beta_1,\beta_2)$ is said to be strongly (resp, weakly) well-posed in the generalized if $Ep(\beta_1,\beta_2)$ has aunique solution $b$, and for every sequence $\{b_n\}$ with $b_n\to b$, every approximating sequence for $Ep(\beta_1,\beta_2)$ converge strongly (resp, weakly) to the unique solution, if $Ep(\beta_1,\beta_2)$ has anonempty solution set $I(s)$, and every approximate solution sequence has a subsequence which strongly (resp, weakly) converge to some Point $I(s)$.

In what follows, we shall establish some characterizations of well-posedness for $Ep(\beta_1, \beta_2)$ for any $c > 0$ we define two sets:

$$\Gamma(c) = \{(s,b) \in K \times A; \beta_1(s,b,e) + \beta_2(a,b,e) + c \parallel b-e \parallel\parallel a \parallel \geq 0 \quad \forall a,e \in A\}$$

$$N(c) = \{(s,b) \in K \times A; \beta_2(a,e,b) + h(b,e) \leq \beta_1(s,b,e) + c \parallel b-e \parallel\parallel a \parallel \qquad \forall a,e \in A\}$$

**Lemma 3.3.** Let A be a nonempty convex, closed subset of Banach space V,and suppose $\beta_1: K \times A \times A \to \mathbb{R}$, $B_2: A \times A \times A \to \mathbb{R}$ and $h: A \times A \to \mathbb{R}$ be three function, Let the following conditions hold:

i. $\beta_1(s,b,b) = 0 \ \forall\, s \in K, \ \ \forall b \in A$

ii. $\beta_2(a,b,e)$ is $\beta h$ -monotone trifunction and hemicontinuous in second argument.

iii. $\beta_1(s,b,.), \beta_2(a,b,.)$ and $h\,(b,.)$ are convex $\forall\, s \in K, a, b \in A$.

Then $\Gamma(c) = N(c)$ for all $c > 0$.

**Proof.** Suppose that $(s,b) \in \Gamma(c)$, then $(s,b) \in K \times A$ such that:

$$\beta_1(s,b,e) + \beta_2(a,b,e) + c \parallel b-e \parallel\parallel a \parallel \geq 0 \qquad \forall a,e \in A$$

Since $\beta_2$ is $\beta h$ -monotone trifunction

$$\beta_2(a,b,e) + \beta_2(a,e,b) + \ h\,(b,e) \leq 0 \qquad \forall\, a,b,e \in A.$$

Then
$$\beta_2(a,e,b) + \ h\,(b,e) \leq -\,\beta_2(a,b,e)$$
$$\leq \beta_1(s,b,e) + c \parallel b-e \parallel\parallel a \parallel$$

Therefore $(s,b) \in N(c)$

Conversely, assume that $(s,b) \in N(c)$ and let $e \in A$ such that $b_r = b - r(b-e)$, $\quad r \in (0,1)$ then$b_r \in A$, because A convex we get

$$\beta_2(a,b_r,b) + \ h\,(b,b_r) \leq \beta_1(s,b,b_r) + c\|b-b_r\| \parallel a \parallel$$
$$\beta_2(a,b_r,b) + \ h\,(b,b_r) - \beta_1(s,b,b_r) \leq c\|b-b_r\| \parallel a \parallel$$
$$= rc \parallel b-e \parallel\parallel a \parallel$$

(5)

$$0 = \beta_2(a,b_r,b_r) \leq \beta_2(a,b_r,b) - r[\beta_2(a,b_r,b) - \beta_2(a,b_r,e)]$$
$$0 \leq \beta_2(a,b_r,b) - r[\beta_2(a,b_r,b) - \beta_2(a,b_r,e)];$$

$$r[\beta_2(a,b_r,b) - \beta_2(a,b_r,e)] \leqslant \beta_2(a,b_r,b) \tag{6}$$

Since $\beta_1(s,b,.)$ convex

$$\beta_1(s,b,b_r) \leq \beta_1(s,b,b) - r\big[\beta_1(s,b,b) - \beta_1(s,b,e)\big] \tag{7}$$

From (6) and (7)

$$r\big[\beta_2(a,b_r,b) - \beta_2(a,b_r,e) + \beta_1(s,b,b) - \beta_1(s,b,e)\big] \leq \beta_2(a,b_r,b) + \beta_1(s,b,b)$$
$$-\beta_1(s,b,b_r)$$

From (5)

$$r\big[\beta_2(a,b_r,b) - \beta_2(a,b_r,e) + \beta_1(s,b,b) - \beta_1(s,b,e)\big] \leq rc \parallel b-e \parallel\parallel a \parallel \ -h(b,b_r)$$
$$+\beta_1(s,b,b)$$

Divided by $(-r)$

$$-\beta_2(a,b_r,b) + \beta_2(a,b_r,e) - \beta_1(s,b,b) + \beta_1(s,b,e) \geq -c \parallel b-e \parallel\parallel a \parallel \ + \frac{h(b,b_r)}{r}$$
$$-\frac{\beta_1(s,b,b)}{r}$$

Since $\beta_2(a,\cdot,e)$ is hemicontinuouse

$$\beta_2(a,b,e)+\beta_1(s,b,e)\geq -c\parallel b-e\parallel\parallel a\parallel$$

Hence $(s,b)\in\Gamma(c)$

Therefore $\Gamma(c)=N(c)$ for all $c>0$.

**Lemma (3.4)**. Let A be a nonempty, Convex, closed, subset of Banach space V, suppose that $\beta_1=K\times A\times A\to\mathbb{R}$ , $\beta_2: A\times A\times A\longrightarrow\mathbb{R}$ and $h: A\times A\to\mathbb{R}$ satisfy in the following conditions:

i. $\beta_1(.,.,e)$ is usc $\forall e\in A$,

ii. $\beta_2(a,e,.)$ and $h(.,e)$ are lsc $\forall a,e\in A$,

Then $N(c)$ is closed in $K\times A$ for every $c>0$

**Proof .** Assume that $\{(S_n,b_n)\}\subseteq N(c)$ is a sequence such that $(S_n,\ b_n)\longrightarrow(s,b)$. Then

$$\beta_2(a,e,b_n)+h(b_n,e)\leq\beta_1(S_n,b_n,e)+c\|b_n-e\|\parallel a\parallel\qquad\forall a,e\in A$$

Since $\beta_2(a,b,.)$ and $h(.,e)$ are lsc and $\beta_1(.,.,e)$ is usc, then

$$\beta_2(a,e,b)+h(b,e)\leq\lim_{n\to\infty}\inf\left[\beta_2(a,e,b_n)+h(b_n,e)\right]\leq\lim_{n\to\infty}\sup\left[\beta_1(S_n,b_n,e)+c\|b_n-e\|\parallel a\parallel\right]\leq B_1(s,b,e)+c||b-e\parallel\parallel a\parallel$$

Then

$$\beta_2(a,e,b)+h(b,e)\leq\beta_1(s,b,e)+c\|b-e\|\parallel a\parallel\qquad\forall a,e\in A$$

which implies that $(s,b)\in N(c)$

Therefore $N(c)$ is closed in $K\times A$ for all $C>0$.

**Theorem 3.5.** Assume that A is a nonempty convex, closed subset of Banach space V . Let $\beta_1: K\times A\times A\to\mathbb{R}$ , $\beta_2: A\times A\times A\to\mathbb{R}$ and $h: A\times A\to\mathbb{R}$ be three function, $Ep(\beta_1,\beta_1)$ is well posed then:

$$\Gamma(c)\neq\varphi\ ,\lim_{c\to 0}\ \mathrm{diam}\ [\Gamma(c)]=0\tag{8}$$

Moreover, if the following assumptions hold:

i. $\beta_1(.,.,e)$ is usc $\forall e\in A$;

ii. $\beta_1(s,b,b)=0$ $\forall(s,b)\in K\times A$;

iii. $\beta_2(a,b,e)$ is βh-monotone trifunction and hemicontinuous in second argument;

iv. $\beta_1(s,b,.),\beta_2(a,b,.)$ and $h(b,.)$ are convex $\forall s\in K,a,b\in A$.

v. $\beta_2(a,e,.),h(.,e)$ are lsc $\forall a,e\in A$.

Then the converse holds.

**Proof .** Assume that $Ep(\beta_1,\beta_2)$ is well - posed, then $Ep(\beta_1,\beta_2)$ admit a unique solution $(s,b)\in K\times A$ such that

$$\beta_1(s,b,e)+\beta_2(a,b,e)\geq 0\quad\forall a,e\in A$$

Clearly $\Gamma(c)\neq\varphi$ for any $c>0$, by contradiction, assume that $\lim_{n\to\infty}\mathrm{diam}\,[\Gamma(C_n)]>p>0$ for some sequence $\{c_n\}>0$. We could find two sequence $\{(s_n,b_n)\}$ and $\{(s_n,e_n)\}\in\Gamma(c_n)$.

$$\|(s_n,b_n)-(S_n,e_n)\|>p\quad\forall n\in N\tag{9}$$

Since $\{(s_n, b_n)\}$ , $\{(s_n, e_n)\}$ are approximating sequence for $Ep(\beta_1, \beta_2)$ . By well - posedness of $Ep(\beta_1, \beta_2)$, they have to converge to unique solution of $Ep(\beta_1, \beta_2)$ a contradiction to (9).
Conversely,suppose that Condition (8) holds.

Let $\{(s_n, b_n)\}$ be an approximating sequence for $Ep(\beta_1, \beta_2)$. Then there exists a nonnegative sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$ such that

$$\beta_1(s_n, b_n, e) + \beta_2(a, b_n, e) + c_n \|b_n - e\| \parallel a \parallel \geq 0 \quad \forall n \in N, a, b \in A \tag{10}$$

Then $\{(s_n, b_n)\} \subseteq \Gamma(c_n)$.

From (8), $\{(s_n, b_n)\}$ is a cauchy sequence and so it converges to a point $(s, b)$.
It follows from (10) and $\beta_2$ is $\beta h -$ monotone, $\beta_2(a, e, .), h(., e)$ are lse and $\beta_1(., ., e)$ usc

$$0 = \lim_{n\to\infty} \inf c_n \|b_n - e\| \parallel a \parallel \geq \lim_{n\to\infty} \inf[-\beta_1(s_n, b_n, e) - \beta_2(a, b_n, e)]$$
$$\geq \lim_{n\to\infty} \inf[-\beta_1(S_n, b_n, e) + \beta_2(a, b_n, e) + h(b_n, e)]$$
$$0 \geq \lim_{n\to\infty} \inf[-\beta_1(S_n, b_n, e) + \beta_2(a, b_n, e) + h(b_n, e)]$$
$$\lim_{n\to\infty} \inf \beta_1(s_n, b_n, e) \geq \lim_{n\to\infty} \inf[\beta_2(a, e, b_n) + h(b_n, e)]$$
$$\beta_1(s, b, e) \geq \beta_2(a, e, b) + h(b, e)$$

From the above, together with theorem (2.6), we establish $(s, b)$ as a solution to $Ep(\beta_1, \beta_2)$.

As for the uniqueness follow immediately from (8).

**Remark 3.6**. Note that the diameter of $\Gamma(c)$ does not tend to Zero, if $Ep(\beta_1, \beta_2)$ has more that one solution. In the next result we consider the Kuratowski non compactness measure of approximating Solution set instead of the diameter.

**Theorem 3.7.**Assume that K and A are anonempty, closed subsets of real Banach space W and V respectivel. If $Ep(\beta_1, \beta_2)$ is strongly well-posed in the generalization sense, then:

$$\Gamma(c) \neq \varphi \quad \forall c, \lim_{c\to 0} \mu[\Gamma(c)] = 0 \tag{11}$$

Moreover, if the following hold

i. $\beta_1(s, b, b) = 0 \ \forall s \in K, b \in A$ ,
ii. $\beta_1(., ., e)$ is usc $\in e \in A$,
iii. $\beta_2(a, b, e)$ is $\beta$h-monotone trifunction and hemicontinuouse in second argument,
iv. $\beta_1(s, b, .), \beta_2(a, b, .)$ and $h\ (b, .)$ are convex $\forall\ s \in K, a, b \in A$ ,
v. $\beta_2(a, e, \cdot)$ and $h(\cdot, e)$ are lsc $\forall a, e \in A$.

Then the converse holds.

**Proof.** Let $Ep(\beta_1, \beta_2)$ be strongly well-posed in a generalized sense. Then the solution set I of $Ep(\beta_1, \beta_2)$ is a nonempty. This indicates that for any $c > 0, \Gamma(c) \neq \varphi$, since $I \subset \Gamma(c)$. Moreover, we claim here that the solution set of $Ep(\beta_1, \beta_2)$ is compact. Indeed, for any sequence $\{(s_n, b_n)\}$ in $I, \{(s_n, b_n)\}$ is an approximating sequence for$Ep(\beta_1, \beta_2)$.

Thus there exists a converging subsequence to some point of I. This implies that I is compact, Now we show that $\lim_{c\to 0}\mu[(\Gamma(c)] \to 0$.It follows from $I \subseteq \Gamma(c)$ that

$$H(\Gamma(c), I) = \max\{e(r(c), I), e(I, \Gamma(c))\}$$

Since the solution set I is compact, one can have

$$\mu(\Gamma(c)) \leq 2H(\Gamma(c), I) + \mu(I) = 2e(\Gamma(c), I)$$

where $\mu(I) = 0$ , since I is compact. To prove $\text{Lim}_{c\to 0}\ \mu(\Gamma(c)) = 0$ . It is sufficient to show that$e(\Gamma(c), I) \to 0$as $c \to 0$. If not, there existconstant $c > 0$and $c_n \to 0$, and $\{(s_n, b_n)\} \subset \Gamma(c_n)$ in which

$$(s_n, b_n) \notin I + B_{\frac{c}{2}}(0)\ \forall n \in N \tag{12}$$

Where $B_{\frac{c}{2}}(0)$ is an open ball with center 0 and radius $\frac{c}{2}$. However $\{(s_n, b_n)\} \subseteq \Gamma(c_n)$ , is an approximating sequence for $Ep(\beta_1, \beta_2)$. It Follows the generalized well-posed of $Ep(\beta_1, \beta_2)$ that there exists asubsequence converges to some Point of $(s, b) \in I$, which contradicts (12).

Conversely, suppose that (11). By Lemma (3,3) and Lemma (3,4),$\Gamma(c)$ is anonempty and closed for all $c > 0$. By the Kuratowski theorem [28] we Can obtain

$$H(\Gamma(c), I) \to 0 \text{ as } c \to 0 \tag{13}$$

Where $I = \bigcap_{c>0} \Gamma(c)$ is a nonempty and compact.

Let $\{(s_n, b_n)\} \subset K \times A$ be any approximate Solution sequence for $Ep(\beta_1, \beta_2)$. Then there exists anonnegative sequence $\{c_n\}$ with $c_n \to 0$as $n \to \infty$, such that

$$\beta_1(s_n\ , b_n, e) + \beta_2(a, b_n, e) + c_n\|b_n - e\|\ \|\ a\ \| \geq 0 \quad \forall n \in N, a, b \in A.$$

This mean that $(s_n, b_n) \in \Gamma(c_n)$. This together with ( 11 ) indicate that

$$d[(s_n, b_n), I] \leq e(\Gamma(c), I) \to 0$$

Since I compact, it follows that there exists $(\bar{s}_n, \bar{b}_n) \in I$ in which

$$\|(s_n, b_n) - (\bar{s}_n, \bar{b}_n)\| = d[(s_n, b_n), I] \to 0$$

Again, by the Compactness of the solution set I, the sequence $(\bar{s}_n, \bar{b}_n)$ has a supsequence $\{(\bar{s}_{n_k}, \bar{b}_{n_k})\}$ converge strongly to $\{\bar{s}_n, \bar{b}_n)\} \in I$. There for the corresponding $\{(s_{n_k}, b_{n_k})\}$ subsequence of$\{(s_n, b_n)\}$ converge strongly to $\{\bar{s}_n, \bar{b}_n)\}$.
Hence $Ep(\beta_1, \beta_2)$ is well-posed in generalized sense

## 4. WELL-POSEDNESS FOR OPTIMIZATION PROBLEMS WITH GENERALIZED PARAMETRIC EQUILIBRIUM CONSTRAINTS

In this section, let us introduce the formulation of optimization problems with equilibrium constraint. The optimization problem with generalized equilibrium constraint denoted by (OPGETC) is formulated as follows:

$$\min H(s,b) \text{ such that } (s,b) \in K \times A,$$

where $b \in I(s), K$ is anonempty closed subset of aparametric space, $H: K \times A \longrightarrow \mathbb{R}, \beta_1: K \times A \times A \to \mathbb{R}$ $\beta_2: A \times A \times A \to \mathbb{R}$, and $I(s)$ is the solution set of the parametric generalized equilibrium $Ep_s(\beta_1, \beta_2)$. definedby, $b \in I(s)$ if and only if:

$$\beta_1(s,b,e) + \beta_2(a,b,e) \geq 0 \quad \forall a, e \in A.$$

**Definition 4.1.** A sequence $\{(s_n, b_n)\} \subset K \times A$ is said to be approximating sequence for (OPGETC) if :

i. there exists a nonnegative sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$ such that:

$$\beta_1(S_n, b_{n,}, e) + \beta_2(a, b_n, e) + c_n \|b_n - e\| \parallel a \parallel \geqslant 0$$
$$\forall n \in N, a, e \in A$$

ii. $\lim_{n\to\infty} \sup H(s_n, b_n) \leq \inf_{m\in k, e\in I(m)} H(m,e)$

**Definition 4.2.** (OPGETC) is said to be strongly (resp., weakly) well-posed (resp., strongly (resp, weakly) well - posed in the generalized sense ) if (OPGETC) has a unique Solution and for every approximating sequence for (OPGETC) converges strongly (resp, weakly) to the unique solution (resp., if $I \neq \varphi$ and every approximating solution sequence has a subsequence which strongly (resp., weakly) converge to some point of I ).

The set of approximating solution of (OPGETC) is defined by:

$$\psi(c,\epsilon) = \begin{cases} (s,b) \in K \times A; H(s,b) \leq \inf_{m\in k, e\in I(m)} H(m,e) + \epsilon \text{ and} \\ \beta_1(s,b,e) + \beta_2(a,b,e) + c_n \|b - e\| \parallel a \parallel \geq 0 \qquad \forall a, e \in A \end{cases}$$

**Theorem 4.3**. Assume that A is a nonempty convex, closed subset of a Banach space V. Let $\beta_1: K \times A \times A \to \mathbb{R}, \beta_2: A \times A \times A \to \mathbb{R}$, $H: K \times A \to \mathbb{R}$ and $h: A \times A \to \mathbb{R}$ be four function. If (OPGETC) is strongly well-posed. Then

$$\psi(c,\epsilon) \neq \varphi \;\; \forall\, c, \epsilon > 0 \text{ , diam } \psi(c,\epsilon) \to 0 \tag{14}$$

where $(c,\epsilon) \to (0,0)$.

Moreover, if the following assumptions hold:

i. $\beta_1(\cdot,\cdot,e)$ is usc $\forall e \in A$;

ii. $\beta_1(s,b,.), \beta_2(a,b,.)$ and $h\,(b,.)$ are convex $\forall\, s \in K, a, b \in A$ ;

iii. $\beta_2(a,b,e)$ is $\beta h$ -monotone trifunction and hemicontinuous in second argument;

iv. $\beta_2(a,e,.), h(\cdot,e)$ are lsc $\forall a, e \in A$;

v. H is lsc.

**Proof .** Assume that (OPGETC ) is strongly well-posed. Then (OPGETC) admits a unique solution ( $s,b) \in K \times A$ such that

$$\begin{cases} H(s,b) = \inf_{m\in k, e\in I(m)} H(m,e) \text{ and} \\ \beta_1(s,b,e) + \beta_2(a,b,e) \geq 0 \qquad \forall\, a, e \in A \end{cases}$$

Obviously, $\psi(c,\epsilon) \neq \varphi$ for any $c, \epsilon > 0$, since $(s,b) \in \psi(c,\epsilon)$ for any $c, \epsilon > 0$. If diam $\psi(c,\epsilon) \nrightarrow 0$ as $c \to$

$0, \epsilon \to 0$ then there exists a constrantp $> 0$ and a nonnegative sequences $\{c_n\}$ and $\{\epsilon_n\}$with $c_n \to 0$, $\epsilon_n \to 0$ as $n \to 0$ and $(s_n, b_n), (s_n, e_n) \in \psi(c, \epsilon)$ in which

$$\|(s_n, b_n) - (s_n, e_n)\| > p \; \forall n \in N \tag{15}$$

since $(s_n, b_n), (s_n, e_n) \in \psi(c, \epsilon) \;\; \forall n \in N$, so both $\{(s_n, b_n)\}$ and $\{(s_n, e_n)\}$ are approximating sequence for (OPGETC). By the well - posedness of (OPGETC), they have to converge strongly to the unique solution of (OPGETC) a contradiction to (15).

Conversely**,** suppose that Condition (14) holds. Let $\{(S_n, b_n)\}$ be an approximating sequence for (OPGETC). Then there exists a nonnegative sequence $\{c_n\}$ with $C_n \to 0$ as $n \to \infty$ in which

$$\begin{cases} \lim\limits_{n\to\infty} \sup H(s_n, b_n) \le \inf\limits_{m\in K, e\in I(m)} H(m, e) \text{ and} \\ \beta_1(s_n, b_n, e) + \beta_2(a, b_n, e) + c_n\|b_n - e\| \parallel a \parallel \ge 0 \; \forall \; n \in N \,, a, e \in A \end{cases} \tag{16}$$

This yields that $(s_n, b_n) \in \psi(c_n, \epsilon_n)$.

It follows from (14), that $\{(s_n, b_n)\}$ is a cauchy sequence and so it converge strongly to a point $(s, b) \in K \times A$ .

It follows from (16) and assumptions (i), (iii), and (v) that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \inf c_n\|b_n - e\| \parallel a \parallel \\ &\ge \lim_{n\to\infty} \inf\left[-\beta_1(s_n, b_n, e) - \beta_2(a, b_n, e)\right] \\ &\ge \lim_{n\to\infty} \inf\left[-\beta_1(s_n, b_n, e) + \beta_2(a, e, b_n) + h(b_n, e)\right] \\ \lim_{n\to\infty} \inf \beta_1(s_n, b_n, e) &\ge \lim_{n\to\infty} \inf\left[\beta_2(a, e, b_n) + h(b_n, e)\right] \\ \beta_1(s, b, e) &\ge \beta_2(a, e, b) + h(b, e) \end{aligned}$$

Also, one can note from (16) and assumption(vii) that

$$\begin{aligned} \inf_{m\in K, e\in I(m)} H(m, e) &\ge \lim_{n\to\infty} \sup H(S_n, b_n) \\ &\ge \lim_{n\to\infty} \inf H(S_n, b_n) \\ &\ge H(s, b) \end{aligned}$$

So by Theorem (2.6), $(s, b)$ solves (OPGETC). The uniqueness follows immediately from (14) . Therefore, we complete the proof.

By a similar proof as that of theorem (3.7), we can obtain the following result for the well-posedness of (OPGETC).

**Theorem 4. 4**. Assume that K and A are a nonempty, closed and convex subset of real Banach spaces W and V respectively. If (OPGETC) is strongly well - posed in the generalized sense, then:

$$\psi(c, \epsilon) \ne \varphi \quad \forall c, \epsilon > 0, \lim_{(c,\epsilon)} \mu[\psi(c, \epsilon)] = 0 \tag{17}$$

Moreover, if the following assumptions hold :

i. $\beta_1(.,.,e)$ is usc, $\forall e \in A$,

ii. $\beta_1(s, b, .), \beta_2(a, b, .)$ and $h\,(b, .)$ are convex $\forall\; s \in K, a, b \in A$,

iii. $\beta_2(a, b, e)$ is $\beta h$ -monotone trifunction and hemicontinuous in second argument,

iv. $\beta_1(s,b,b)=0\ \ \forall s\in K, b\in A$,
v. $\beta_2(a,e,.)$ and $h(.,e)$ are lsc $\forall a,e\in A$,
vi. H is lsc

Then the converse holds.

To investigate the uniqueness of solutions to (OPGETC) we show that under suitable Conditions, in the next result the well-posedness of (OPGETC)is equivalent to the existence and uniqueness of solution.

**Theorem 4.5.** Let K and A be a nouempty, closed and Convex subsets of finte dimensional Banach, space W and V respectively. Let $\beta_1=K\times A\times A\to\mathbb{R}, B_2: A\times A\times A\to\mathbb{R}$,
$h: A\times A\to\mathbb{R}$ and $H: K\times A\longrightarrow\mathbb{R}$ be four functions.
Suppose that:
i. $\beta_1(.,.,e)$ is usc $\forall e\in A$, covex,
ii. $\beta_1(s,b,b)=0\ \ \forall s\in K, b\in A$,
iii. $\beta_2(a,b,e)$ is $\beta h$ -monotone and hemicontinuous in second argument,
iv. $\beta_2(a,e,.)$, $h(e,.)$are lsc , $\forall a,e\in A$,
v. $\beta_1(s,b,.), \beta_2(a,b,.)$ and $h\,(b,.)$ are convex $\forall\ s\in K, a,b\in A$,
vi. H is convex and lsc.

Then, (OPGPEC) is a strongly well-posed if and only if it has a unique solution.

**Proof** .The necessity is obvious. For the sufficiency, Suppose that (OPGETC) has a unique solution $(S^*,b^*)$. It follow that:

$$\begin{cases} H(s^*,b^*)=\inf\limits_{m\in K,e\in I(m)} H(m,e) \\ \beta_1(s^*,b^*,e)+\beta_2(a,b^*,e)\geq 0 \quad \forall\, a,e\in A \end{cases} \tag{18}$$

Let $\{(s_n,b_n)\}\subset K\times A$ be approximating sequence for (OPGETC). Then there exists $c_n>0$ with $c_n\to 0$ as $n\to\infty$ such that:

$$\begin{cases} H(s_n,b_n)=\inf\limits_{m\in K,e\in I(m)} H(m,e) \\ \beta_1(s_n,b_n,e)+\beta_2(a,b_n,e)+C_n\|b_n-e\|\ \|a\|\geq 0 \quad \forall n\in N, a,e\in A \end{cases} \tag{19}$$

we claim that $\{(s_n,b_n)\}$ is bounded. If not without loss of generality, one can assume that $\|(s_n,b_n)\|\to+\infty$ . Let $r_n=\frac{1}{\|(s_n,b_n)-(s^*,b^*)\|}$
and

$$\begin{aligned} (t_n,z_n) &= r_n(s_n,b_n)+(1-r_n)(s^*,b^*) \\ &= [r_n s_n+(1-r_n)s^*, r_n b_n+(1-r_n)b^*]. \end{aligned}$$

without loss of generality, one can assume that $r_n\in(0,1)$ and $(t_n,z_n)\to(t,z)$ with $(t,z)\neq(s^*,b^*)$ since $W\times V$ is finite -dimensional, Taking into account the closedness and Convex of K and A, one has $(t_n,z_n)\in K\times A$, Thus, by assumption (vi) and (18), (19)for any $(t,z)\in K\times A$, we have

$$\begin{aligned} H(s^*,b^*) &= \lim_{n\to\infty}\sup r_n H(S_n,b_n)+\lim_{n\to\infty}\sup(1-r_n)H(s^*,b^*) \\ &\geq \lim_{n\to\infty}\sup[r_n H(s_n,b_n)+(1-r_n)H(s^*,b^*)] \end{aligned}$$

$$\geq \lim_{n\to\infty} \sup H(t_n, z_n)$$
$$\geq \lim_{n\to\infty} \inf H(t_n, z_n)$$
$$\geq$$
$$H(t, z) \quad (20)$$

Moreover, following from conditions (i), (ii), (19) and (18), we have

$$0 = \lim_{n\to\infty} \inf r_n c_n \|b_n - e\| \parallel a \parallel$$
$$\geq \lim_{n\to\infty} \inf -r_n\left[\beta_1(S_n, b_n, e) + \beta_2(a, b_n, e)\right] - (1 - r_n)\left[\beta_1(s^*, b^*, e) + \beta_2(a, b^*, e)\right]$$
$$\geq \lim_{n\to\infty} \inf\left[-r_n\beta_1(s_n, b_n, e) - (1 - r_n)\beta_1(s^*, b^*, e)\right] - r_n\beta_2(a, b_n, e) - (1 - r_n)\beta_2(a, b^*, e)\Big]$$
$$\geq \lim_{n\to\infty} \inf\left[-\beta_1(t_n, z_n, e) - \beta_2(a, z_n, e)\right]$$
$$\geq \lim_{n\to\infty} \inf\left[-\beta_1(t_n, z_n, e) + \beta_2(a, e, z_n) + h(z_n, e)\right]$$
$$\lim_{n\to\infty} \inf \beta_1(t_n, z_n, e) \geq \lim_{n\to\infty} \inf\left[\beta_2(a, e, z_n) + h(z_n, e)\right]$$
$$\beta_1(t, z, e) \geq \beta_2(a, e, z) + h(z, e) \quad (21)$$

Applying Theorem (2. 6 ) implies that

$$\beta_1(t, z, e) + \beta_2(a, z, e) \geq 0$$

Hence, from (20), (21), that $(t, z)$ solves (OPGETC), which is a contradiction. So $\{(s_n, b_n)\}$ is bounded. Let $\{(s_{n_i}, b_{n_i})\}$ be anysubsequence of $\{(s_n, b_n)\}$ in which $(s_{n_i}, b_{n_i}) \longrightarrow (s_0, b_0)$ as $i \longrightarrow \infty$ It follows that:

$$0 \geq \lim_{n\to\infty} \inf c_{n_i}\|b_{n_i} - e\| \parallel a \parallel$$
$$\geq \lim_{n\to\infty} \inf\left[-\beta_1(s_{n_i}, b_{n_i}, e) - \beta_2(a, b_{n_i}, e\right]$$
$$\geq \lim_{n\to\infty} \inf\left[-\beta_1(s_{n_i}, b_{n_i}, e) + \beta_2(a, e, b_{n_i}) + h(b_{n_i}, e)\right)$$
$$\geq -\beta_i(s_0, b_0, e) + \beta_2(a, e, b_0) + h(b_0, e) \quad (22)$$

$\forall a, e \in A$. Applying Theorem (2.6) implies that

$$\beta_1(s_0, b_0, e) + \beta_2(a, b_0, e) \geq 0 \quad \forall a, e \in A \quad (23)$$

It follows from (19) and lower semicontinuity of H that

$$\lim_{m\in K, e\in I(m)} \inf H(m, e) \geq \lim_{i\to\infty} \sup H(s_{n_i}, b_{n_i})$$
$$\geq \lim_{i\to\infty} \operatorname{ing} H(s_{n_i}, b_{n_i})$$
$$\geq H(s_0, b_0) \quad (24)$$

From (23) and (24), $(s_0, b_0)$ solves ( OPGETC). Taking into account the uniqueness of the solution, we have $(s_0, b_0) = (s^*, b^*)$ and hence $(s_n, b_n)$ converges to $(s^*, b^*)$.
Therefore, (OPGPEC) is strong well-posed.

**Theorem 4. 6**. Let K and A be a nonempty, closed and convex subsets of finite dimensional Banach space W and V respectively. Let $\beta_1: K \times A \times A \to \mathbb{R}, \beta_2 = A \times A \times A \to \mathbb{R}, h: A \times A \to \mathbb{R}$ and $H: K \times$

$A \to \mathbb{R}$ be four functions. If there exists some $\epsilon > 0$ such that $\psi(\epsilon, \epsilon)$ is a nonempty and bounded and suppose that the following assumptions holds:

i. $\beta_1(.,.,e)$ is usc $\forall\, e \in A$,
ii. $\beta_1(s,b,b) = 0\ \ \forall\, s \in K, b \in A$,
iii. $\beta_2(a,b,e)$ is $\beta h -$ monotone and hemicontinuous in second argument,
iv. $\beta_1(s,b,.), \beta_2(a,b,.)$ and $h\,(b,.)$ are convex $\forall\, s \in K, a, b \in A$,
v. $\beta_2(a,e,.), h(e,.)$ are lsc $\forall a, e \in A$,
vi. H is convex and lsc.

Then (OPGPEC) is strongly well-posed in the generalized sense.

**Proof.** Let $\{(s_n, b_n)\} \subseteq K \times A$ be an approximating sequence for (OPGETC). Then there exists a nonnegative sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$ such that

$$\begin{cases} H(s_n, b_n) = \inf\limits_{m \in K, e \in I(m)} H(m, e) \\ \beta_1(s_n, b_n, e) + \beta_2(a, b_n, e) + c_n \|b_n - e\| \parallel a \parallel \geq 0 \qquad \forall n \in N, a, e \in A \end{cases} \tag{25}$$

Since $\psi(\epsilon, \epsilon)$ is anonempty and bounded. Then there exists $n_0$ such that $(s_n, b_n) \in \psi(\epsilon, \epsilon)$ for all $n \geq n_0$. Taking into account the boundedness of $\psi(\epsilon, \epsilon)$, there exists subsequence $\{(s_{n_i}, b_{n_i})\}$ of $\{(s_n, b_n)\}$ in which $(s_{n_i}, b_{n_i}) \to (s_0, b_0)$ as $i \to \infty$.

Consequently, as proved in theorem (4.5), $(s_0, b_0)$ solves (OPGETC). Then (OPGETC) is strongly well-posed in the generalized sense.

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