Turkish Journal of Computer and Mathematics Education Vol.12 No.12 (2021), 429-445

Research Article

Rayleigh-Bénard Convection, Dynamic Bifurcation And Stability

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 23 May 2021

Abstract.
In this paper, we look at the bifurcation and stability of Boussinesq equation solutions, as well as the onset of Rayleigh-Bênard convection. nonlinear theory,was developed based on a new concept of bifurcation called attractor bifurcation and its corresponding theorem . The three aspects of this principle are as follows. First, regardless of the multiplicity of the eigenvalue R_c for the linear problem, the problem bifurcates from the trivial solution attractor $A_R$when the Rayleigh number R exceeds the first critical Rayleigh number R_c for all physically sound boundary conditions. Second, the asymptotically stable bifurcated attractor A_R. Third, bifurcated solutions are structurally stable and can be classified when the spatial dimension is two. Furthermore, the technical approach developed offers a recipe that can be used to solve a variety of other bifurcation and pattern forming problems.

1. Introduction

When a fluid is heated from below, it causes convection, which is a well-known phenomenon of fluid motion caused by buoyancy. It's well-known for being the driving force behind atmospheric and oceanic phenomena. as well as in the kitchen! The famous experiments performed by Bênard in 1900 gave rise to the Rayleigh-Bénard convection problem. -Bénardlooked into a fluid that had a free surface and was heated from below in a dish. and saw a hexagonal convection cell pattern that was fairly regular. Lord Rayleigh [12] proposed a hypothesis to explain the phenomenon of –Bénard experiments in 1916. To model Bênard 's experiments, he chose the Boussinesq equations with some boundary conditions and linearized them using normal modes. He then was that convection would only occur if the non-dimensional parameter, known as the Rayleigh number, was greater than one. and exceeds a critical threshold T_0on the lower surface and T_1on the upper surface, h the depth of the fluid layer, κ the thermal diffusivity, and ν the kinematic viscosity

There have been extensive studies of this problem since Rayleigh's pioneering work; see, for example, Chandrasekhar [1] and Drazin and Reid [2] for linear theories, and Kirchgassner [5], Rabinowitz [11], and Yudovich [13, 14], as well as the references. Nonlinear hypotheses can be found there. The majority, if not all, of Rayleigh-Bénard problem, 's known bifurcation and stability analysis findings are limited to the bifurcation and stability analysis when the Rayleigh number reaches a simple eigenvalue in some cases. By imposing symmetry on the entire phase space, subspaces of the entire phase space can be obtained.

It is self-evident that a complete nonlinear bifurcation and stability theory for this problem must involve at the This condition:

a)When the Rayleigh number reaches the first critical number for all mechanically sound boundary conditions, the bifurcation theorem applies.

b) bifurcated solution asymptotically stable, and

c) the physical space's structure/patterns, as well as their continuity and transitions.

The key obstacles to such a comprehensive theory are twofold. The first is due to the problem's high nonlinearity, which is common in fluid problems. The second is due to the lack of a theory to deal with bifurcation and stability because the linear problem's eigenvalue has even multiplicity. The main goal of this paper is to develop a nonlinear theory for Rayleigh-Bénard convection based on a new concept of bifurcation called attractor bifurcation. , as well as the authors' recent development of a related theory in [6]. [9] announces a portion of the findings presented in this report. Following the three aspects of a complete theory for the problem just described, as well as the key concept and methods used, we now discuss each aspect of our findings in this article. The Boussinesq equations bifurcate from the trivial solution attractor $A_R$as the Rayleigh number R crosses the first critical value R_C. with a dimension ranging from m to $m + 1$. The first eigenvalue of the linear eigenvalue problem is defined as the first critical Rayleigh number R_C. and m + 1 is the multiplicity of this eigenvalue R_C. The bifurcation theorem obtained in this article is for all cases with the multiplicity $m + 1$ of the critical eigenvalue R_C for the –Bénard problem under any set of physically sound boundary conditions, in contrast to known results. As the Rayleigh number approaches the critical value R_C, the trivial solution becomes unstable, and $A_R$does not contain this trivial solution.

Second, the bifurcated attractor A_R has asymptotic stability as an attractor, attracting all solutions with initial data in the phase space outside of the stable manifold, with co-dimension $m + 1$, of the trivial solution An ideal stability theorem, as Kirchgassner pointed out in [5], would involve all physically meaningful perturbations and determine the local stability of a selected class of stationary solutions[4], but we are still a long way from that goal. Fluid flows, on the other hand, are typically time-dependent. As a result, bifurcation analysis for steady-state problems only offers partial solutions in most cases. and is insufficient to address the issue of stability. As a result, it appears that the attractor close can better represent the correct notion of asymptotic stability after the first bifurcation. However, the trivial condition is excluded. One of our key reasons for implementing attractor bifurcation is to contribute to an ideal stability theorem via the stability of the bifurcated attractor obtained in this article.

Third, classifying the structure/pattern of the solutions after the bifurcation is an essential feature of a full nonlinear theory for Rayleigh-Bénard convection. The structural stability of the solutions in physical space is a natural tool for attacking this problem. The writers have been working on this aim since 1997. , and developed a systematic theory for the structural stability and bifurcation of 2-D divergence-free vector fields (see the authors' survey article in [8]). We demonstrate in this article that in the two-dimensional case, for any initial data outside of the stable manifold of the trivial solution, using the structural stability theorem proved in [7], As long as t is large enough, the solution of the Boussinesq equations will have the roll form. The above results for Rayleigh-Bénard convection are obtained using a new concept of dynamic bifurcation called attractor bifurcation, which was recently introduced by the authors in [6]. The key theorem relating to attractor bifurcation states that when the control parameter crosses a critical value and there are $+1$ $(m \geq 0)$ eigenvalues crossing the imaginary axis, attractor bifurcation occurs. If the critical state is asymptotically stable, the system bifurcates from a trivial steady-state solution to an attractor with a dimension between m and $m + 1$. This current bifurcation definition is a generalization of the previously discussed bifurcation definitions. Attractor bifurcation has a few distinguishing characteristics. First, the bifurcation attractor is physically significant since it excludes the trivial steady-state and is stable. Second, the attractor holds a set of evolution equation solutions, probably including steady states. Periodic orbits, as well as homoclinic and heteroclinic orbits, are all examples of periodic orbits. Third, it offers a unified perspective on dynamic bifurcation that can be applied to a wide range of physics and dynamics problems. Fourth, from the standpoint of application, The Krasnoselskii-Rabinowitz theorem requires an odd integer for the number of eigenvalues $m + 1$ crossing the imaginary axis, and the Hopf bifurcation is for $m + 1 = 2$. The new attractor bifurcation theorem obtained in this paper, on the other hand, can be generalized to all $m \geq 0$ instances. Moreover, as previously said, the bifurcated attractor is stable, which is a subtle problem for other established bifurcation theorems. Of course, the cost is the verification of the critical state's asymptotic stability, as well as the analysis needed for the eigenvalues problems in the linearized problem. For problems with symmetric linearized equations, it provides a way to achieve asymptotic stability of the critical state. The asymptotic stability of the trivial solution to the Rayleigh-Bénard problem can be easily defined from to this theorem. This theorem can be useful in many problems involving symmetric linearized equations in mathematical physics. The following is a breakdown of how this article is structured. In Section 2, we review the Boussinesq equations, their mathematical setting, and their applications. as well as some known solution existence and unique outcomes. Section 3 summarizes the key attractor bifurcation principle from [6], as well as a theorem, for the asymptotic stability of the critical state for problems involving symmetric linearized equations in an evolution system. . Section 4 states and shows that the Rayleigh-Bénard convection produces the main attractor bifurcation. Section 5 discusses examples and the topological structure of bifurcated solutions. Section 6 contains the corresponding results for the two-dimensional problem. In Section 7 Appendix, the definition and main results on the structural stability of 2-D divergence-free vector fields are recalled, as well as the concept and main results on the structural stability of 2-D divergence-free vector fields.

2. Mathematical solutions to Boussinesq equations

2.1. Boussinesq equations

The Boussinesq equations can be used to model the B enard experiment; see, for example, Rayleigh [12], Drazin and Reid [2], and Chandrasekhar [1]. They were reading.

$$\frac{\partial_u}{\partial_t} + (u \cdot \nabla)\, u - v\, \Delta\, u + p_0^{-1}\, \nabla\, p = -\, gk\; [1 - \alpha(T - T_0)], \qquad (1)$$

$$\frac{\partial \mathrm{T}}{\partial_t} + (\, u \cdot \nabla)\, T - k\, \Delta\, T = 0, \qquad (2)$$

$$\mathrm{div}\; u = 0\;\;, \qquad (3)$$

where v, k, g are the constants, $u = (u_1,\, u_2,\, u_3)$ is the velocity area, p is the pressure function, T is the temperature function, T_0 is a constant representing the lower surface temperature at $x_3 = 0$, and k is the unit vector in the x_3 −direction;

Enable the equations to be non-dimensional using thr following scale .

$x = hx^{'}$

$t = h^2 t\, /\kappa,$

$u = ku^{'}\; / \,h,$

$T = \beta\, h\, (\mathrm{T}\; / \sqrt{}\, \mathrm{R}) + T_0 - \beta h x^{'}{}_3$

$p = p_0 k^2 p^{'}\, / h^2 + p_0 - g p_0 (h x^{'}{}_3 + \alpha\beta h^2 (x_3^{'}) / 2),$

$p_r = v/\kappa.$

The equations $(2) - (4)$ can be rewritten as follows without the primes , The Rayleigh number R is defined by (1), and the Prandtl number $p_r = v/\kappa,$

$$\frac{1}{\mathrm{Pr}}\;\left[\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + (u\, \cdot\, \nabla) u + \nabla \mathrm{p0}\right] - \Delta u - \sqrt{\mathrm{R}}\mathrm{T}\; \mathrm{k} = 0, \qquad (4)$$

$$\frac{\partial \mathrm{T}}{\partial \mathrm{t}} + (u\, \cdot\, \nabla)\mathrm{T} - \sqrt{\mathrm{R} u_3} - \Delta\, \mathrm{T} = 0, \qquad (5)$$

$$div\; u = 0. \qquad (6)$$

The area of non-dimensionality is $\Omega = D\, \times\, (0, 1) \subset \mathbb{R}^3$ where $D \subset \mathbb{R}^2$ is a set that is open to interpretation. The coordinate system is defined as follows $x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$.

The Boussinesq equations $(4) - (6)$ are the fundamental equations used in this article in order to analyze the Rayleigh –Bénard problem. The following initial value conditions are added to them.

$$(u, T\,) = (\, u_0,\, T_0) \text{ at } t = 0. \qquad (7)$$

At the top and bottom, as well as at the lateral boundary, boundary conditions are needed

$\partial D \times (0, 1)$. At the top and bottom of the slope (x_3 = 0, 1), either rigid or free boundary conditions are defined.

T= 0, u = 0 (rigid boundary),

T= 0, u_3= 0, $\frac{\partial(u1,u2)}{\partial x_3}$ (free boundary). (9)

In various physical settings, such as rigid-rigid, rigid-free, free-rigid, and free-free, different combinations of top and bottom boundary conditions are commonly used. Each of the following boundary conditions exists on the lateral boundary ∂D ×[0, 1].

1. Occasional condition:

$$(u,\ T)(x_1 + k_1L_1, x_2 + k_2L_2, x_3) = (u, T)(x_1, x_2, x_3)), \qquad (10)$$

For any k_1, $k_2 \in Z$.

2.Boundary state of Dirichlet:

$$u = 0, T = 0(\text{or } \frac{\partial \mathrm{T}}{\partial \mathrm{n}} = 0) \qquad (11)$$

Free boundary condition:

$$T = 0, u_n = 0, \quad \frac{\partial u_\tau}{\partial \mathrm{n}} = 0, \qquad (12)$$

Where n and τ the unit normal and tangent vectors, respectively, on $\partial D \times [0, 1]$, and

$u_n = u \cdot n,\ u_{\tau} = u \cdot \tau.$

For the sake of convenience, we'll use the following set of boundary conditions in this article; however, all of the results apply to other combinations of boundary conditions as well.

$$\begin{cases} T = 0, u = 0 \text{ at } x_3 = 0, 1, \\ (\mathrm{u}, T)(x_1 + k_1L_1, x_2 + k_2L_2, x_3, t) = (u, T)(x, \mathrm{t}), \end{cases} \qquad (13)$$

for any k_1, $k_2 \in Z$.

2.2. Solution properties and functional setting

We refer interested readers to Foias, Manley, and Temam [3] for information on the practical setting of equations (4)-(6) with initial and boundary conditions (7) and (13). Let us proceed in this direction.

$$H = \{(u,\ ) \in L^2(\Omega)^3 \times L^2(\Omega) \mid \text{div } u = 0, u_3|_{x_3=0,1} = 0, \qquad (14)$$

where u_i is periodic in the x_i direction (i = 1, 2)},

$$V = \{(u,\ ) \in H_0^1(\Omega)^4 \quad | \quad \text{div } u = 0,$$

(15)

where u_i is periodic in the x_i direction (i = 1, 2)},

where H_0^1 (Ω) is the space of H^1(Ω functions that vanish at x_3 = 0, 1 and are periodic in the x_i −directions (i = 1, 2). H^1(Ω is the standard Sobolev space. The classical results for the existence of a solution for (4)-(6)

with initial and boundary conditions (7) and (13) are then obtained. For every $(\emptyset_0, T_0) \in H$, (4)- (6) with (7) and (13) possesses a weak solution

$$(u, \mathrm{T}) \in L^\infty([0,\tau]; H) \cap L^2(0,\tau; V)\ \forall\tau > 0. \tag{16}$$

If $(u_0, T_0) \in$ V , (4)-(6) with (7) and (14) has a one-of-a-kind solution at some interval $[0, \tau_1]$,

$$(u, \mathrm{T}) \in C([0,\tau_1]; V) \cap L^2(0, \tau_1; H^2(\Omega)^4 \cap V), \tag{17}$$

where $\tau_1 = \tau_1$ (M) depends on a bound of the V norm of (T_0, T_0):

$\|(u_0,, T_0)\| \leq.$ (M)

is determined by a V norm bound of $(\emptyset_0, T_0)$: $\|(u_0,, T_0)\| \leq. M$

Furthermore, for every $\|(\emptyset_0, T_0)\| \leq \delta$ small, (4)-(6) with (7) and (13) has a one-of-a-kind global (in-time) solution

$$(u, \mathrm{T}) \in C([0,\tau]; V) \cap L^2, (0,\tau; H^2\ (\Omega)^4 \cap V),\ \forall\, \tau > 0. \tag{18}$$

We may describe a semi-group based on these existing results.

$$S(\mathrm{t}):(u_0, T_0) \rightarrow (u\,(\mathrm{t}), T\,(\mathrm{t})),$$

which benefits from the properties of a semi-group

3. Nonlinear evolution equations with dynamic bifurcation

In this section, we'll go through some of the authors' findings on the dynamic bifurcation of abstract nonlinear evolution equations, which is crucial for understanding the –Bénard problem. Indeed, we'll give you a recipe for proving dynamic bifurcations for problems involving symmetric linear operators in this section.

3.1. Bifurcation of the attractor

H and H_1 is a compressed and compact insertion, assuming H and H_1 are two Hilbert spaces. The nonlinear evolution equations that resulted were used in this study.

$$\frac{du}{dt} = L_{\beta Y} + G(u, \lambda) \tag{19}$$

$$u\,(0) = u_0 \tag{20}$$

where u: $(0,\infty) \longrightarrow H$ and represent the unknown function, $\beta \in \mathbb{R}$ is the machine parameter, and L_β: $H_1 \rightarrow H$ are parameterized linear, absolutely continuous fields that are constantly dependent on $\lambda \in R^1$, and satisfy the equations below.

$$\begin{cases} L_\lambda = -A + \mathrm{B}_\lambda \text{ is a business} - \text{to} - \text{ business operator} \\ \mathrm{A}: H_1 \rightarrow H \text{ a homeomorphism that is linear,} \\ \mathrm{B}_\lambda : H_1 \rightarrow H \text{ the linear compact operators that are parameterized} \end{cases} \tag{21}$$

It is useful to note that L_λ gives an analytic semi-group as in some previous studies [20,21] $\{e^{tL\lambda}\}t \geq 0$ .As a result, fractional power operators $L_\lambda^\propto$ can be defined for any $0 \leq \propto \leq 1$ with its domain Domain $H_\propto = D(L_\lambda^\propto)\ in$ such that $H_{\propto 1} \subset H_{\propto 2}\ if\ \alpha 1 > \alpha_2$, and $H_0 = H$.

Similarly, the nonlinear terms will be considered in this study. $G(.,\lambda)$: $H_\propto \rightarrow H$ for some $0 \leq \alpha < 1$ belongs to the parameterized C^rbounded operator $(r \geq 1)$ family, which is continuously dependent on the parameter $\lambda \in \mathbb{R}^1$.

$$G(y,\beta) = 0(\|y\|H_{\propto}) \qquad \forall\ \lambda \in \mathbb{R}^1 \qquad (22)$$

This analysis has to do with the sectorial operator $L_\beta = -A + B_\lambda$ in these diagrams. for which there is a real eigenvalue sequence $\{\rho k\} \subset \mathbb{R}^1$. and an eigenvector sequence $\{e_k\} \subset H_1\ of\ A$:

$$A_{ek} = \rho k^e k$$
$$0 < \rho_1 < \rho_2 < \cdots \qquad (23)$$
$$\rho k \to \infty\ (k \to \infty)$$

such that $\{e_k\}$ is an orthogonal basis of H .
So, in the case of the compact operator $B_\lambda : H_1 \to H$, this work will continue to assume the presence of a constant $0 < \theta < 1$ in such that

$$B_\lambda: H_\theta \to H \text{ Bounded}, \forall\ \lambda \in \mathbb{R}^1 \qquad (24)$$

Let $\{S_\lambda(t)\}\ t \geq 0$ be an operator semi-group generated by the equation (19) that enjoys the properties.
1.$S_\lambda(t): H \to H$ is a linear continuous operator for any $t \geq 0$,
2.$S_\lambda(0) = I : H \to H$ for the purpose of determining H's identity, and
3.Then, for whatever reason $t, s \geq 0,\ S_\lambda(t + s) = S_\lambda(t) \cdot S_\lambda(s)$
We can assume that the solution to equations (19) and (20) can be expressed as

$$y(t) = S_\lambda(t)y_0, \qquad t \geq 0.$$

Definition 3.1:
If $\Sigma \subset H$ for any $S\ (t) = \Sigma$, a set H is considered an invariant set of (19). It is compact and there exists a neighborhood $U \subset H\ \Sigma$ such that for any $\phi \in U$ we have, an invariant set $t \geq 0$ of (19) is said to be an attractor.

$$lim_{t\to\infty} \mathrm{dist}_H(u\ (\mathrm{t},\varphi\ ) = 0 \qquad (25)$$

The basin of attraction of Σ is the largest open set U satisfying (25)

Definition 3.2.
1.The equation (19) is said to bifurcate from $(u,\lambda) = (0, \lambda_0)$ an invariant set Ω_λ if a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (19), $0 \notin \Omega_{\lambda_n}$exists,

$$lim_{n\to\infty}\lambda_n = \lambda_0$$
$$lim_{n\to\infty} = \max |x|_{\ x\in\Omega_{\lambda_n}} = 0.$$

2.The bifurcation is known as attractor bifurcation if the invariant sets Ω_λ are attractors of (19).

3. The bifurcation is known as S^m–attractor bifurcation if $\Omega_\lambda$are attractors and are homotopy identical to a m –dimensional sphere S^m,

If there are $xy \in H_1$ complex numbers with the same eigenvalue as $\beta = \alpha_1 + i\alpha_2 \in \mathbb{C}$, it is considered an eigenvalue of L_λ.

$$L_{\lambda x} = \alpha_1 x - \alpha_{2\ y}$$
$$L_{\lambda y} = \alpha_2 x - \alpha_{1\ y}$$

Let the eigenvalues of L_λ be given by (counting the multiplicity).

$\beta_1(\lambda), \beta_{12}(\lambda), \cdots, \beta_K(\lambda) \in \mathbb{C}$,

The complex space is denoted by the letter ℂ. Assume the

$$R_e\beta_i \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases} \qquad (1 \leq i \leq m+1) \tag{26}$$

$$R_e\beta_i\,(\lambda_0) < 0, \qquad \forall m + 2 \leq j \tag{27}$$

Consider the eigenspace of L_λ at L_0 as follows.

$$E_0 = U_1 \leq i \leq m + 1\{u\epsilon H_1 \big| (L_{\lambda 0} - \beta_i(\lambda_0\,))^k\} \quad u = 0,\ K = 1.2\ldots\ldots\}$$

It is well understood that dim $E_0 = m + 1$. proved the following complex bifurcation theorems for the (19).

Theorem 3.3 (Attractor Bifurcation) Assume that (21), (22), (26) and (27) are real and that $u = 0$ is a locally asymptotically stable equilibrium point of (20) at $\lambda - \lambda_0$. Then the following statements are correct.

1. (19) bifurcates from $(u,\lambda) = (0,\lambda_0)$ with $m \leq \dim A_\lambda \leq m + 1$, which is an attractor$A_\lambda$ for $\lambda > \lambda_0$,.$m > 0$ is connected;

2.If the attractor A_λ is a limit of a sequence of $(m+1)$-dimensional annulus M_k with $M_k + 1 \subset M_k$; particularly if A_λ is a finite simplicial complex, then A_λ has the homotopy form S^m;

3 . Any $u_\lambda \in A_\lambda$ can be represented as u_λ.

$$u_\lambda = v_\lambda + o(\|v_\lambda\| H_1), \qquad v_\lambda \in E_0:$$

4.If $G : H_1 \rightarrow H$ is compact and (19) in A_λ has finite equilibrium points, we have the index formula.

$$\sum_{u_{i\in A_\lambda}} Ind[-(L_\lambda + G), u_i] = \begin{cases} 2, & if\ m = odd \\ 0, & if\ m\ =\ even \end{cases}$$

5. If u =0 is globally stable for (19) at $\lambda = \lambda_0$, then there is an $\varepsilon > 0$ such that $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, for any bounded open set $U \subset$ H with $0 \in U$, the attractor A_λ bifurcated from $(0, \lambda_0)$ attracts U / T in H, where T is the stable manifold of u = 0 with co-dimension m +1. If (19) has a global attractor for all λ near λ_0, then ε here can be chosen without regard to U.

3.2. At critical states, asymptotical stability is essential.

It is important to check the asymptotic stability of the critical states in order to apply the above dynamic bifurcation theorems. In this section, we prove a theorem that ensures the necessary asymptotic stability for equations with symmetric linear sections. Assume that the linear operator L_λ in (19) is symmetric.

$$\langle L_\lambda u, v\rangle_H = \langle u, L_\lambda v\rangle_H , \qquad \forall u, v \in H$$

All of L_λ eigenvalues are then real numbers. Allow the eigenvalues $\{ \beta_k\}$ and L_λ at $\lambda = \lambda_0$ to satisfy each other.

$$\begin{cases} \beta_i = 0, \ \ 1 \le i < m+1 \quad (m \ge 0) \\ \beta_j < 0, \ \ \ m+2 \le j < \infty \end{cases} \tag{28}$$

$$E_0\{u \in H_1 | L_{\lambda 0} u = 0\}$$
$$E_1 = E_0^{\perp} = u \in H_1 | \langle u, v\rangle_H = 0 \qquad \forall v \in E_0$$

$p_1; H \to E_1$ the projected image

By (28), $\dim E_0 \ = \ m + \ 1$.

Theorem (3.4): Assume that L_λ in (21) is symmetric, and that the spectrum given by (28) is real, and that $G_{\lambda 0}; H_1 \to H$ meets the orthogonal condition:

$$\langle G_{\lambda 0} u, u\rangle_H = 0, \forall u \in H_1 \tag{29}$$

1.Then one and only one of the following two statements is correct:

At $\lambda = \lambda_0$, there is a sequence of invariant sets $\{T_n\} \subset E_0$ of (19) such that

$$0 \notin T_n, lim_{n\to\infty} dist(T_n, 0) = 0$$

1. Under the H–norm, the trivial steady state solution $u = 0$ for (19) is locally asymptotically stable at $\lambda = \lambda_0$.
2. Furthermore, $u = 0$ is globally asymptotically stable if (19) has no invariant sets in E_0 except for the trivial one $\{0\}$.

Proof. The measures are as follows: steps one, two, three, and four.

Step 1: It is self-evident that Assertions (1) and (2) of Theorem (3.4) cannot both be valid.

We will always work on the case where $\lambda = \lambda_0$ is present in this proof. Direct energy estimates in this case mean that the solutions u of (19) satisfy that.

$$\frac{d}{dt} \|u\|_H^2 = 2 < \langle L_{\lambda 0} u, u\rangle = \sum_{n=m+2}^{\infty} \beta_i \, |u_i|^2 \le 0. \tag{30}$$

$$\|u\|_H^2 \le \|u_0\|_H^2 - 2|\beta_{m+2}| \int_0^t \|v\|_H^2 \, d\tau \, . \tag{31}$$

where

$$u = w + v \in H = E_0 \oplus E_0^{\perp}$$

$$v = \sum_{i=m+2}^{\infty} u_i \in E_0^{\perp}$$

$$w = \sum_{i=1}^{m+1} u_i \in E_1 = E_0^{\perp}$$

It is clear that the solution $u(t,\varphi)$ is non-increasing for any $\varphi \in H_1$, $i.e.$

$$\|u(t_2,\varphi)\| \leq \|u(t_1,\varphi)\|, \forall t_1 < t_2 \text{and } \varphi \in H_1 \tag{32}$$

As a result, $lim_{t\to\infty}\|u(t,\varphi)\|$ exists.

Step 2: For any $\varphi \in H_1$, we've got

$lim_{t\to\infty}\|u(t,\varphi)\| = lim_{t\to\infty}\|v(t,\varphi)\| + w(t,\varphi) = \delta \leq |\varphi|$.

The w-limit set, which is an invariant set, then satisfies this requirement.

$w\ (\varphi) \subset S_\delta = \{u \in H | \|u\| = \delta\}$.

Since $w(\varphi)$ is an invariant sequence, we have $\psi \in\ w(\varphi)$

$$u(t,\psi) \subset\ w(\varphi)\ \subset S_\delta\ \ \forall\ t \geq 0$$

As a result, if $\psi = \bar{\upsilon} + \tilde{\omega} \in E_0^{\perp}$ is multiplied by $E_0^{\perp} \neq 0$, the result is (31), for any $t > 0$.

$$\|u(t,\psi\| < \|\psi\| = \delta$$

This is a contradiction. For example, for any $\varphi \in H_1$

$$w\ (\varphi) \subset E_0 \tag{33}$$

Step 3: If Assertion (2) is false, then $u_n \in H_1$ has $u_n \to 0$ as $n \to \infty$, resulting in $0 \notin w(u_n) \subset E_0$, and

$$lim_{n\to\infty} = \|u(t,\varphi)\| = 0$$

Assertion (1), in particular, is right.

Step 4: If Assertion (1) is false, there exists a neighborhood $U \subset H$ of 0 such that for any $\emptyset \in U$, there exists a neighborhood $U \subset H$ of 0.

$$lim_{t\to\infty} = \|u(t,\varphi)\| = 0$$

Assertion (2): in particular, is right. The remainder of the proof is straightforward, and the proof is finished.

4. The –Bénard problem's attractor bifurcation

The Rayleigh number is denoted by R. The same boundary conditions (13) as the nonlinear Boussinesq scheme are added to these equations. The Rayleigh number R has a symmetric eigenvalue problem.

4.1 Principal theorems.

$$\begin{cases} -\Delta u + \nabla p - \sqrt{R}\mathrm{Tk} = 0 \\ -\Delta T - \sqrt{R_{\boldsymbol{u_3}}} = 0 \\ \mathrm{div} = 0 \end{cases} \tag{34}$$

As a result, we know that all eigenvalues R_k are equal.

True numbers $M_k$of (34) with (13) are multiplicities, and

$$0< R_1 < \cdots < R_k < R_{k+1}< \cdots . \quad (35)$$

The critical Rayleigh number is the first eigenvalue R_1, which is also denoted by $R_c = R_1$. Let the multiplicity of R_c be $m_1 = m$ +1 ($m \geq 0$), and the (34) orthonormality of the first eigenvectors $\Psi_1 = (e_1\ (x), T_1), \cdots , \Psi_m +1 = (e_{m+1}, T_{m+1})$:

$$\langle\Psi_i, \Psi_j\rangle_H = \int_\Omega [e_i \cdot e_j + T_i T_j]\, dx = \delta_{iJ} .$$

Let E_0be the first eigenspace of (34) for the sake of simplicity. (13)

$$E_0 = \left\{\sum_{k=1}^{m+1} \alpha_k, \Psi_k, \mid \alpha_k \in R, 1 \leq K \leq m + 1\right\}. \quad (36)$$

The following theorems are the most important findings in this section.

Theorem 4.1 The following assertions are valid for the –Bénar dproblem (4) − (6) with boundary condition(2.13).

1. The steady-state (u, T)=0 is a globally asymptotically stable equilibrium point of the equations when the Rayleigh number is less than or equal to the critical Rayleigh number: $R \leq R_c$,
2. The equations split at ((u, T), R) = (0, R_c) an attractorA_R for $R > R_c$, with $m \leq$ dim $A_R \leq m$ +1 being related when $m > 0$ is used.
3. The velocity field u can be expressed for any $(u,\ T) \in A_R$.

$$u = \sum_{k=1}^{m+1} \alpha_k e_k + o\left(\sum_{k=1}^{m+1} \alpha_k e_k\right) \quad (37)$$

4. If A_R is a finite simplicial complex, the attractor $A_R$has the homotopy form of an m −dimensional sphere S^m.
5. An open neighborhood $U \subset H$ of (u, T) and an $\varepsilon > 0$, such as s$R_c < R <R_c + \varepsilon$, are accessible. Where T is the stable manifold of (u, T) with co-dimension m + 1, the attractor A_R attracts U/T in H

Theorem 4.2. If the first eigenvalue of $L_{\lambda 0}$ is simple, i.e. dim E_0 = 1, then the bifurcated attractor A_R of the –Bénard problem (4) − (6) with boundary condition(2.13). has exactly two points,$\emptyset_1, \emptyset_2 \in H_1 = V \cap H^2 (\Omega)^4$ given by

$$\emptyset_1 = \alpha\Psi_1 + o(|\alpha|),\ \emptyset_2 = -\alpha\Psi_1 + o(|\alpha|),$$

for some $\alpha \neq 0$, where Ψ_1 is the first eigenvector producing E_0 in (36). Furthermore, there is a $\varepsilon > 0$as $R_c < R < R_c + \varepsilon$, for any bounded open set $set\ U \in H\ with\ 0 \in U$.
U can be broken down into two open sets, U_1 and U_2, with the result that

1. $U = U_1 + U_2, U_1 \cap U_2 = \emptyset$ and $0 \in \partial U_1 \cap \partial U_2$,
2. $\emptyset_i \subset U_i$ ($i = 1, 2$), and
3. $S_\lambda\ (t)\emptyset_0$ is the solution of the –Bénard problem (4)-(6) with (13) with initial data $\emptyset_0 = (u_0,\ T_0)$. for any $\emptyset_0 \in U_i$ $(i = 1, 2)$, $\lim_{t\to\infty} S_\lambda\ (t)\emptyset_0 = \emptyset_i$,. we like to make a few observations now.

Remark 4.3: (37) in Theorem 4.1 is important for studying the topological structure of the Rayleigh-Bénard convection, as we'll see in the next section.

Remark 4.4: The classic pitchfork bifurcation is described by Theorem 4.2. The key benefit of this theorem is that we know that these bifurcated steady states are stable.

Remark 4.5: Both theorems are valid for Boussinesq equations (4)-(6) with various boundary conditions, as defined in Section 2.

Proof of Theorem 4.1. we'll use the abstract results from Section 3 and follow the steps below.

Step 1: Without sacrificing generality, we assume the Prandtl amount.

$$Pr = 1; \tag{38}$$

Otherwise, we only need to take the form (4)-(6), and the proof remains the same.

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla - P_r\Delta u - \sqrt{R}\sqrt{P_r}\theta k = 0 \\ \frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \sqrt{R}\sqrt{P_r}u3 - \Delta\theta = 0, \\ div\, u = 0, \end{cases} \tag{39}$$

in which $\theta = \sqrt{P_r}T$

Now consider H, the function space defined by (14), and H_1, the intersection of H and the H^2 Sobolev space, $i.e.$

$$H_1 = H \cap (H^2.(\Omega))^4.$$

Then let $: H_1 \to H$, and $L_\lambda = -\mathrm{A} + B_\lambda : H_1 \to H$ be defined by

$$\begin{cases} G(\emptyset) = (-P[(u \cdot \nabla)u], -(u \cdot \nabla)\mathrm{T}), \\ A\emptyset = (-P(\Delta u), -\Delta\mathrm{T}), \\ B\lambda\emptyset = \lambda(P(\mathrm{T\,k}), u_3) \end{cases} \tag{40}$$

Any $\emptyset \in H$ will suffice. The Leray projection is $\lambda = \sqrt{R}$, and $: L^2\ (\Omega)^3 \to H$. Then it's clear that these operators have the following characteristics:

1. The symmetric linear operators A, B_λ, and L_λ

   2. G is an orthogonal nonlinear operator, $i.e.$

$$\langle G_{(\emptyset)}, \emptyset\rangle_H = 0 \tag{41}$$

3 For the operators described in (21)—(24), the conditions (21)—(24) hold valid (40). The Boussinesq equations (4) can then be rewritten in the operator form shown below.

$$\frac{dt}{dt} = L_\lambda\emptyset + \mathrm{G}(\emptyset), \qquad \emptyset = (u, T). \tag{42}$$

Step 2: Now we'll look at the conditions (26) and see if they're right (27). Consider the issue of eigenvalue.

$$L_\lambda\emptyset = \beta(\lambda)\,\emptyset, \qquad \emptyset = (u, T) \in \mathrm{H1}. \tag{43}$$

This eigenvalue issue is the same as

$$\begin{cases} -\Delta u + \nabla\mathrm{p} - \lambda\mathrm{T\,k} + \beta(\lambda)u = 0, \\ -\Delta\mathrm{T} - \lambda_{\mathrm{u3}} + \beta(\lambda)\mathrm{T} = 0, \\ div\, u = 0. \end{cases} \tag{44}$$

The eigenvalues β_k $(k = 1, 2, \cdots)$ of (44) are considered to be real numbers satisfying

$$\begin{cases}\beta_1(\lambda) \geq \beta_2(\lambda) \geq \cdots \geq \beta_k(\lambda) \geq \cdots, \\ \lim_{k\to\infty} \beta_k(\lambda) = -\infty, \end{cases} \tag{45}$$

The relationship between the first eigenvalue $\beta_1(\lambda)$ of (44) and the first eigenvalue $\lambda_1 = \sqrt{R_c}$ of (34) is:

$$\beta_1(\lambda)\begin{cases}< 0 \text{ as } 0 \leq \lambda < \lambda_1, \\ = 0 \text{ as } \lambda = \lambda_1. \end{cases} \tag{46}$$

Step3: To prove (26) and (27), it is sufficient to show that (45) and (46).

$$\beta_1(\lambda) > 0 \text{ as } \lambda > \lambda_1. \tag{47}$$

The first eigenvalue $\beta_1(\lambda)$ of (44) is known to have minimal property.

$$-\beta_1(\lambda) = min_{(u,T)\in H_1} \frac{\int_\Omega [|\nabla u|^2 + |\nabla T|^2 - 2\lambda T_{u_3}]d_x}{\int_\Omega [T^2 + u^2]dx} \tag{48}$$

The first eigenvectors $(e, \varphi) \in H_1$ are clearly satisfied.

$$\int_\Omega [|\nabla e|^2 + |\nabla\varphi|^2 - 2\lambda_{e3}\varphi]dx \begin{cases}= 0, & \text{for} \quad \lambda = \lambda_1 \\ < 0, & \text{for} \quad \lambda > \lambda_1. \end{cases} \tag{49}$$

We can deduce (48) and (49) from. (47). As a result, the conditions (26) and (27) are met.

Step 4:Finally, we must demonstrate that $(u, T) = 0$ is a globally asymptotically stable equilibrium point of (4)- (6) at the critical Rayleigh number$\lambda_1 = \sqrt{Rc}$ to prove Theorem 4.1 using Theorems 3.3. Theorem 3.4 states that the equations (4)-(6) have no invariant sets except the steady-state $(u, T) = 0$ in the first eigenspace E_0

Since the Boussinesq equations (4)-(6) have abounded absorbing set in H, all invariant sets in H have the same bound as the absorbing set. Assume that (4)-(6) have a $B = \{0\}$ at $\lambda_1 = \sqrt{Rc}$ invariant $\subset E_0$. The Boussinesq equations (4)-(6) can then be rewritten in B, which includes eigenfunctions of the linear component corresponding to the eigenvalue 0.

$$\begin{cases}\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \\ \frac{\partial T}{\partial t} + (u \cdot \nabla)T = 0, \end{cases} \tag{50}$$

It's clear that solutions $(u, T) \in B$ and , $(u, T) = \alpha(u(\alpha t), T(\alpha t)) \in \alpha B \subset E_0$of (51), respectively, are also solutions of (50). The set $\alpha B \subset E_0$ is an invariant set off for any real number $\alpha \in R$ As a result, we conclude that (4)-(6) has an unbounded invariant set, which contradicts the presence of an absorbing set. As a result, the invariant set B can only contain $(u, T) = 0$ elements.The proof is finished.

Proof of Theorem 4.2. It is sufficient to demonstrate that the stationary equations of (34) can bifurcate exactly two singular points in H_1 as $R > Rc$ by Theorem 4.1. To prove this claim, we use the Lyapunov–Schmidt process.

H_1 can be decomposed into two parts since the operator $L_\lambda : H_1 \to H$ described by (39) is a symmetric absolutely continuous field.

$$H_1 = E_1^\lambda \oplus E_2^\lambda,$$
$$E_1^\lambda = \{\alpha\Psi_1(\lambda) \mid \alpha \in \mathbb{R}, \Psi_1(\lambda) \text{ the first eigenvector of } L_\lambda + G\}$$
$$E_2^\lambda = \{\emptyset \in H_1 \mid \langle\emptyset, \Psi_1\rangle_H = 0\}.$$

Furthermore, the subspaces E_1^{λ} and E_2^{λ} of $L_{\lambda} + G$ are invariant. Let the canonical projection be $P_1 : H_1 \rightarrow E_1^{\lambda}$, and

$$\emptyset = x\Psi_1 + y, \quad x \in \mathrm{R},\ y \in E_2^{\lambda}$$

The $L_{\lambda\emptyset} + G(\emptyset) = 0$ equations can then be decomposed into

$$\beta(\lambda)x + \langle G(\emptyset), \Psi_1(\lambda)H \rangle_H = 0, \qquad (51)$$

$$L_{\lambda y} + P_1 G(u) = 0. \qquad (52)$$

The eigenvalues $\beta_i(\lambda)$ of $L_{\lambda\emptyset} = \beta(\lambda)\emptyset$ satisfy that $\beta_i(\lambda_1) = 0$ for $j \geq 2$, and $\lambda_1 = \sqrt{Rc}$, according to the assumption. As a result, the constraint.

$L_{\lambda|E_2^{\lambda}} : E_2^{\lambda} \longrightarrow E_2^{\lambda}$,

can be inverted. From (52), the implicit function theorem shows that y is a function of x:

$$y = y(x, \lambda), \qquad (53)$$

which fulfills (52). The function (53) is also analytic since $G(u) = G(x\Psi_1 + y)$ is an analytic function of u. As a result, the function

$$f(x, \lambda) = \langle G(x\Psi_1 + y(x, \lambda)), \Psi_1 \rangle_H \qquad (54)$$

is analytical in nature. As a result, the expansion (51) is found in the equation.

$$\beta(\lambda)x + f(x, \lambda) = \beta(\lambda)x + \alpha(\lambda)x^k + o(|x|^k) = 0, \qquad (55)$$

For some $\alpha(\lambda) \in R$, such as $\alpha(\lambda_1) \neq 0$ and $k > 1$, the critical Rayleigh number is $\lambda_1 = Rc$. On the premise

$$\beta(\lambda)\begin{cases} < 0 \text{ as } \lambda < \lambda_1, \\ = 0 \text{ as } \lambda = \lambda_1, \\ > 0 \text{ as } \lambda > \lambda_1. \end{cases}$$

Furthermore, since $\lambda \leq \lambda_1$ $(i.e. R \leq Rc)$ and $\lambda_1 - \lambda$ are small, the equations (51) and (52) There are no non-zero solutions., implying that $\alpha(\lambda_1) < 0$ and $k = odd$ are equal.

As a result, we can deduce that the equation (55) has only two solutions.

$$\pm x = \pm\left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/K} + o\left(\left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/k}\right)$$

With $\lambda > \lambda_1$ with $\lambda - \lambda_1$ is a good compromise. We have shown that the stationary equations of (4)-(6) bifurcate from $(\emptyset, \lambda) = (0, \lambda_1)$ exactly two solutions as $\lambda > \lambda_1$ or $R > Rc$, with $\lambda - \lambda_1$ sufficiently small.

$$\emptyset_{\lambda} = x \pm \Psi_1 + o(|x \pm |).$$

As a result, this theorem is established.

5. Remarks on the topological structure of Rayleigh-Bénard problem solutions. The structure of the eigenvectors of the linearized problem (34), as previously discussed, is significant in studying the onset of the Rayleigh- -Bénard convection. The eigenspace E_0 is dimension m +1 also defines the dimension of the bifurcated attractor A_R. As a result, in this section, we look at the first eigenspace in detail for various spatial domain geometry and boundary conditions.

5.1. The eigenvalue problem and its solutions.

In the following, we will always consider the –Bénard problem on a rectangular region: $\Omega = (0, L_1) \times (0, L_2) \times (0,1)$, with the free boundary condition as the boundary condition.

$$u.n = 0, \quad \frac{\partial u.t}{\partial n} = 0 \ on\ \partial\Omega \qquad (56)$$

$$T = 0 \text{ at } \quad x_3 = 0,1, \qquad (57)$$

$$\frac{dT}{dn} = 0 \ \text{at} \quad x_1 = 0, L_1 \quad \text{or} \quad x_2 = 0, L_2 \tag{58}$$

We separate the variables as follows for the eigenvalue equations (34) with the boundary condition (56)—(58).

$$\begin{cases} (u_1.u_2) = \frac{1}{a^2}\left(\frac{\partial f(x_1.x_2)}{\partial x_1}, \frac{\partial f(x_1.x_2)}{\partial x_2}\right)\frac{dH(X_3)}{dx_3} \\ u_3 = f(x_1,x_2)H(x_3) \\ T = f(x_{1,},x_{2)}\alpha(x_3) \end{cases} \tag{59}$$

where $a^2 > 0$ is an undefined constant

The functions f, H, α satisfy (56)—(58) as a result of (34) with (56)—(58).

$$\begin{cases} -\Delta_1 f = a^2 f, \\ \frac{\partial f}{\partial x_1} = 0 \qquad \text{at } x_1 = 0, L_1 \\ \frac{\partial f}{\partial x_2} = 0 \qquad \text{at } x_2 = 0, L_2 \end{cases} \tag{60}$$

and

$$\begin{cases} \left(\frac{d^2}{dz^2} - a^2\right)^2 \quad H = a^2 \ \lambda\alpha, \\ \left(\frac{d^2}{dz^2} - a^2\right)\alpha = -\lambda H, \end{cases} \tag{61}$$

in addition to the boundary conditions

$$\begin{cases} \varphi(0) = \varphi(1) = 0, \\ H(0) = H(1) = 0, H''(0) = H''(1) = 0 \end{cases} \tag{62}$$

It is obvious that the solutions to (60) come from

$$\begin{cases} f(x_1,x_2) = \cos(a_1x_1)\cos(a_2x_2) \\ a_1^2 + a_2^2 = a^2, \ (a_1,a_2) = k_1\pi/L_1, k_2\pi/L_2 \end{cases} \tag{63}$$

,

for any $k_1, k_2 = 0,1 \ldots..$

Let $a_1^2 + a_2^2$ be your guide. It is clear that the first eigenvalue λ_0(and the eigenvectors of (61) and (62) are given by for each given a^2.

$$\begin{cases} \lambda_0(a) = \frac{(\pi^2+a^2)^{3/2}}{a} \\ (H,\alpha) = (sin \ \pi x_3, \frac{1}{a}\sqrt{\pi^2+a^2} sin \, \pi x_3 \end{cases} \tag{64}$$

The first eigenvalue $\lambda_1 = \sqrt{R_C}$ of (35 with (61)—(62) is clearly the minimum of λ_0 (a):

$$R_C = min_{a^2=a_1^2+a_2^2} \ \lambda_0^2(a) \tag{65}$$

$$= min_{k_{1,}k_2\in\mathbb{Z}}\left[\pi^4 + \left(1\frac{k_1^2}{L_1^2} + \frac{K_1^2}{L_2^2}\right)^3 / \left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right)\right]$$

Thus, (59), (63), and (64) can be used to derive the first eigenvectors of (34) with (56)—(58):

$$\begin{cases} u_1 = -\frac{a_1\pi}{a^2}\sin(a_1x_1)\cos(a_2x_2)\cos(\pi x_3), \\ u_2 = -\frac{a_1\pi}{a^2}\sin(a_1x_1)\cos(a_2x_2)\cos(\pi x_3), \\ u_3 = \cos(a_1x_1)\cos(a_2x_2)\sin(\pi x_3), \\ T = -\frac{1}{a}\sqrt{\pi^2+a^2}\cos(a_1x_1)\cos(a_1x_2)\sin(\pi x_3), \end{cases} \tag{66}$$

where $a^2 = a_1^2+a_2^2$ satisfies (65)

By Theorem 4.1 The topological structure of the bifurcated solutions of the –Bénard problem (4)–(6) with (56)—(58) is defined by (66), which is dependent on the horizontal length scales, according to (65). $L_1$and L_2 are two numbers that can be used together. In the –Bénard problem, the pattern of convection is determined by the size and shape of the fluid containers. The rest of this section will show you how to do that.

6. Asymptotic and structural stabilities of bifurcated solutions in two-dimensional Rayleigh-Bénard convection

The primary goal of this section is to investigate the dynamic bifurcation and structural stability of bifurcated solutions of the $2-D$ Boussinesq equations that are linked to Rayleigh- -Bénard convection. It is clear that both Theorems 4.1 and 4.2 are true This holds true for any combination of boundary conditions in the 2D Boussinesq equations, as stated in Section 2. As a result, in this section, we concentrate on structural stability in the physical space of bifurcated solutions, which justifies the creation of roll patterns in Rayleigh-Bénard convection.

Technically, we can see from (65) that since L_1/L_2 is a small number, the wavenumber $k_2 = 0$ is used. As a result, the three-dimensional –Bénard problem is reduced to a two-dimensional problem. Furthermore, the three-dimensional –Bénard convection can be well understood by the two-dimensional version due to the symmetry of the honeycomb arrangement of the –Bénard convection on the xy–plane.

We always assume that the domain $\Omega = [0, L] \times [0, 1]$ is in coordinate system $x = (x_1, x_3)$. for accuracy. The 3-D Boussinesq equations have the same structure as the 2-D Boussinesq equations for 2-D –Bénard convection (4)-(6)

$$\begin{cases} \frac{1}{\mathrm{Pr}}\left[\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + (u \cdot \nabla)\mathrm{u} + \nabla \mathrm{p}\right] - \Delta \mathrm{u} - \sqrt{\ }\mathrm{RT\,k} = 0, \\ \frac{\partial \mathrm{T}}{\partial \mathrm{t}} + (u \cdot \nabla)\mathrm{T} - \sqrt{\ }\mathrm{Ru3} - \Delta \mathrm{T} = 0, \\ \mathrm{div\,u} = 0, \end{cases} \tag{67}$$

In the $x = (x_1, x_3)$ coordinate system, the velocity field is replaced by $u = (u_1, u_3)$, and the operators are the corresponding 2-D operators. For the sake of convenience, we will just consider the free-free boundary conditions:

$$\begin{cases} u \cdot \mathrm{n} = 0, \quad \frac{\partial \mathrm{u}\tau}{\partial \mathrm{n}} = 0, \mathrm{on}\ \partial\Omega, \\ T = 0 \text{ at } x_3 = 0, 1, \quad \frac{\partial \mathrm{T}}{\partial \mathrm{x1}} = 0, \mathrm{at}\ x_1 = 0, \mathrm{L}. \end{cases} \tag{68}$$

The function space H defined by (14), in this case, is replaced by

$$H = \{(u,T) \in L^2(\Omega)^3 \mid div\, u = 0, u_3\, |_{x3=0,1} = 0, u_1\, |_{x1=0,L} = 0\}$$

The wavenumber k and the critical Rayleigh number are determined by (65) and (66) for the equation (67) with the free boundary condition.

$$k \cong a_c \frac{L}{\pi} = \frac{L}{\sqrt{2}},$$

$$R_C = \frac{\pi^4 (k^2+L^2)^3}{L^4},$$

$E_0$is a one-dimensional eigenspace, and it is given by

$$\begin{cases} E_0 = \text{Span}\{\Psi_1 = (e_1, T_1)\}, \\ e_1 = \left(-\frac{\mathrm{L}}{\mathrm{k}} \sin\frac{k\pi x_1}{\mathrm{L}} \cos\pi x_3, \cos\frac{k\pi x_1}{\mathrm{L}} \sin\pi x_3\right) \\ T = \frac{1}{K}\sqrt{L^2K^2} \cos\frac{k\pi x_1}{L} \sin\pi x_3. \end{cases} \quad (69)$$

Since the first eigenvectors (69) are structurally stable, we can derive the following result from Theorem 4.2.

Theorem 6.1. As the Rayleigh number $Rc < R < Rc + \varepsilon, U$ can be decomposed into two open sets U_1 and U_2 depending on R, there is a $\varepsilon > 0$ for every bounded open set $U \subset H$ with $0 \in U$.

1. $U = U_1 + U_2, \quad U_1 \cap U_2 = \emptyset, \qquad 0 \in \partial U_1 \cap \partial U_2;$

2. There exists a time $t_0 > 0$ for any initial value $\emptyset_0 \in U_i \quad (i = 1, 2)$ such that the solution $SR(t)\emptyset_0$ of (67) with (68) is topologically identical to either the structure is shown in Figure 6.1(a) or the structure is shown in (b) for all $t > t_0$.

7. Structural Stability for Divergence-Free Vector Fields

Let $C^r(\Omega, \mathbb{R}^2)$ be the space on Ω that contains all $C^r(r \geq 1))$ vector fields. We consider a $C^r(\Omega, \mathbb{R}^2)$ subspace:

$$B^r = (\Omega, \mathbb{R}^2) = \left\{v \in C^r(\Omega, \mathbb{R}^2) \,\middle|\, div\, v = 0, v_n = \frac{\partial v_\tau}{\partial n} = 0 \ on\ \partial\Omega\right\}$$

Definition 7.1 If a homeomorphism of $: \Omega \to \Omega$ exists that takes the orbits of u to orbits of v and preserves their orientation, two vector fields $u. v \in B^r = (\Omega, \mathbb{R}^2)$ are said to be topologically equivalent.

Definition 7.2. In $B^r = (\Omega, \mathbb{R}^2)$ a vector field $v \in B^r = (\Omega, \mathbb{R}^2)$is referred to as structurally stable.

If there is a neighborhood $\subset B^r = (\Omega, \mathbb{R}^2)$ of v such that u and v are equal for any $u \in U$

Topologically, they're the same.

Following that, we'll go through some fundamental facts and definitions about divergence-free vector fields.

Let $v \in B^r = (\Omega, \mathbb{R}^2)$ be the variable.

If $v(p) = 0$; a point $p \in \Omega$ is called a singular point of v; if the Jacobian matrix $Dv(p)$ is invertible, a singular point p of v is called non-degenerate; v is called normal if all singular points of v are non-degenerate.

A non-degenerate boundary singularity must be a saddle, and an interior non-degenerate singular point of v may be either a center or a saddle.

V's saddles must be attached to saddles. $p \in \Omega$ is the name of an interior saddle self–awareness if p is connected only to itself.

[7] proved the following theorem, which stipulates all necessary and sufficient conditions.

For a divergence-free vector field's structural stability

Theorem 7.3. Let $v \in B^r = (\Omega, \mathbb{R}^2)$ $(r \geq 1)$ be your guide. Then v in $B^r = (\Omega, \mathbb{R}^2)$is structurally stable if and only if

v is regular;

All of v's interior saddles are self-contained; and Each boundary saddle point is linked to other boundary saddle points on the same boundary-connected component.

In addition, in $B^r = (\Omega, \mathbb{R}^2)$, the set of all structurally stable vector fields is open and dense.

Remark 7.4. The theorems of structural stability for divergence-free vector fields on a torus with the Dirichlet boundary state and Hamiltonian vector fields T^2 has been established.

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