Research Article

Mean Inequalities For The Arguments In Index And Conjugate Index Set

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Abstract: The inequalities for positive arguments which is analogous to the concept of Ky- Fan type inequalities are introduced by the use of conjugate index arguments and established the inequalities involving the arithmetic mean, geometric mean, harmonic mean and contraharmonic means.

1. Introduction

The applications of means and their inequalities are investigated by various researchers and scholars. Consider an n-tuple (positive) $y = (y_{1,}\ y_2,\ \ldots\ldots\ldots . y_n)$. The unweighted harmonic, arithmetic and geometric means of y are given by

$H_n = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{y_i}\right)}$ $\quad A_n = \frac{1}{n}\sum_{i=1}^{n} y_i$ $\quad G_n = (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}$ respectively. Assume that each $y_i \leq \frac{1}{2}$, $(i = 1, 2, \ldots\ldots . . n)$ an $y' = 1 - y = (1 - y_{1,}\ 1 - y_2\ \ldots\ldots\ldots 1 - y_n)$.Then the corresponding unweighted harmonic, arithmetic and geometric means of y' will be denoted by $H_n',\ A_n'$ and G_n' respectively.

$$H'_n = H'_n(1 - y_{1,}\ 1 - y_2\ \ldots\ldots\ldots 1 - \ y_n)$$
$$= \frac{n}{\sum_{i=1}^{n}(\frac{1}{1-y_i})}$$
$$A'_n = A'_n(1 - y_{1,}\ 1 - y_2\ \ldots\ldots\ldots 1 - y_n)$$
$$= \frac{1}{n}\sum_{i=1}^{n} 1 - y_i$$
$$G'_n = G'_n(1 - y_{1,}\ 1 - y_2\ \ldots\ldots\ldots 1 - y_n)$$
$$= (\prod_{i=1}^{n} 1 - y_i)^{\frac{1}{n}}$$

The classical Ky-Fan inequality reads as follows $\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n}$

A companion inequality to equation has been obtained by Wang and Wang and it states that

$$\frac{H_n}{H'_n} \leq \frac{G_n}{G'_n}$$

In the articles [2, 3, 5, 9, 10, 13, 14, 15, 16] authors discussed about Mathematical inequalities and in the articles [1, 4, 7, 17, 18], authors discussed on the Ky Fan inequalities and their refinements. In the papers and books the detailed information of Mathematical means and their importance are given in the book [19], interested researches are referred to [2, 6, 8, 11, 12].

Let $a \in R^+$, $a \neq 1$, the conjugate index of a is denoted by a' and is defined by $a' = \frac{a}{a-1}$ and $b' = \frac{b}{b-1}$. It is clear that for $a = 1, b = 1$, a', b' are not defined so we study for $a, b\ \mathcal{E}\ R^+ - \{1\}$. Further for $a, b \in (0,1), a'$and b' are negative and the mean definition does not hold. Therefore, we shall consider $a, b\ \in (1, \infty)$.

2. Definitions and propositions

In this section, we shall recall some definitions and properties which are essential to develop this paper.

Definition 2.1. For any $a, b\ \in (1, \infty)$, the $a' = \frac{a}{a-1}$ and $b' = \frac{b}{b-1}$ are the conjugates of a and b.

Definition 2.2. For any $a,b \in (1,\infty)$, the Arithmetic, Geometric, Harmonic and Contrahormonic means are respectively given by A= $\frac{a+b}{2}$, G = $\sqrt{ab}$, H= $\frac{2ab}{a+b}$, C = $\frac{a^2+b^2}{a+b}$ and Arithmetic, Geometric, Harmonic and Contraharmonic mean of conjugates of a,b are given by

$A' = \frac{a'+b'}{2} = \frac{2ab-(a+b)}{2(a-1)(b-1)}$,

$G' = \sqrt{a'b'} = \sqrt{\frac{ab}{(a-1)(b-1)}}$,

$H' = \frac{2a'b'}{a'+b'} = \frac{2ab}{a(b-1)+b(a-1)}$,

$$C' = \frac{a'^2 + b'^2}{a' + b'} = \frac{2a^2b^2 + a^2 + b^2 - 2a^2b - 2ab^2}{(2ab-a-b)(ab-a-b+1)}$$

3. Propositions

For each $a,b \in (1,\infty)$ we have

(3.1) $\frac{1}{a} + \frac{1}{a'} = 1$

(3.2) $(a')' = a$

(3.3) $a > 1 \Leftrightarrow a' > 1$

(3.4) $a > 0$ and $a' > 0 \Leftrightarrow a > 1$ and $a' > 1$

(3.5) $aa' = a + a'$

4. Results

In this section, we discuss some inequalities involving the arithmetic, geometric, harmonic, contraharmonic means and its conjugate index set.

Theorem 4.1. For the real numbers $a,b \geq 1$, the Harmonic mean(H), Arithmetic mean(A) and for the corresponding conjugate index arguments the Arithmetic mean (A') and the Harmonic mean (H') satisfy the inequalities.

1. $\frac{A}{A'} \leq \frac{H}{H'}$ if $a,b \in (1,2]$
2. $\frac{A}{A'} \geq \frac{H}{H'}$ if $a,b \in (2,\infty]$

Proof: Consider the Arithmetic Mean(A) and Harmonic Mean(H) and its conjugate index set. The conjugate index set

A= $\frac{a+b}{2}$ and $H = \frac{2ab}{a+b}$

$A' = \frac{a'+b'}{2} = \frac{2ab-a-b}{2(a-1)(b-1)}$

$H' = \frac{2a'b'}{a'+b'} = \frac{2ab}{a(b-1)+b(a-1)}$

Consider AH'-$A'H$

$= \frac{a+b}{2}\left\{\frac{2ab}{a(b-1)+b(a-1)}\right\} - \frac{2ab-a-b}{2(a-1)(b-1)}\left\{\frac{2ab}{a+b}\right\}$

$= ab\left\{\frac{a+b}{a(b-1)+b(a-1)} - \frac{2ab-a-b}{(a-1)(b-1)(a+b)}\right\}$

$= ab\left\{\frac{a+b}{2ab-a-b} - \frac{2ab-a-b}{(a-1)(b-1)(a+b)}\right\}$

$= ab\left\{\frac{(a+b)(a-1)(b-1)(a+b)-(2ab-a-b)^2}{(2ab-a-b)(a-1)(b-1)(a+b)}\right\}$

Consider the numerator of above equation

$(a+b)(a-1)(b-1)(a+b) - (2ab\ a-b)^2$

$= (a+b)^2\,(ab-a-b+1) - \{\,4a^2b^2 + (a+b)^2 - 4ab(a+b)\,\}$

$= (a+b)^2\,ab - (a+b)^3 + (a+b)^2\,4a^2b^2 - (a+b)^2 + 4ab(a+b)\,\}$

$= (a+b)^2\,ab - (a+b)^3 - 4a^2b^2 + 4ab(a+b)$

$= (a+b)^2\,\{ab-a-b\} - 4ab\{ab-a-b\}$

$= \{(a+b)^2 - 4ab\,\}\,\{ab-a-b\}$

$= (a-b)^2\{G^2 - 2A\,\}$

Case.1 For all $a,b \in (1,2]$ $(a-b)^2 > 0$ and

$G^2 - 2A = \frac{ab}{2} + \frac{ab}{2} - a - b = \frac{a}{2}(b-2) + \frac{b}{2}(a-2) \leq 0$ for all $a,b \in (1,2]$

Therefore $(a+b)(a-1)(b-1)(a+b) - (2ab-a-b)^2 \leq 0$
Denominator $(2ab-a-b) \geq 0$,
$(a-1)(b-1) > 0, \quad (b-1) > 0.$
Hence AH'-$A'H \leq 0$. Equivalently $\frac{A}{A'} \leq \frac{H}{H'}$

Case.2 $(a-b)^2 > 0$ and
$G^2 - 2A = \frac{ab}{2} + \frac{ab}{2} - a - b = \frac{a}{2}(b-2) + \frac{b}{2}(a-2 \geq 0$ for all $a, b \in (2, \infty]$
Therefore

$$(a+b)(a-1)(b-1)(a+b) - (2ab-a-b)^2 \geq 0$$

Denominator $(2ab-a-b) \geq 0$,
$(a-1)(b-1) > 0, \quad (b-1) > 0.$
Hence $AH' - A'H \geq 0$ or

$$\frac{A}{A'} \geq \frac{H}{H'}$$

The following figure 1, represents the graphical interpretation of theorem(4.1) for $a, b \in (1,2]$ and $a, b \in (2, \infty]$. The interval (1, 3) can be chosen
for clarity purpose even this interval can be extended further.

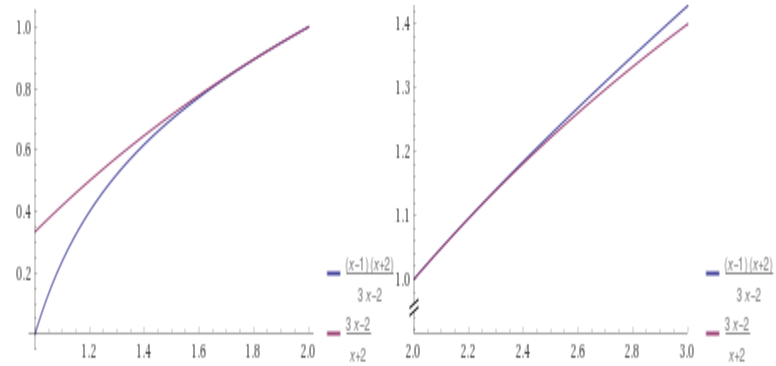

Figure 1. Graph of $\frac{A}{A'}$ and $\frac{H}{H\prime}$

Theorem 4.2. For the real numbers $a, b \geq 1$, the Arithmetic mean(A), Contraharmonic mean(C) and for the corresponding conjugate index arguments the Arithmetic mean (A') and the Contraharmonic (C') satisfy the inequalities .

1. $\frac{A}{A'} \geq \frac{C}{C'}$ if $a, b \in (1,2]$
2. $\frac{A}{A'} \leq \frac{C}{C'}$ if $a, b \in (2, \infty]$

Proof: Consider the Arithmetic Mean(A) and Contraharmonic Mean(C) and its conjugate index set. The conjugate index set A= $\frac{a+b}{2}$ and C= $\frac{a^2+b^2}{a+b}$

$A' = \frac{a'+b'}{2} = \frac{2ab-a-b}{2(a-1)(b-1)}$

$$C' = \frac{{a'}^2 + {b'}^2}{a' + b'} = \frac{a^2(b-1)^2 + b^2(a-1)^2}{(a-1)(b-1)(2ab-a-b)}$$

Consider AC'- $A'C$=

$= \frac{a+b}{2}\left\{\frac{a^2(b-1)^2+b^2(a-1)^2}{(a-1)(b-1)(2ab-a-b)}\right\} - \frac{2ab-a-b}{2(a-1)(b-1)}\left\{\frac{a^2+b^2}{a+b}\right\} = \frac{a+b}{2(a-1)(b-1)}\left\{\frac{a^2(b-1)^2+b^2(a-1)^2}{(2ab-a-b)} - (2ab-a-b)\frac{a^2+b^2}{(a+b)^2}\right\}$

$= \frac{a+b}{2(a-1)(b-1)}\left\{\frac{(a+b)^2[a^2(b-1)^2+b^2(a-1)^2]-(2ab-a-b)^2(a^2+b^2)}{(2ab-a-b)(a+b)^2}\right\}$

Consider $(a+b)^2[a^2(b-1)^2 + b^2(a-1)^2] - (2ab-a-b)^2(a^2+b^2)$
$= (a+b)^2[a^2b^2 + a^2 - 2a^2b + b^2a^2 + b^2 - 2b^2a]$- $[4a^2b^2(a+b)^2 - 4ab(a+b)](a^2+b^2)$
$= (a+b)^2[2a^2b^2 + a^2 - 2a^2b + b^2 - 2b^2a]$- $[4a^2b^2 + (a+b)^2 - 4ab(a+b)](a^2+b^2)$
$= (a+b)^2[2a^2b^2 - 2ab(a+b)] - [4a^2b^2 - 4ab(a+b)](a^2+b^2)$
$= (a+b)^2[2ab(ab-a-b)] - [4ab(ab-a-b)](a^2+b^2)$
$= [2ab(ab-a-b)][(a+b)^2 - 2(a^2+b^2)]$
$= [2ab(ab-a-b)](2ab-a^2-b^2)$
$= [2ab(2A-G^2)](a-b)^2$

Case.1 $(a-b)^2 > 0$ and

$(2A-G^2) = a+b-ab = a+b-\frac{ab}{2}-\frac{ab}{2}$

$= \frac{a}{2}(2-b)+\frac{b}{2}(2-a) \geq 0$ for all $a,b \in (1,2]$

Therefore $(2A-G^2)\,(a-b)^2 \geq 0$

Denominator $(2ab-a-b) \geq 0$,

$(a-1) > 0,\ (b-1) > 0$

Therefore AC'- $A'C \geq 0$ or $\frac{A}{A'} \geq \frac{C}{C'}$

Case.2 $(a-b)^2 > 0$ and $(2A-G^2) = a+b-ab = a+b-\frac{ab}{2}-\frac{ab}{2} = \frac{a}{2}(2-b)+\frac{b}{2}(2-a) \leq 0$ for all $a,b \in (2,\infty]$

Therefore $(2A-G^2)\ (a-b)^2 \leq 0$

Denominator $(2ab-a-b) \geq 0$,

$(a-1) > 0,\ (b-1) > 0$

Therefore AC'- $A'C \leq 0$ or $\frac{A}{A'} \leq \frac{C}{C'}$

Hence the proof of the theorem (4.2) completes

The following figure 2, shows the graph of $\frac{A}{A'}$, and $\frac{C}{C'}$ in the interval $a,b \in (1,2]$ and $a,b \in (2,\infty]$.

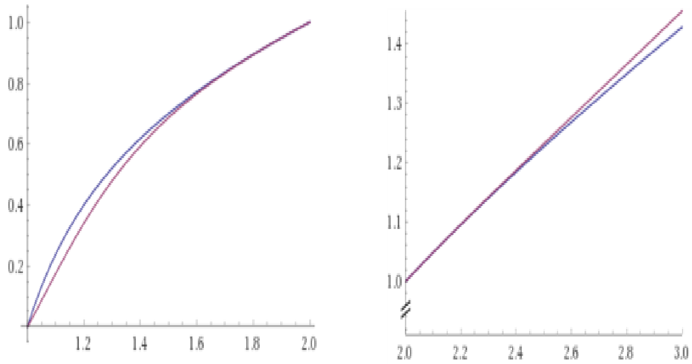

Figure 2. Graph of $\frac{A}{A'}$ and $\frac{C}{C'}$

Theorem.4.3. For the real numbers $a,b \geq 1$, the Harmonic mean(H), Geometric(G) and for the corresponding conjugate index arguments the Harmonic mean (H') and the Geometric (G') satisfy the inequalities.

1. $\frac{H}{H'} \geq \frac{G}{G'}$ if $a,b \in (1,2]$
2. $\frac{H}{H'} \leq \frac{G}{G'}$ if $a,b \in (2,\infty]$

Proof: Consider the Harmonic mean, Geometric mean and the conjugate index set

G$=\sqrt{ab}$ $\quad H = \frac{2ab}{a+b}$

$G' = \sqrt{a'b'} = \sqrt{\frac{ab}{(a-1)(b-1)}}$

$H' = \frac{2a'b'}{a'+b'} = \frac{2ab}{a(b-1)+b(a-1)}$

Consider $HG'-H'G$

$= \frac{2ab}{a+b}\sqrt{\frac{ab}{(a-1)(b-1)}} - \frac{2ab}{2ab-(a+b)}\sqrt{ab}$

$= 2ab\ \sqrt{ab}\ \left\{\frac{1}{(a+b)\sqrt{(a-1)(b-1)}} - \frac{1}{2ab-(a+b)}\right\}$

$= 2ab\sqrt{ab}\ \left\{\frac{(2ab-(a-b)) - (a+b)\sqrt{(a-1)(b-1)}}{(a+b)\ (2ab-(a+b))\sqrt{(a-1)(b-1)}}\right\}$

$= \frac{2ab\sqrt{ab}}{(a+b)(2ab-a-b)(\sqrt{(a-1}(b-1))}\ \{\ 2ab-(a+b)-(a+b)\sqrt{(a-1)(b-1)}$

Consider the numerator

$2ab-(a+b)-\ (a+b)\sqrt{(a-1)(b-1)} > 0$

$2ab-(a+b) >\ (a+b)\sqrt{(a-1)(b-1)}$

$(2ab-(a+b))^2 > [(a+b)\sqrt{(a-1)(b-1)}\ ]^2$

$$4a^2b^2 + (a+b)^2 - 4ab(a+b) > (a+b)^2(a-1)(b-1)$$
$$4a^2b^2 + (a+b)^2 - 4ab(a+b) > (a+b)^2ab - (a+b)^3 + (a+b)^2$$
$$4a^2b^2 - 4ab(a+b) > (a+b)^2ab - (a+b)^3$$
$$4ab(ab-a-b) > (a+b)^2(ab-a-b)$$
$$4ab(a+b-ab) < (a+b)^2(a+b-ab)$$
$$4ab < (a+b)^2$$
$$ab < (\frac{a+b}{2})^2$$

Therefore $G^2 < A^2$. Hence $\frac{H}{H'} \geq \frac{G}{G'}$ for $a, b \in (1, 2]$

Similarly

$$2ab - (a+b) - (a+b)\sqrt{(a-1)(b-1)} \leq 0$$

for $a, b \in (2, \infty]$ implies $G^2 > A^2$

Therefore $\frac{H}{H'} \leq \frac{G}{G'}$

Hence the proof of the theorem (4.3) completes.

The following figure 3, shows the graph of $\frac{H}{H'}$ and $\frac{G}{G'}$ in the interval (1, 2] and [2, 3).

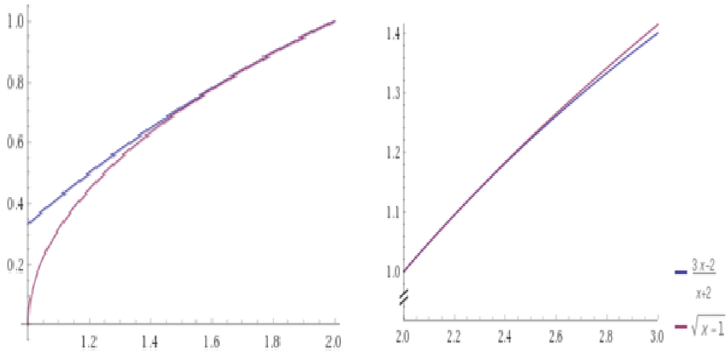

Figure 3. Graph of $\frac{H}{H\prime}$ and $\frac{G}{G'}$

Theorem.4.4. For the real numbers $a, b \geq 1$, the Arithmetic mean (A), Geometric mean (G) and for the corresponding conjugate index arguments the Arithmetic mean (A') and the Geometric mean (G') satisfy the inequalities.

1. $\frac{A}{A\prime} \leq \frac{G}{G}$ if $a, b \in (1,2]$
2. $\frac{A}{A\prime} \geq \frac{G}{G}$ if $a, b \in (2, \infty]$

Proof: Consider the Arithmetic mean, Geometric mean and the conjugate index set.

The conjugate index set

$$A' = \frac{a'+b'}{2} = \frac{2ab-a-b}{2(a-1)(b-1)}$$

$$G' = \sqrt{a'b'} = \sqrt{\frac{ab}{(a-1)(b-1)}}$$

Consider $AG' - A'G$

$$= \frac{a+b}{2}\sqrt{\frac{ab}{(a-1)(b-1)}} - \frac{2ab-a-b}{2(a-1)(b-1)}\sqrt{ab}$$

$$= \frac{\sqrt{ab}}{2}\left[\frac{a+b}{\sqrt{(a-1)(b-1)}} - \frac{2ab-a-b}{(a-1)(b-1)}\right]$$

Assume $\frac{a+b}{\sqrt{(a-1)(b-1)}} - \frac{2ab-a-b}{(a-1)(b-1)} < 0$

$$\frac{a+b}{\sqrt{(a-1)(b-1)}} < \frac{2ab-a-b}{(a-1)(b-1)}$$

Squaring on both sides

$$\frac{(a+b)^2}{(a-1)(b-1)} - \frac{(2ab-a-b)^2}{(a-1)^2(b-1)^2}$$
$$(a+b)^2(a-1)(b-1) < (2ab-a-b)^2$$
$$(a+b)^2(ab-a-b+1) < (2ab-a-b)^2$$
$$(a+b)^2(ab-a-b+1) < 4a^2b^2 + (a+b)^2 - 4ab(a+b)$$
$$(a+b)^2ab - (a+b)^3 + (a+b)^2 < 4a^2b^2 + (a+b)^2 - 4ab(a+b)$$

$(a+b)^2ab-(a+b)^3+(a+b)^2 < \; 4a^2b^2+(a+b)^2-4ab(a+b)$

$4ab(a+b)-4a^2b^2 < (a+b)^3-(a+b)^2ab$

$4ab(a+b-ab) < (a+b)^2\,(a+b-ab)$

$4ab < (a+b)^2$

$ab < \frac{(a+b)^2}{4}$ implies $G^2 < \; A^2$ for the $a, b \in (1, 2]$ and the denominator is greater than zero.

Therefore $AG' - A'G < 0$ or

$\frac{A}{A\prime} \leq \frac{G}{G}$

Similarly, if $\frac{a+b}{\sqrt{(a-1)(b-1)}} \geq \frac{2ab-a-b}{(a-1)(b-1)}$ then $G^2 \geq \; A^2$ for the interval $[2, \infty)$ the denominator is greater than zero.

Therefore $AG' - A'G \geq 0$ or $\frac{A}{A\prime} \geq \frac{G}{G}$

Hence the proof of the theorem (4.4) completes.

The following figure 4, shows the graph of $\frac{A}{A'}$ and $\frac{G}{G'}$ in the interval (1, 2] and [2, 3).

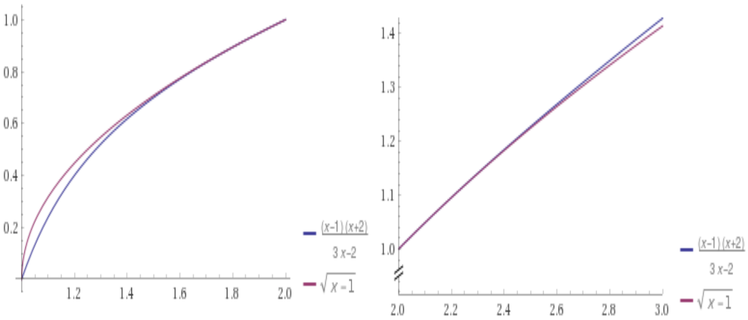

Figure 4. Graph of $\frac{A}{A\prime}$ and $\frac{G}{G'}$

5. Conclusion

Summarizing all the results in this work analogous to KY-Fan inequalities we get the inequalities involving the arithmetic, geometric, harmonic and contraharmonic means and their conjugates is obtained as given below.

$$\frac{H}{H'} > \frac{G}{G'} > \frac{A}{A'} > \frac{C}{C'}$$

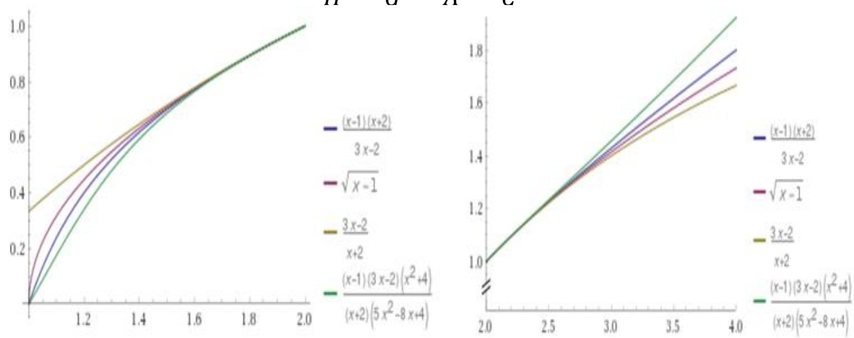

Figure 5. Graph of $\frac{A}{A\prime}$, $\frac{G}{G'}$ $\frac{H}{H\prime}$ and $\frac{C}{C\prime}$

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