Fuzzy Projective-Injective Modules and Their Evenness Over Semi-Simple Rings

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Abstract: In this paper, we investigate the fuzzy aspects of split exact sequences, projective-injective and semi simple modules, theoretical results connecting them are supported by appropriate examples. Remarkable equivalence conditions are established to unveil the evenness of fuzzy projective-injective modules over semi simple rings. Also, towards the conclusion we have developed a procedure to compute injective dimension of a fuzzy module based on the possible length of injective resolution and illustrate its working by means of an example.

Keywords: Fuzzy modules, injective dimensions, semi-simple fuzzy module, fuzzy projective module, fuzzy injective module

1. **Introduction and Review of Related Studies**

Projective and Injective Modules were first explored by (Cartan & Eilenberg 1956). Trailing the same, research done in the area resulted in topical concepts like pseudo projective-injective modules, psuedo semi-projective modules, small psuedo projective modules, M - injective modules and many more. In 1972 concept of finite Goldie dimension in modules (Goldie.A.W.1972) drew the attention of researchers like (Satyanarayana.B. et al.,2006) and (Yenumula & Satyanarayana 1987). Some work on projective dimension over various interesting rings for example Weyl algebra, polynomial ring and Laurent polynomial ring have been studied and analyzed in (Greuel & Pfister 2008), (Mishra.R.K. et al.,2011) and (Vargas.J.G.2003). From 1965 onwards various algebraic structures were fuzzified, when the critically vital concept "fuzzy" came into existence. Concepts like fuzzy projective – injective modules, fuzzy G-modules injectivity and quasi injectivity of fuzzy G-modules came into picture ensuing the above. Then (Michielsen.J. 2008) defined simple and semi-simple modules. And here in this paper we have studied and explored few of the concepts mentioned above in their fuzzy context and analyzed their relations with semi simple fuzzy modules and split exact sequences. Also, towards the end an interesting theorem is discussed showing the evenness of both fuzzy projective and injective modules over semi simple rings.

Also, the current study can be used to characterize global dimension of a ring in terms of projective-injective dimension of fuzzy modules, which can be extended from rings to internally graded rings. Authors also encourages the readers to study fuzzy modules over semi simple lie algebras which plays central role in many fields of mathematics and can give novel contributions in the field of differential geometry.

2.Significance of the study

In this paper apart from analysing the fuzzy version of split exact sequences, projective-injective and semi simple modules, their evenness is proved over semi simple rings as one of the substantial results of the paper, same is supported with suitable examples. Also, we have developed a procedure to calculate injective dimension of a fuzzy module and illustrated its working by means of an example. Study done here can further be extended to finite Gorenstein injective dimension and can be used to establish fuzzy version of Bass formula which associates the injective dimension to the depth of the module. Using this, global dimension of a ring in terms of projective-injective dimension of fuzzy modules can also be characterized, which can then be extended from rings to internally graded rings.

3.Objectives of the study

- To investigate and analyse the fuzzy aspects of split exact sequences, projective-injective and semi simple modules where in the theoretical results are supported by appropriate examples.
- To establish remarkable equivalence conditions to unveil the evenness of fuzzy projective-injective modules over semi simple rings.
- To calculate injective dimension of a fuzzy module and illustrated its working by means of an example.

4.Preliminaries

The basic definitions and elementary results used in the paper are detailed below.

Terminology and Symbols: Used throughout this paper:

- R is a ring with identity.
- Each module mentioned is a unitary R-module where ${}_{\mathrm{R}}M$ and M_{R} denote the left and right R – modules, respectively.
- μ_M means fuzzy module over the module M.
- pd(M) is projective dimension of the module M.
- id(M) is injective dimension of the module M.
- $\in$ Means belongs to.

Definition 4.1[(Kumar, Bhambri, Pratibha, 1995)**]:** A fuzzy subset μ_M is called fuzzy submodule of module M if the following conditions are satisfied:

(i) $\mu(m + n) \geq \{\mu(m), \mu(n)\}$ for all $m, n \in M$
(ii) $\mu(xm) \geq \mu(m)$ for all $m \in M$ and $x \in$ R
(iii) $\mu(-x) = \mu(x)$ for all $x \in M$
(iv) $\mu(0) = 1$

Definition 4.2[(Liu, 2014)**]:** For two fuzzy R-modules μ_A and ν_B. A function $\overline{f} : \mu_A \rightarrow \nu_B$ is called fuzzy R–homomorphism if:

(i) f is a R-homomorphism
(ii) $\nu(f(\mathrm{a})) \geq \mu(\mathrm{a})$ for all a $\in$ A.

Definition 4.3[(Liu, 2014)**]:** A fuzzy R-module μ_P is called projective if and only if for every surjective fuzzy R-homomorphism $\overline{f} : \mu_A \rightarrow \mu_B$ and for every fuzzy R-homomorphism $\overline{g}: \mu_P \rightarrow \mu_B$ there exist a fuzzy R-homomorphism $\bar{h}$:$\mu_P \rightarrow \mu_A$ such that figure 1

$$\begin{tikzcd} & \mu_P \arrow[dl, "\bar{h}"'] \arrow[d, "\bar{g}"] \\ \mu_A \arrow[r, "\bar{f}"'] & \mu_B \end{tikzcd}$$

Figure1. Fuzzy projective module

Commutes that is, $\overline{f}\,\bar{h} = \overline{g}$.

Definition 4.4[(Liu, 2014)**]:** A fuzzy R-module μ_P is called injective if and only if for every injective fuzzy R-homomorphism $\overline{f}$: $\mu_B \to \mu_A$ and for every fuzzy R-homomorphism $\overline{g}$: $\mu_A \to \mu_P$ there exist a fuzzy R-homomorphism $\bar{h}$: $\mu_B \to \mu_P$ such that figure 2

$$\begin{tikzcd} & \mu_P \\ \mu_B \arrow[ur, "\bar{h}"] \arrow[r, "\bar{f}"'] & \mu_A \arrow[u, "\bar{g}"'] \end{tikzcd}$$

Figure2. Fuzzy injective module

Commutes that is, $\bar{g}\bar{f}$ = $\bar{h}$.

Definition 4.5[(Pan, 1987)**]:** A fuzzy R-homomorphism $\bar{f}$ ∈ Hom (μ_A, ν_B) is called fuzzy split if there exist some
$\bar{g}$ ∈ Hom (ν_B, μ_A) such that $\bar{f}\bar{g}$ =$\bar{1}_B$.

Definition 4.6[(Zahedi, Ameri, 1995)**]:** The sequence…. → $\mu_{n-1_{\lambda_{n-1}}} \xrightarrow{\overline{f_{n-1}}} \mu_{n_{\lambda_n}} \xrightarrow{\overline{f_n}} \mu_{n+1_{\lambda_{n+1}}} \to \cdots$ of R-fuzzy module homomorphism is said to be fuzzy exact if and only if Im$\bar{f}_{n-1}$= Ker$\overline{f_n}$ for all n, where Im$\bar{f}_{n-1}$ and Ker$\overline{f_n}$ isμ_n| Imf_{n-1} and μ_n | kerf_n which means $\mu_n$is restricted to image and kernel respectively.

Definition 4.7[(Zahedi, Ameri, 1995)**]:** A fuzzy exact sequence of the form $0 \to \mu_A \overset{f}{\to} \eta_B \overset{g}{\to} \nu_C \to 0$ is called fuzzy short exact sequence.

Definition 4.8[(Isaac, 2004)**]:**$\mu_M$is said to be simple fuzzy left module if it has no proper sub modules.

Definition 4.9[(Isaac, 2004)**]:** $\mu_M$is said to be semi-simple fuzzy left module if whenever for ν_N, a strictly proper fuzzy submodule of $\mu_M$there exist a strictly proper fuzzy submodule $\eta_P$of μ_M such that μ_M= $\nu_N \oplus \eta_P$.

NOTE: A ring is said to be semi-simple if, every left-module over it is semi-simple.

Definition 4.10[(Liu, 2014)**]:** If M is R-module, 0_M represents the fuzzy R-module 0 : $M \to [0, 1]$ satisfyin

$$0(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

Definition 4.11[(Satyanarayana, Godloza., Mohiddin, 2004)**]:** Let M is a module and $\mu_M$is a fuzzy submodule of M. Let x_1, x_2,….,x_n ∈ M are said to be fuzzy linearly independent with respect to $\mu_M$if it satisfies the following two conditions:

(i) x_1, x_2,….,x_n are linearly independent and
(ii) $\mu(y_1, y_2,....,y_n)$ = min{$\mu(y_1)$ …., $\mu(y_n)$} for any $\mu(y_i)$ ∈ Rx_i, $1 \leq i \leq n$.

Definition 4.12[(Satyanarayana, Godloza., Mohiddin, 2004)**]:** Let $\mu_M$is a fuzzy submodule of M. A subset B of M is said to be a fuzzy pseudo basis for μ_M if B is a maximal subset of M such that x_1, x_2,….,$x_k$are fuzzy linearly independent for any finite subset {x_1, x_2,….,x_k} of B.

Definition 4.13[(Satyanarayana, Godloza., Mohiddin, 2004)**]:** The fuzzy pseudo basis of μ is called as fuzzy basis for M.

Definition 4.14[(Pan, Fu-Zheng1988)**]:** For an arbitrary fuzzy linear map $\overline{f}$: $\mu_M \to \nu_A$, $\mu_S$where S = {m ∈ M : $\nu(f(m))$ = 1} is called the kernel of $\bar{f}$.

Lemma 4.15[(Mishra, Kumar, Behara, 2011)**]:** Let M_1, M_2, M_3 be R-modules and
$0 \to M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \to 0$ be a split short exact sequence. Suppose$\beta_1$and $\beta_2$is the splitting corresponding to α_1 and α_2 respectively. Then the following sequence is an exact sequence$0 \to M_3 \xrightarrow{\beta_2} M_2 \xrightarrow{\beta_1} M_1 \to 0$.

Example 4.16[(Zahedi, Ameri, 1995] [Example 2.6]**:** Let $\bar{f}$: $\mu_M \to \eta_N$ be a fuzzy homomorphism. Then the fuzzy sequence $0 \to ker\bar{f} \xrightarrow{\bar{\iota}} \mu_M \xrightarrow{\bar{f}} \eta_N \xrightarrow{\bar{g}} coker\bar{f} \to 0$ is exact where $\bar{\iota}$ is inclusion map and $\bar{g}$ is canonical map.
Lemma 4.17 [(Zahedi, Ameri, 1995)] [Lemma 2.11]**:** Let $\overline{g}'$:$\eta_C \to \rho_B$ be a fuzzy splitting for the fuzzy short exact sequence $0 \to \mu_A \xrightarrow{\bar{f}} \rho_B \xrightarrow{\bar{g}} \eta_C \to 0$ of fuzzy R-modules. Then $\rho_B \cong \mu_A \oplus \eta_C$.
Theorem 4.18[(Zahedi, Ameri, 1995)] [Theorem 3.8]**:** Let μ_I is fuzzy R- module then the following conditions are equivalent:

(i) μ_I is injective
(ii) For each fuzzy short exact sequence $\overline{0} \to \rho_A \xrightarrow{\bar{f}} \eta_B \xrightarrow{\bar{g}} \theta_C \to \overline{0}$
The induced sequence $\overline{0} \to \mathrm{hom_R}(\theta_C,\ \mu_I) \xrightarrow{\bar{g}_*} \mathrm{hom_R}(\eta_B,\ \mu_I) \xrightarrow{\bar{f}_*} \mathrm{hom_R}(\rho_A,\ \mu_I) \to \overline{0}$ is exact.
(iii) If $\bar{\beta}$: $\mu_I \to \rho_B$ is a fuzzy epimorphism then there exists a fuzzy homomorphism $\bar{\alpha}$: $\rho_B \to \mu_I$ such that $\bar{\alpha}\bar{\beta}$ = $\overline{1}_I$
(iv) μ_I is a fuzzy direct summund in every fuzzy module which contains μ_I as a submodule.

Theorem 4.19[(Isaac, 2004)**]:** Let L be a complete distributive lattice. Let μ_M be a fuzzy left module then the following are equivalent:

(i) μ_M is semi simple
(ii) μ_M is a sum of a family of strictly proper simple fuzzy submodules μ_{i_M} of μ_M.
(iii) μ_M is a direct sum of a family of strictly proper simple fuzzy submodules μ_{i_M} of μ_M.

Theorem 4.20[(Zahedi, Ameri, 1995)] [Theorem 3.3]**:** Let η_i be the family of fuzzy R-modules. Then the direct product $\Pi\eta_i$ is injective in the category fuzzy R-modules if and only if each η_i is injective.
Theorem 4.21[(Isaac, 2004)] [Theorem 5.2.4]**:** Every free L-module is a projective L-module.

5.Fuzzy projective module

Let R be a ring and μ_M be a fuzzy finitely generated R-module. A fuzzy exact sequence
$\ldots . \to \mu_{i+1} \xrightarrow{\overline{f_{i+1}}} \mu_i \ldots . \mu_1 \xrightarrow{\overline{f_1}} \mu_0 \xrightarrow{\overline{f_0}} \mu_M \to 0$ with only fuzzy free (resp. projective) modules μ_i $\{i$ = 0, 1, 2...} is called a free (resp. projective) resolution of μ_M. The minimum length of which, is called as projective dimension of μ_M.
Lemma 5.1: Let θ_P is fuzzy R-module then the following conditions are equivalent:

(i) θ_P is fuzzy projective
(ii) Any short exact sequence of the form $\overline{0} \to \mu_A \xrightarrow{\bar{f}} \mu_B \xrightarrow{\bar{g}} \theta_P \to \overline{0}$ splits.
(iii) θ_P is a direct summund of a free fuzzy R-module.

Proof: (i)⇒(ii): Let $\overline{0} \to \mu_A \xrightarrow{\bar{f}} \mu_B \xrightarrow{\bar{g}} \theta_P \to \overline{0}$ be an exact sequence, since θ_P is fuzzy projective we have $\overline{g}'$: $\theta_P \to \mu_B$ in figure 3 such that it commutes.

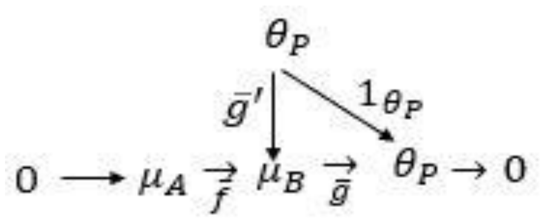

Figure3. Commutativity is shown forthefuzzy projective module θ_P
Then $\overline{g}\bar{g}'$ = 1_{θ_P} thus, the short exact sequence mentioned splits.
(ii)⇒(iii): Since any fuzzy submodule is the homomorphic image of free fuzzy submodule, therefore, we have short exact sequence $\overline{0} \to \theta'_P \xrightarrow{\bar{f}} \nu_B \xrightarrow{\bar{g}} \theta_P \to \overline{0}$ where ν_B is the free fuzzy submodule. If θ_P satisfy (ii) then the sequence splits and hence $\nu_B \cong \theta_P \oplus \theta'_P$ by Lemma 4.17.

(iii)⇒(i): $\bar{0} \to \theta'_P \xrightarrow{\bar{f}} \nu_B \xrightarrow{\bar{g}} \theta_P \to \bar{0}$ where ν_B is free. Now let us consider figure 4

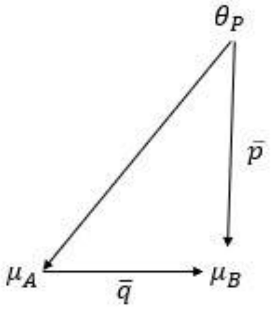

Figure4. For showing the commutativity of the sequence in (ii)

Combining the given sequence and figure 4 we obtain figure 5

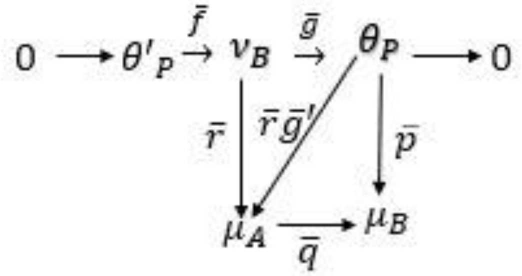

Figure5. Combination of figure 4 and short exact sequence in (ii)

and since the sequence splits, we have $\bar{g}'$: $\theta_P \to \nu_B$ such that $\bar{g}\bar{g}' = 1_{\theta_P}$.

Now since ν_B is free, it is projective by theorem 4.21, we can now have $\bar{r}$: $\nu_B \to \mu_A$ which implies

$\bar{p}\bar{g} = \bar{q}\bar{r}$. Consider $\bar{p} = \bar{p}\ 1_{\theta_P}$

$= \bar{p}\bar{g}\bar{g}'$

$= \bar{q}(\bar{r}\bar{g}')$, where $\bar{r}\bar{g}'$: $\theta_P \to \mu_A$. Hence θ_P is fuzzy projective.

Theorem 5.2: If μ_M is fuzzy projective then the fuzzy module $\mu'_{M/N}$ defined on its quotient module M/N is also fuzzy projective.

Proof: Letif μ_M is fuzzy projective then figure 6 commutes:

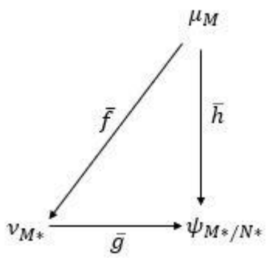

Figure6. Showing the commutativity of fuzzy module μ_M

Here $\bar{f}$, $\bar{g}$ and $\bar{h}$ are defined as $\bar{f}[\mu(m)] = \nu(m*)$, $\bar{g}[\nu(m*)] = \psi(m*+N*)$ and $\bar{h}[\mu(m)] = \psi(m*+N*)$ for all $m \in M$ and $m* \in M*$ respectively. Now, for showing $\mu'_{M/N}$ to be fuzzy projective we need to show $\bar{g}\bar{f}' = \bar{h}'$. Thus, considering figure 7 for the same.

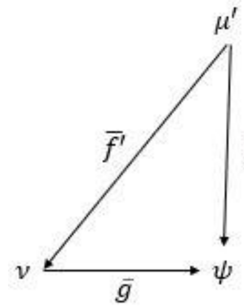

Figure7. Showing the commutativity of fuzzy module$\mu'_{M/N}$

Whereμ', $\bar{f}'$ and $\bar{h}'$ are defined as $\mu'(m+N) = \mu(m)$, $\bar{f}'[\mu'(m+N)] = \nu(m^*)$ or$\bar{f}'[\mu(m)]= \nu(m^*)$ and $\bar{h}'[\mu'((m+N)] = \psi(m^*+N^*)$ or $\bar{h}'[\mu(m)] = \psi(m^*+N^*)$. Consider, $\bar{g}[\bar{f}'(\mu(m))]$

$= \bar{g}[\nu(m^*)]$

$= \psi(m^*+N^*)$

$= \bar{h}'[\mu(m)]$ hence the result.

Example 5.3: Let μ, ν and ψ be the fuzzy modules defined on Q$\sqrt{2}$, Q$\sqrt{3}$ and Q$\sqrt{3}$/Q respectively as

$$\mu(a+b\sqrt{2}) = \begin{cases} 1, \text{ if } a, b = 0 \\ 1/5, \text{ if } a \neq 0, b = 0 \\ 1/6, \text{ if } b \neq 0 \end{cases}$$

$$\nu(a+b\sqrt{3}) = \begin{cases} 1, \text{ if } a, b = 0 \\ 1/2, \text{ if } a \neq 0, b = 0 \\ 1/3, \text{ if } b \neq 0 \end{cases}$$

and $\psi[(a+b\sqrt{3})+Q] = \nu(a+b\sqrt{3})$ for all a, b $\in$ Q. Let μbe fuzzy projective then figure 8 commutes,

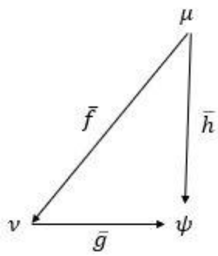

Figure8. Showing μ is fuzzy projective

where $\bar{f}$, $\bar{g}$ and $\bar{h}$ are $\bar{f}[\mu(a+b\sqrt{2})] = \nu(a+b\sqrt{3})$, $\bar{g}[\nu(a+b\sqrt{3})] = [\psi(a+b\sqrt{3})+Q]$ and $\bar{h}[\mu(a+b\sqrt{2})] = [\psi(a+b\sqrt{3})+Q]$ respectively. Then to show μ' defined on Q$\sqrt{2}$/Q as $[\mu'(a+b\sqrt{2})+Q] = \mu(a+b\sqrt{2})$ to be fuzzy projective we need to show figure 9 commutes:

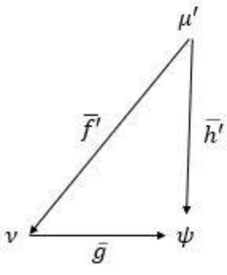

Figure9. For showing the commutativity of μ'

Here$\bar{f}'[\mu'(a+b\sqrt{2})+Q] = \nu(a+b\sqrt{3})\}$ and $\bar{h}'[\mu'(a+b\sqrt{2})+Q] = [\psi(a+b\sqrt{3})+Q]$. Now

$\bar{g}\bar{f}'[\mu'(2+6\sqrt{2})$ +Q] = $\bar{g}[\nu(2+6\sqrt{3})]$
= $[\psi(2+6\sqrt{3})$+Q]
= $\bar{h}'[\mu'(2+6\sqrt{2})$+Q]. Hence, the result.

Example 5.4: Consider, the two fuzzy projective modules of example 5.3. Define a map say $\bar{\phi}$: $\mu_M \rightarrow \mu'_{M'}$ where M = Q$\sqrt{2}$ and M' = Q$\sqrt{2}$/Q. Then $ker\bar{\phi}$ = [(a+b$\sqrt{2}$) $\in$ M $|\mu'(\phi(m))$ = 1] = [(a+b$\sqrt{2}$) $\in M$ $|\mu'$((a+b)$\sqrt{2}$+Q) = 1] = [(a+b$\sqrt{2}$) $\in$ M | μ(a+b$\sqrt{2}$) = 1]. Which means there is a single element, ``zero" in $ker\bar{\phi}$ [definition of μ in example 5.3]. Thus, the following chain stops and the projective dimension in this specific example is zero.

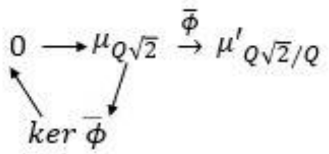

Figure10. Projective dimension using two fuzzy projective modules

Remark: From example 5.3 we have M' = Q$\sqrt{2}$/Q. Any subset B = {1, $\sqrt{2}$} of M' is the fuzzy pseudo basis for $\mu_{M'}$ since B is the maximal subset of M'. Also

(i) {1, $\sqrt{2}$} is the linearly independent and
(ii) $\mu(1+\sqrt{2})$= *min* [$\mu(1)$, $\mu(\sqrt{2})$]
= *min* [1/5, 1/6]
= *min* [0.2, 0.1]

which are equal by the definition of μ. Thus, by 4.12 and 4.13 we can say that {1, $\sqrt{2}$} is the fuzzy pseudo basis of $\mu_{M'}$ or fuzzy basis of Q$\sqrt{2}$/Q.

6.Fuzzy injective module

Let R be a ring and μ_M be a fuzzy finitely generated R-module. A fuzzy exact sequence of the form$\ldots\rightarrow \mu_{i+1} \xrightarrow{\overline{f_{i+1}}} \mu_i \ldots \mu_1 \xrightarrow{\overline{f_1}} \mu_0 \xrightarrow{\overline{f_0}} \mu_M \rightarrow 0$ with only fuzzy injective modulesμ_i {i = 0, 1, 2...} is called a injective resolution of μ_M. The minimum length of the same is termed as injective dimension ofμ_M.

Theorem 6.1: If μ_M is fuzzy injective then the fuzzy module $\mu'_{M/N}$ defined on its quotient module M/N is also fuzzy injective.

Proof: The proof is same as that of theorem 5.2 and can easily be proved by using the definition of fuzzy injective module and imposing suitable modifications.

Example 6.2: Let μ, ν and η be the fuzzy modules defined on Q$\sqrt{3}$, Q$\sqrt{2}$ and Q$\sqrt{2}$/Q as

$$\mu(a+b\sqrt{3}) = \begin{cases} 1, \text{ if a, b} = 0 \\ 1/2, \text{ if a} \neq 0, \text{ b} = 0 \\ 1/3, \text{ if b} \neq 0 \end{cases}$$

$$\nu(a+b\sqrt{2}) = \begin{cases} 1/4, \text{ if a, b} = 0 \\ 1/5, \text{ if a} \neq 0, \text{ b} = 0 \\ 1/6, \text{ if b} \neq 0 \end{cases}$$

and η[(a+b$\sqrt{2}$)+Q] = ν(a+b$\sqrt{2}$) respectively. Let μ be fuzzy injective then figure 11 commutes:

$$\begin{tikzcd} & \mu \\ \nu \arrow[ur, "\bar{g}"] \arrow[r, "\bar{f}"'] & \eta \arrow[u, "\bar{h}"'] \end{tikzcd}$$

Figure11. Showcasing thefuzzy injectivity of μ

where $\bar{f}$, $\bar{h}$and $\bar{g}$ are $\bar{f}[\nu(a+b\sqrt{2})] = \eta[(a+b\sqrt{2})+Q] = \nu(a+b\sqrt{2})$, $\bar{g}[\nu(a+b\sqrt{2})] = \mu(a+b\sqrt{3})$ and $\bar{h}[\eta[(a+b\sqrt{2})+Q] = \mu(a+b\sqrt{3})$ respectively. Then for showing $\mu^{'}$ defined on $Q\sqrt{3}/Q$ as $[\mu^{'}(a+b\sqrt{3})+Q] = \mu(a+b\sqrt{3})$ to be fuzzy injective, we need to show commutativity in figure 12:

$$\begin{tikzcd} & \mu' \\ \nu \arrow[ur, "\bar{g'}"] \arrow[r, "\bar{f}"'] & \eta \arrow[u, "\bar{h'}"'] \end{tikzcd}$$

Figure12. For showing the commutativity of $\mu^{'}$

Here$\bar{g}'[\nu(a+b\sqrt{2})] = \mu'[(a+b\sqrt{3})+Q]$ and $\bar{h}'[\eta(a+b\sqrt{2})+Q] = \mu'[(a+b\sqrt{3})+Q]$.
Consider $\bar{h}'\bar{f}[\nu(6+2\sqrt{2})] = \bar{h}'[\eta(6+2\sqrt{2}) +Q]$
$= \mu'(6+2\sqrt{3})+Q$
$= \mu(6+2\sqrt{3})$. Also, $\bar{g}'[\nu(6+2\sqrt{2})] = [\mu'(6+2\sqrt{3})+Q] = \mu(6+2\sqrt{3})$ and hence the result.

Lemma 6.3: Let $\theta_P$is fuzzy R- module then the following conditions are equivalent:
(i) θ_P is fuzzy injective
(ii) Any short exact sequence of the form $\bar{0} \to \theta_P \overset{\bar{f}}{\to} \mu_B \overset{\bar{g}}{\to} \mu_A \to \bar{0}$ splits.
(iii) θ_P is a direct summund of a fuzzy R-module of which it is a submodule.

Proof: (i) ⇒ (ii): Let $\bar{0} \to \theta_P \overset{\bar{f}}{\to} \mu_B \overset{\bar{g}}{\to} \mu_A \to \bar{0}$ be an exact sequence and since θ_P is fuzzy injective there exist $\overline{f'}$: $\mu_B \to \theta_P$ such that figure 13 commutes.

$$\begin{tikzcd} \bar{0} \arrow[r] & \theta_P \arrow[r, "\bar{f}"] \arrow[dr, "1_{\theta_P}"'] & \mu_B \arrow[r, "\bar{g}"] \arrow[d, "\overline{f'}"] & \mu_A \arrow[r] & \bar{0} \\ & & \theta_P & & \end{tikzcd}$$

Figure13. Using equation in (ii), showing the fuzzy injectivity of θ_P

Thus$\bar{f}\overline{f'} = 1_{\theta_P}$that is, the short exact sequence mentioned in part (ii) splits.

(ii) ⇒ (iii): Let θ_P be a fuzzy submodule of say μ_B and$\overline{f}$: $\theta_P \to \mu_B$be a fuzzy homomorphism. Since the sequence mentioned in (part (ii)) splits, by example 2.16 we have the sequence $\theta_P \overset{\bar{f}}{\to} \mu_B \overset{\bar{g}}{\to} coker\bar{f}$ as short exact. Thus, by lemma 4.17 we have $\mu_B \cong \theta_P \oplus coker\bar{f}$.

(iii) ⇒ (i): can trivially be derived from theorem 4.18.

Lemma 6.4: Let $\mu_1, \eta,\ \mu_2$ be the fuzzy R- modules over M_1, M_2, M_3respectively and
$\bar{0} \to \mu_1 \xrightarrow{\overline{\alpha_1}} \eta \xrightarrow{\overline{\alpha_2}} \mu_2 \to \bar{0}$. Be a fuzzy split short exact sequence. Suppose $\bar{\beta}_1$ and $\bar{\beta}_2$are the fuzzy splitting corresponding to$\bar{\alpha}_1$and $\bar{\alpha}_2$ respectively, then the following sequence is fuzzy exact sequence
$\bar{0} \to \mu_2 \xrightarrow{\overline{\beta_2}} \eta \xrightarrow{\overline{\beta_1}} \mu_1 \to \bar{0}$.   .

Proof: For the above we need to prove Im$\bar{\beta}_2$= ker$\bar{\beta}_1$. Since Im$\bar{\beta}_2$ = η|Im$\beta_2(x)$ which is equal to $\eta(x)$ for all $x \in$ Imβ_2. Also, ker$\bar{\beta}_1$= η| ker$\beta_1(y)$ which equals $\eta(y)$ for all $y \in$ kerβ_1. Then from Lemma 4.15, we have kerβ_1= Imβ_2. Therefore, Im$\bar{\beta}_2$= ker$\bar{\beta}_1$implying given sequence is fuzzy exact.

6.5.Procedure for fuzzy injective dimension

STEP 1- For a given fuzzy injective module η_P choose a fuzzy injective module $\mu_A$and define a fuzzy monomorphism map say from $\bar{\phi}$:$\eta_P \ \rightarrow \mu_A$such that the following is a fuzzy exact sequence$\overline{0} \rightarrow \eta_P \xrightarrow{\bar{\phi}} \mu_A \xrightarrow{\bar{\psi}} co\ ker\ \overline{\phi} \rightarrow \overline{0}$ where $coker\ \overline{\phi}$= $\mu_A/$Im$\bar{\phi}$

STEP 2- If $coker\ \overline{\phi} \neq 0$ then embed it to a new fuzzy injective module generated by the number of elements in $coker\ \overline{\phi}$ say μ_B. Define a map say $\overline{\phi_1}$: $coker\ \overline{\phi} \rightarrow \mu_B$implies the following is a fuzzy exact sequence $\overline{0} \rightarrow \eta_P \xrightarrow{\bar{\phi}} \mu_A \xrightarrow{\bar{\psi}} co\ ker\ \overline{\phi} \xrightarrow{\bar{\phi}_1} \mu_B \xrightarrow{\overline{\psi_1}} co\ ker\ \overline{\phi_1} \rightarrow \overline{0}$. Similarly, again if $coker\ \overline{\phi_1} \neq 0$ we have $\overline{0} \rightarrow \eta_P \xrightarrow{\bar{\phi}} \mu_A \xrightarrow{\bar{\psi}} co\ ker\ \overline{\phi} \xrightarrow{\bar{\phi}_1} \mu_B \xrightarrow{\overline{\psi_1}} co\ ker\ \overline{\phi_1} \xrightarrow{\bar{\phi}_2} \mu_C \xrightarrow{\overline{\psi_2}} co\ ker\ \overline{\phi_2} \rightarrow \overline{0}$ as fuzzy exact sequence. Continuing the steps, we have

$\overline{0} \rightarrow \eta_P \xrightarrow{\overline{\alpha_0}} \mu_A \xrightarrow{\overline{\alpha_1}} \mu_B \ldots\ldots \xrightarrow{\overline{\alpha_{k-1}}} \mu_{k-1} \xrightarrow{\overline{\alpha_k}} \mu_k \rightarrow \overline{0}$ ……. (1)

STEP3- For k = 1 we have $\overline{0} \rightarrow \eta_P \xrightarrow{\overline{\alpha_0}} \mu_A \xrightarrow{\overline{\alpha_1}} \mu_B \rightarrow \overline{0}$ .... (2)

Since η_P is fuzzy injective module, equation (2) splits by lemma 6.3. Thus, there exists $\overline{\beta}_0$:$\mu_A \rightarrow \eta_P$ such that $\bar{\beta}_0\bar{\alpha}_0 = I_{\eta_P}$.

STEP 4- From Lemma 6.4 the fuzzy exact sequence in equation (2) splits, giving rise to another exact sequence $\overline{0} \rightarrow \mu_B \xrightarrow{\overline{\beta_1}} \mu_A \xrightarrow{\overline{\beta_0}} \eta_P \rightarrow \overline{0}$ implying $\mu_B \oplus \eta_P = \mu_A$ by [Lemma 4.17].

Since$\overline{\alpha_0}$: $\eta_P \rightarrow \mu_A$ is a fuzzy monomorphism we have a fuzzy exact sequence

$\overline{0} \rightarrow \eta_P \xrightarrow{\overline{\alpha_0}} \mu_A \xrightarrow{\bar{\pi}} co\ ker\ \overline{\alpha_0} \rightarrow \overline{0}$ …….......... (3).

Here η_P is fuzzy injective so the above equation (3) splits, implying $coker\bar{\alpha}_0$ = Im$\bar{\pi}$ is fuzzy injective and hence id(η_P) = 0.

STEP 5- Suppose η_P is not fuzzy injective then equation (3) does not split. Therefore $coker\bar{\alpha}_0$ = Im$\bar{\pi}$ is not fuzzy injective, continuing in this way all Im$\bar{\pi}$, Im$\bar{\pi}_1$, Im$\bar{\pi}_2$,.......,Im$\bar{\pi}_{k-1}$ are not injective but Im$\bar{\pi}_{k-1}$ will be injective. Thus, after finite number of steps id(η_P) = k.

Example 6.6: Defines a map $\bar{\phi}$ between two fuzzy injective modules $\mu_M$and $\mu'_{M/N}$ of example 6.2. And since, this map is surjective in nature, there will be only single element "zero" in the $cokernel$ of $\bar{\phi}$. Because of which the chain in figure 14 stops and the injection dimension of this specific example will zero.

$$\begin{tikzcd}
\overline{0} \arrow[r] & \mu_M \arrow[r, "\bar{\phi}"] & \mu'_{M/N} \arrow[r] \arrow[dr] & \overline{0} \\
 & & & coker\ \overline{\phi} \arrow[u]
\end{tikzcd}$$

Figure14. Injection dimension using two fuzzy injective modules

7.Equivalence of fuzzy projective-injective modules

In this section along with an example of semi-simple fuzzy module, we extend the concept of equivalence of projective and injective modules over semi-simple rings to fuzzy settings.

Example 7.1: Let M = Q$\sqrt{2}$ over Q. Then M is semi-simple R-module. Here M = Q$\sqrt{2}$ = Q $\oplus$ $\sqrt{2}$Q. Let μ be same as was in example 5.3, that is

$$\mu(a+b\sqrt{2}) = \begin{cases} 1, & \text{if a, b = 0} \\ 1/5, & \text{if a} \neq 0\text{, b = 0} \\ 1/6, & \text{if b} \neq 0 \end{cases}$$

Let $\mu_1$be defined over Q as

$$\mu_1(x) = \begin{cases} 1, & \text{if } x = 0 \end{cases}$$

1/5, if $x \neq 0$
And μ_2 over Q as

$$\mu_2(x) = \begin{cases} 1, \text{ if } x = 0 \\ 1/5, \text{ if } x \neq 0 \end{cases}$$

Then both $\mu_1$and μ_2 are fuzzy modules over Q and $\sqrt{2}$Q respectively. Also $\mu = \mu_1 \oplus \mu_2$ thus, μ is a semi-simple R-module over M.

NOTE: The following theorem shows the equivalence of fuzzy projective and fuzzy injective modules.

Theorem 7.2: For each ring R following properties are equivalent:
(i) R is semi-simple as a fuzzy left R- module
(ii) Every fuzzy left ideal of R is a direct summund of R
(iii) Every fuzzy left ideal of R is fuzzy injective
(iv) All fuzzy left modules over R are semi-simple
(v) All fuzzy exact sequences $\overline{0} \to \mu \to \nu \to \eta \to \overline{0}$ of fuzzy left R-modules split
(vi) All the fuzzy left R- modules are fuzzy projective
(vii) All the fuzzy left R- modules are fuzzy injective

Proof: Equivalence of (i) - (ii) and (iv) - (v) can trivially be derived from lemma 4.15. (v) - (vi), (v) - (vii) are equivalent from lemma 5.1 and 6.3. Thus, (iv) - (vii) are equivalent. The implication (vii) ⇒ (iii) is obvious. If the fuzzy left ideal is fuzzy injective then by lemma 6.3 it is a direct summund giving (iii) ⇒ (ii). Finally, since we know every fuzzy ideal is fuzzy R-module (ii) ⇒ (vii) is a direct implication of (iv) ⇒ (i) of theorem 4.18.

Example 7.3: Let μbe a semi simple fuzzy module defined in example 7.1. For it to be fuzzy projective we need to show figure 15 commutes:

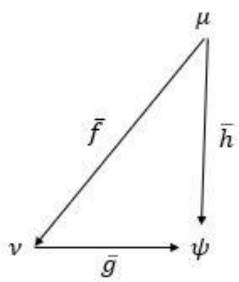

Figure15. For proving thefuzzy projectivity of μ

Where $\nu$and ψ are same as were in example 5.3. Consider,$\bar{g}\bar{f}[\mu(\text{a+b}\sqrt{2})] = \bar{g}[\nu(\text{a+b}\sqrt{3})]$
$= \psi[(\text{a+b}\sqrt{3})\text{+Q}] = \bar{h}[\mu(\text{a+b}\sqrt{2})]$. Thus, it is fuzzy projective. Also, for it to be fuzzy injective figure 16 must commute:

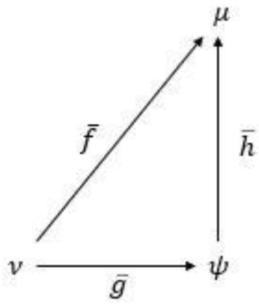

Figure16. For proving thefuzzy injectivity of μ

we have$\bar{h}\bar{g}[\nu(\text{a+b}\sqrt{3})] = \bar{h}\bar{\psi}[(\text{a+b}\sqrt{3})\text{+Q}] = \mu(\text{a+b}\sqrt{2}) = \bar{f}[\nu(\text{a+b}\sqrt{3})]$.

9. Recommendations

The present work can further be extended to fuzzy version of Gorenstein injective modules and Ext finite modules having finite cosyzygy and finite Gorenstein injective dimension. One can always use local co-homology to study the fuzzy Gorenstein injective modules over Gorenstein rings, which when developed over local rings can be useful to prove Auslander-Bridger formula. Also, this finite Gorenstein injective dimension can be used to establish fuzzy version of Bass formula which associates the injective dimension to the depth of the module.

8.Conclusions

Our contributions to the current research work are:

(i) We proved the equivalence of fuzzy projective and injective modules over semi simple rings.

(ii) We have discussed few basic analogues of theorems on fuzzy projective and injective modules, motivated by Hilton and Chiang's work in [(Hilton, Wu, 1974)].

(iii) We have described the procedure along with an example to find the injective dimension of a fuzzy module motivated by the technique discussed in [(Mishra, Kumar, Behara, 2011)].

References

1. Cartan, H. Eilenberg, S. (1956). Homological Algebra, Princeton University Press,N.J.
2. Goldie, A.W., (1972). “The structure of Noetherian rings", Lectures on rings and modules, 213--321.
3. Greuel, G.M., Pfister, G. (2008). “A Singular Introduction to Commutative Algebra", Springer, Berlin, Germany.
4. Hilton, P., Wu, Y.C. (1974). “A Course in Modern Algebra", Springer Verlag, New York.
5. Isaac, P. (2004). “Studies in fuzzy commutative algebra", Ph.D thesis.
6. Kumar, R., Bhambri, S.K., Pratibha. (1995). “Fuzzy submodules: some analogues and deviations", Fuzzy sets
7. and systems, 70(1), 125—130.
8. Liu, H.X. (2014). “Fuzzy projective modules and tensor products in fuzzy module categories", Iranian Journal of Fuzzy Systems, 11(2), 89-101.
9. Michielsen, J. (2008). “Semisimplicity and finite groups", achelor thesis, Mathematisch Instituut, Universiteit Leiden.
10. Mishra, R.K, Kumar, S.D., Behara, S. (2011). “On Projective Modules and Computation of Dimension of a Module over Laurent Polynomial Ring", ISRN Algebra, 1—11.
11. Pan, Fu-Zheng. (1987). “Fuzzy finitely generated modules", J. Fuzzy Sets and Systems, 21, 105—113.
12. Pan, Fu-Zheng. (1988). “Exact Sequences of fuzzy linear maps", J. Fuzzy Sets and Systems, 27, 317—325.
13. Satyanarayana, B., Godloza, L., Mohiddin, S.S. (2004). “On fuzzy Dimension Of a module with DCC on submodules", Acharya Nagarjuna International Journal of Mathematics and Information Technology,1(1), 13-32.
14. Satyanarayana, B., Prasad, K.S., Nagaraju, D. (2006). “A Theorem on Modules with Finite Goldie Dimension", Soochow Journal of mathematics, 32(2), 311--315
15. Vargas, J.G. (2003). “Bases for projective modules in A_n(k)", Journal of Symbolic Computation, vol. 36, no.6, pp. 845–853.
16. Yenumula, V.R., Satyanarayana, B. (1987). “A note on modules", Proc. Japan Acad. A, 63, 208—211.
17. Zahedi, M.M., Ameri, R. (1995). “On fuzzy projective and injective modules", J. Fuzzy Math, 3(1), 181—190.