Tow-Dimensional Monic Polynomials For Solving Linear And Nonlinear Partied Differential Equations

Marina Shirwan [1], Ahmed Farooq Qasim [2],
[1,2] College of Computer Sciences and Mathematics, University of Mosul, Republic of Iraq.
E-mail[1]: Marina.csp104@student.uomosul.edu.iq
E-mail[2]: **ahmednumerical@uomosul.edu.iq**
https://orcid.org/0000-0002-2019-8769

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Abstract: In this paper, two dimensional monic polynomial technique is present for solving linear and nonlinear partied differential equations. The application of the method to boundary value problems leads to algebraic systems. The procedure in handling solutions of differential equations using Monic polynomial is to express the derivatives of a function in terms of its values by operational matrices. The suggested method can be used to facilitate greatly the setting up of the algebraic systems to be obtained solving differential equations. The effective application of the method is demonstrated by three examples.

Keywords: monic polynomial, nonlinear differential equations, differentiation matrix.

1. Introduction

It is well known that the numerical methods have played an important role in solving (PDEs). Several applications have been developed for numerical solutions of PDEs. Some of the most known numerical methods are finite difference methods, finite element methods, Adomian decomposition [1,2] , Homotopy perturbation method [3,4], differential transform [5], and many others. Approximation method have always been the subject of intense investigation because they have been for most of the times inescapable in the resolution to some partial differential equations [6]. In recent years, the monic polynomials have the advantage that they provide the best approximation in solve differential equations and integral equations. Borwein P. B., and al. [7], studied the problem of minimizing the supremum norm by monic polynomials with integer coefficients. El-Kady M. And El-Sawy N. [8], presented a new formula of the spectral differentiation matrices is. therefore, the numerical solutions for higher-order differential equations are presented by expanding the unknown solution in terms of monic Chebyshev polynomials. Azim Rivaz, and al. [9], presented a new method to gain the numerical solution of the straight two-dimensional Fredholm and Volterra Integro-differential equations (2d-fide and 2d-vide) by two-dimensional Chebyshev polynomials and construct their operational matrices of integration. Abdelhakem M., and al. [10], formulate a technique for discovering a new approach to solve ordinary differential equations (DEs) by using Galerkin spectral method. The Galerkin approach relies on Monic Gegenbauer polynomials (MGPs). Abdelhakem M., and al. [11], concentrated on carrying out a new approach for solving linear and nonlinear higher-order boundary value problems (HBVPs). The trial function of this method is the Monic Chebyshev polynomials (MCPs). This approach was depending on inflective of MCPs which explicit in the series expansion. Shoukralla E. S. and M. A. Markos [12], presents a numerical method for solving a specific class of Fredholm integral equations of the first kind, using the economized monic Chebyshev polynomials of the identical degree, the given possibility function is closed by monic Chebyshev polynomials of the same degree.

in this paper, a new formula for solving linear and nonlinear partied differential equations using two-dimensional Monic polynomial. In section 2, the basic ideas of monic polynomial are described. Section 3, a new differentiation matrices of two-dimensional monic polynomial are presented. Section 4, a new formula used for solving partied differential equations based on two-dimensional Monic technique. The results and comparisons of the numerical solutions are presented in section 5, and concluding remarks are given in section 6.

2. Function approximation of by monic polynomials

The monic polynomials have the advantage that they provide the best approximation in the minimax sense to arbitrary, continuous linear functions with integral and integrodifferential problems in any given finite intervals. The monic polynomials of degree $n(n = 1,2,\dots)$ on $[-1,1]$ are defined by the formula [13,14]:

$$Q_n(x) = 2^{1-n}\cos(n\cos^{-1}x) \qquad (1)$$

where

$$Q_0(x) = 1, \qquad Q_1(x) = x, \qquad Q_2(x) = x^2 - \frac{1}{2}$$
$$and \qquad Q_n(x) = xQ_{n-1}(x) - 1/4Q_{n-2}(x), \qquad n > 2$$

Clearly $|Q_n(x)| \le 1 \; for \; x\epsilon[-1,1]$.

A relationship between the monic polynomials $Q_n$and the Chebyshev polynomials T_n of the first kind is:

$$Q_n(x) = \frac{1}{2^{n-1}} T_n(x) \qquad n = 1,2, \dots \tag{2}$$

The monic approximations of a given function $f(x) \in C^{\infty}[-1,1]$ using $(N+1)$ Chebyshev Gauss-Lobatto (CGL) points $x_i = -\cos\left(\frac{i\pi}{N}\right)$, $i = 0,1, \dots N$, are

$$f(x) \cong \sum_{n=0}^{N} c_n a_n Q_n(x), \tag{3}$$

where $Q_n(x)$ is the monic polynomials, $c_n = 1, n = 0,1, \dots N-1, c_N = \frac{1}{2}$, and

$$a_n = \begin{cases} \frac{1}{N}\sum_{j=0}^{N} \theta_j f(x_j), & n = 0 \\ \frac{2}{N}\sum_{j=0}^{N} \theta_j f(x_j) x_j, & n = 1 \\ \frac{1}{2^{1-2n}N}\sum_{j=0}^{N} \theta_j f(x_j) Q_n(x_j), & n = 2, \dots, N \end{cases} \tag{4}$$

Where $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_j = 1, for\ j = 0,1, \dots N-1.$ Now, the exact relation between Chebyshev functions and its first derivatives is expressed as [15]:

$$T_n'(x) = \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2n}{c_k} T_k(x) \tag{5}$$

$$T_n''(x) = \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{1}{c_k} n(n^2-k^2) T_k(x) \tag{6}$$

Then, from relationship (2) between the monic polynomials $Q_n$and the Chebyshev polynomials T_n of the first kind:

$$\frac{1}{2^{1-n}} Q_n'(x) = \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2n}{c_k} \frac{1}{2^{1-n}} Q_n(x)$$

$$Q_n'(x) = \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2n}{c_k} Q_n(x) \tag{7}$$

Similarly

$$Q_n''(x) = \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{1}{c_k} n(n^2-k^2) Q_k(x) \tag{8}$$

Where $c_0 = 2$ and $c_i = 1$ for $i \geq 1$. In general:

$$Q_n^{(m)}(x) = \sum_{\substack{k=0 \\ (n+k+m)even}}^{n-m} \prod_{\substack{i=2-m \\ m>1}}^{m-2} (n^2-(k+i)^2) \frac{n}{c_k} \frac{1}{(m-1)!\, 2^{(m-2)}} Q_k(x), \quad m \geq 1 \tag{9}$$

From equation (7) and by differentiation the series in equation (3) term by term

$$f'(x) = \frac{2}{N}\sum_{j=0}^{N} \theta_j f(x_j) x_j + \frac{1}{N}\sum_{n=2}^{N}\sum_{j=0}^{N}\sum_{k=0}^{n-2} C_n \theta_j \frac{2n}{c_k} \frac{1}{2^{1-n}} f(x_j) Q_n(x_j) Q_k(x), \tag{10}$$

Also

$$f''(x) = \frac{1}{N}\sum_{n=2}^{N}\sum_{j=0}^{N}\sum_{\substack{k=0 \\ (n+k)even}}^{n-2} C_n \theta_j \frac{1}{2^{1-n}} \frac{1}{c_k} n(n^2-k^2) f(x_j) Q_n(x_j) Q_k(x) \tag{11}$$

Now, rewrite equations (10) and (11) by the following relations:

$[f'] = D_1[f], \quad [f''] = D_2[f],$

where $D_1$and $D_2$are square matrices of order (N+1) and the elements of the column matrices $[f'']$, $[f']$, $[f]$ are given by $f_i'' = f''(x_i)$, $f_i' = f'(x_i)$, $f_i = f(x_i)$,$i = 0,1, \dots . . N$ respectively. The first and second derivatives of the function $f(x)$at the point x_k are given by

$$f'(x_k) = \sum_{j=0}^{N} d_{kj}^1 f(x_j) \tag{12}$$

$$f''(x_k) = \sum_{j=0}^{N} d_{kj}^2 f(x_j) \tag{13}$$

the coefficients d_{kj}^{1} and d_{kj}^{2} , $j = 0,1, \dots N$ are the elements of the kth row of the matrices D_1 and D_2 respectively.

$$d_{k,j}^{(1)} = \frac{2}{N}\theta_j x_j + \frac{1}{N}\sum_{n=0}^{N}\sum_{\substack{l=0\\(n+l)odd}}^{n-1} C_n\theta_j \frac{2n}{c_k}\frac{1}{2^{1-n}} Q_n(x_j)Q_l(x_k), \quad k,j = 0,1, \dots N, \qquad (14)$$

$$d_{k,j}^{(2)} = \frac{1}{N}\sum_{n=0}^{N}\sum_{\substack{l=0\\(n+l)even}}^{n-2} C_n\theta_j \frac{1}{2^{1-n}}\frac{1}{c_k} n(n^2-k^2)Q_n(x_j)Q_k(x_k), \quad k,j = 0,1, \dots N, \qquad (15)$$

3. Differentiation matrices of Two-dimensional monic polynomial

Let $u\colon[-1,1]\times[-1,1]$ a continuous function and of bounded variation in the interval $I = [-1,1]\times[-1,1]$, if one of its partial derivatives exists and is bounded in I, the function f has a bivariate tow dimension Chebyshev expansion

$$\mathrm{u}\,(\mathrm{x},\mathrm{y}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e_i e_j a_{ij}\, Q_i(x)\, Q_j(y) \qquad (16)$$

for the truncated polynomial at degrees n and m with respect to x and y respectively

$$\mathrm{u}\,(\mathrm{x},\mathrm{y}) = \sum_{i=0}^{n}\sum_{j=0}^{m} e_i e_j a_{ij}\, Q_i(x)\, Q_j(y) \qquad (17)$$

Where a_{ij} is defend by the following case :

$$a_{00} = \frac{1}{NM}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)$$

$$a_{01} = \frac{2}{NM}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)y_L$$

$$a_{0j} = \frac{1}{N2^{1-2n}M}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)Q_j(y_L) \qquad for\ j = 2, \dots M$$

$$a_{10} = \frac{2}{NM}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)x_K$$

$$a_{11} = \frac{4}{NM}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)x_K y_L$$

$$a_{i0} = \frac{1}{M2^{1-2n}N}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)Q_i(x_K) \qquad for\ i = 2, \dots N$$

$$a_{1j} = \frac{2}{N2^{1-2m}M}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)x_K Q_j(y_L) \qquad for\ j = 2, \dots M$$

$$a_{i1} = \frac{2}{M2^{1-2n}N}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)y_L Q_i(x_K) \qquad for\ i = 2, \dots N$$

$$a_{ij} = \frac{2}{2^{1-2n}2^{1-2m}NM}\sum_{K=0}^{N}\sum_{L=0}^{M}\theta_K\theta_L f(x_K, y_L)Q_i(x_K)Q_j(y_L) \qquad for\ i \geq 2, j \geq 2$$

Where $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_j = 1 \quad for\ j = 0,1, \dots N-1$. $c_n = 1,\ n = 0,1, \dots N-1 \quad and\ c_N = \frac{1}{2}$.

Now, suppose that N=M and

$$b_i = \sum_{j=0}^{M} e_i e_j\, a_{ij}Q_j(y) \qquad i = 0,1, \dots N \qquad (18)$$

$$c_i = \sum_{i=0}^{N} e_i e_j\, a_{ij}Q_i(x) \qquad j = 0,1, \dots N \qquad (19)$$

Then, equation (17) can be interpreted in the form:

$$u(x,y) = \sum_{i=0}^{N} b_i\, Q_i(x) \qquad (20)$$

or

$$u(x,y) = \sum_{j=0}^{M} c_j\, Q_j(y) \qquad (21)$$

Extensions of the results of sections (2) together with the extension of the above notation to series for the partial derivatives of u, are simple when finding the partial derivative with respect to x , b_i are constant terms.
By using equation (7) with (20), we obtain:

$$u_x = \sum_{i=0}^{N} b_i \sum_{\substack{k=0\\(i+k)odd}}^{i-1} \frac{2i}{c_k} Q_k(x) \qquad (22)$$

Rewrite the equation (22) and by consider that b_i have constant terms

$$u_x(x_L,y) = \frac{2}{N}\sum_{L=0}^{N} b_1 \theta_L u(x_L,y) x_L + \sum_{i=2}^{N} b_i \sum_{L=0}^{N} \sum_{k=0}^{i-1} c_i \theta_L \frac{2i}{c_k} \frac{1}{2^{1-i}} u(x_L,y) Q_i(x_L) Q_k(x)$$

$$u_x(x_L,y) = \sum_{i=0}^{N} d_{k,i}^{1}\, u(x_i,y) \qquad L = 0,1,\dots N \qquad (23)$$

then

$[u_x] = [D_x^1]\ [u]$

Where u is a square matrix of order $(N+1)\times(N+1)$

$$[u] = \begin{bmatrix} u(x_0,y_0) & u(x_0,y_1) & \dots & u(x_0,y_N) \\ u(x_1,y_0) & \dots & & \dots \\ u(x_N,y_0) & \dots & & u(x_N,y_N) \end{bmatrix}_{(N+1)\times(N+1)}$$

Similarly

$$u_y = \sum_{j=0}^{N} c_j \sum_{\substack{k=0\\(j+k)odd}}^{j-1} \frac{2j}{c_k} Q_k(y) \qquad (24)$$

This leads to

$[u_y] = [u]\ [D_y^1]^T$

Where $[D_y^1]^T = [D_x^1]$ are defined in equation (14). Subsituiting equation (8) in equation (20), the second derivative for $u(x,y)$:

$$u_{xx} = \sum_{i=0}^{N} b_i \sum_{\substack{k=0\\(i+k)even}}^{n-2} \frac{1}{c_k} n(n^2-k^2) Q_k(x) \qquad (25)$$

Rewrite equation (25)

$$u_{xx}(x_L,y) = \frac{1}{N}\sum_{i=2}^{N}{}'' b_i \sum_{L=0}^{N}{}'' \sum_{k=0}^{i-2} c_i \theta_L \frac{1}{c_i} i(\,(i^2-k^2) \frac{1}{2^{1-i}} u(x_L,y) Q_i(x)\, Q_k(x)$$

$$u_{xx}(x_L,y) = \sum_{i=0}^{N} d_{k,i}^{2}\, u(x_i,y) = [D_x^2]\ [u] \qquad L = 0,1,\dots N \qquad (26)$$

Similarly

$$u_{yy}(x,y_L) = \sum_{j=0}^{N} d_{k,j}^{2}\, u(x,y_j) = [u]\ [D_y^2]^T \qquad L = 0,1,\dots N \qquad (27)$$

Where $[D_y^2]^T = [D_x^2]$ are defined in equation (15). Now

$$u_{xy}(x_L,y_L) = \sum_{i=0}^{N} d_{k,i}^{1} u_y(x_i,y) \qquad L = 0,1,\dots N \qquad (28)$$

By subsituation equation (24) in (28)

$$u_{xy}(x_L,y_L) = \sum_{i=0}^{N} d_{k,i}^{1} \sum_{j=0}^{N} d_{k,j}^{1}\, u(x_i,y_j) \qquad L = 0,1,\dots N \qquad (29)$$

$$u_{xy}(x_L,y_L) = \sum_{i=0}^{N} d_{k,i}^{1} \sum_{j=0}^{N} d_{k,j}^{1}\, u(x_i,y_j) = [D_x^1]\ [u]\ [D_y^1]^T \qquad (30)$$

4. Two- dimensional monic polynomial for solving non- linear partial differential equations

The general form of a second order non-liner and non-homogeneous partial differential equation is:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g(u) = h(x,y) \qquad (31)$$

Where a,b,c,d,e,f,h are functions of independent variables x,y or constants and g(u) are linear and non-linear terms. The numerical solution for equation (31) using the differentiation operational matrices for two-dimensional Monic polynomial by equations (22) -(30) , equation (31) becomes:

$$A\,[D_{xx}^2]\,U_{s+1} + B\,[D_x^1]\,U_{s+1}\,[D_y^1] + C\,U_{s+1}\,[D_{yy}^2] + D\,[D_x^1]\,U_{s+1} + E\,U_{s+1}\,[D_y^1] + F\,U_{s+1} + G(u_s) = H \qquad (32)$$

Where $D_{xx}^2 \ldots D_y^1$ are calculate from equations (14) and (15).

$$A = \begin{bmatrix} a(x_0,y_0) & a(x_0,y_1) & \ldots & a(x_0,y_N) \\ & a(x_1,y_1) & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ a(x_N,y_0) & & \cdots & a(x_N,y_N) \end{bmatrix}_{(N\times 1)\times(N\times 1)}$$

Similarly, B, C, D, E, F, H are defined. we convert the nonlinear equation into a linear system of equations using the initial condition in the approximation of nonlinear terms. using Kronecker product [16] , the equation (32) divides in the form

$$\left[\left(A[D_{xx}^2] + D[D_x^1] + F + G(u_s)\right) \otimes I\right)]\,U_{s+1} + [I \otimes \left(B[D_x^1][D_y^1] + C[D_{yy}^2] + E\,[D_y^1]\right)^T\,]\,U_{s+1}$$

$$= H \qquad (33)$$

Where the capacity of the matrices doubles to$(N \times 1)^2 \times (N \times 1)^2$. H becomes the vector $(N \times 1)^2 \times 1$ as well U_{s+1}.

Equation (33) It produces a linear system that solves by one of the methods for solving linear equations, such as the Gaussian elimination or Gauss -Gordon method, to get the new repetition U_{s+1}. Repeat the steps until we get a minimum step.

$$|U_{s+1} - U_s| < \in$$

Where $\in$ a very small value. The maple program was used in solving types of partial differential equations, as described in Numerical examples.

5. Numerical examples

In this section, three linear and nonlinear partial differential problems are solved by Two- dimensional monic method mentioned above.

Example 1: Let us have non homogenous linear partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + 10\,u = 12e^{x+y} \qquad -1 \le x \le 1\ , -1 \le y \le 1 \qquad (34)$$

With initial condition $u(-1, y) = e^{y-1}$.

Solution:

By application the equations (24) and (26), We obtain a system of linear equations as follows:

$$D_{xx}^2\,U + U\,{D_y^1}^T + 10\,I\,U = F \qquad (35)$$

I is identity matrix, F is a nonhomogeneous term matrix:

$$F = \begin{bmatrix} f(x_0,y_0) & f(x_0,y_1) & \ldots & f(x_0,y_1) \\ f(x_1,y_0) & \ldots & & \ldots \\ f(x_N,y_0) & \ldots & & f(x_N,y_N) \end{bmatrix}$$

$$U = \begin{bmatrix} u(x_0,y_0) & \ldots & \ldots & u(x_0,y_N) \\ u(x_1,y_0) & \ldots & & \ldots \\ u(x_N,y_0) & \ldots & & u(x_N,y_N) \end{bmatrix}$$

using Kronecker tensor products, denoted by $\otimes$, and a Lexicographic reordering, or reshaping of U and F [16], we may write equation (35) as:

$$(D_{xx}^2 + 10\,I) \otimes I + I \otimes {D_y^1}^T)U = F \qquad (36)$$

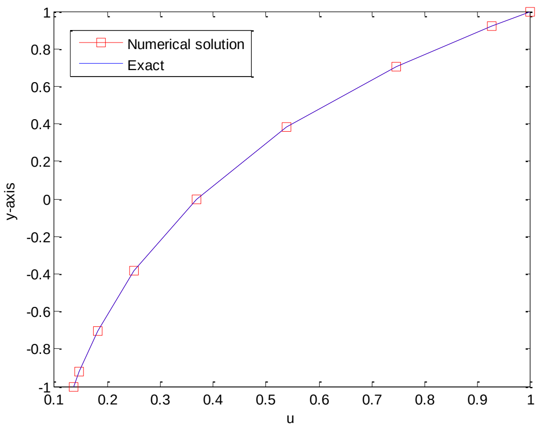
The solution to equation (36) produces a linear system contains $(N+1) \times (N+1)$equations and $(N+1) \times (N+1)$ of the unknown variables and solving this system we get on The numerical solution U is explain in table (12) where the exact solution is:

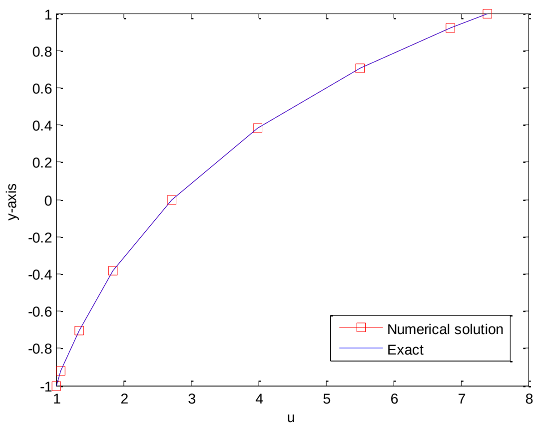
$u = e^{x+y}$

Table (1): the comparison between numerical solution using Monic polynomial and the exact solution for equation (34) When N=4,8.

X	Y	Numerical solution	Exact	Absolute error N=4	Absolute error N=8
-1	-1	0.135379	0.135335	1.848545×10^{-4}	4.417800×10^{-5}
-1	-0.92388	0.146062	0.146039		2.338526×10^{-5}
-1	-0.7071	0.181390	0.181399	7.368398×10^{-4}	3.124276×10^{-6}
-1	-0.382683	0.250892	0.250904		1.266115×10^{-5}
-1	0	0.367860	0.367879	1.275875×10^{-3}	1.895931×10^{-5}
-1	0.382683	0.539362	0.53939		2.782946×10^{-5}
-1	0.7071	0.746063	0.746102	3.013476×10^{-3}	3.853095×10^{-5}
-1	0.92388	0.926657	0.926705		4.756205×10^{-5}
-1	1	0.999948	1	3.708234×10^{-3}	5.123444×10^{-5}
1	-1	1.000081	1	4.235145×10^{-3}	8.062928×10^{-5}
1	-0.92388	1.079142	1.079093		4.894825×10^{-5}
1	-0.7071	1.340308	1.34030	1.099981×10^{-3}	8.171712×10^{-6}
1	-0.382683	1.853942	1.853946		4.435004×10^{-6}
1	0	2.718274	2.718282	3.982650×10^{-4}	7.739769×10^{-6}
1	0.382683	3.985570	3.985582		1.145497×10^{-5}
1	0.7071	5.512972	5.5130	3.934866×10^{-3}	1.593591×10^{-5}
1	0.92388	6.847452	6.847472		1.955793×10^{-5}
1	1	7.3890352	7.389056	2.817195×10^{-3}	2.093001×10^{-5}
MSE				4.577032×10^{-5}	1.191513×10^{-9}

Figure (1): illustrates comparing the numerical solution with the exact solution using monic polynomial for equation (34) when N=8, $x = -1$ and $x = 1$ respectivaly.





From table (1), we see that the results using monic polynomial approach from the exact solution when N is increasing.

Example 2: If we have the non-homogeneous and nonlinear partial difference equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}+10\,u+u^2=12e^{x+y}+e^{2(x+y)} \quad -1\le x,y\le 1 \qquad (37)$$

With initial and boundary conditions

$$u(-1,y)=e^{y-1}$$
$$u(x,-1)=e^{x-1} \qquad (38)$$
$$u(x,1)=e^{x+1}$$

By application the equations (24) and (26), we convert the nonlinear equation into a linear system of equations using the initial condition (38) in the approximation of nonlinear terms. Then the equation (37) becomes:

$$D_{xx}^2 U_{s+1}+U_{s+1}\,D_y^{1^T}+10\,I\,U_{s+1}+u_s U_{s+1}=F \qquad s=0,1,\dots \qquad (39)$$

Such that U_s when $s=0$ is the initial condition (38):

$$u_0=\begin{bmatrix} u(-1,y_0) & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & u(-1,y_N)\end{bmatrix}$$

Then, using Kronecker tensor products, we have:

$$\left((D_{xx}^2+10\,I+u_s)\otimes I+I\otimes D_y^{1^T}\right)U_{s+1}=F \qquad s=0,1,\dots \qquad (40)$$

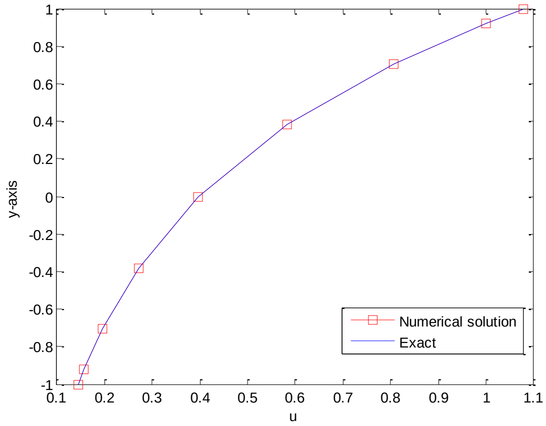
The exact solution is:

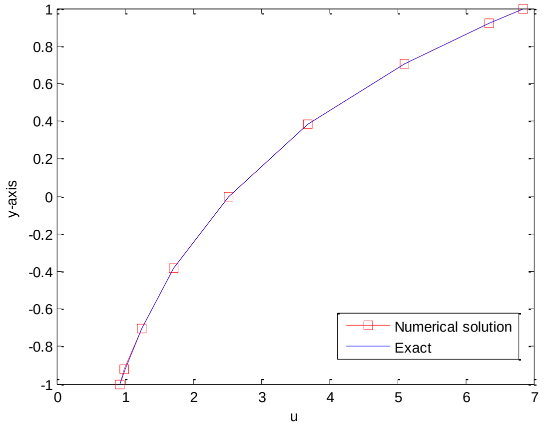
$$u=e^{x+y}$$

Table (2): the comparison between numerical solution using Monic polynomial and the exact solution for equation (37) When N=4,8.

X	**Y**	**Numerical solution**	**Exact**	**Absolute error N=4**	**Absolute error N=8**
-0.92388	-1	0.146005	.1460392976	3.719274×10^{-3}	3.396977×10^{-5}
-0.92388	-0.92388	0.157557	.1575899197		3.315623×10^{-5}
-0.92388	-0.7071	0.195715	.1957364213	8.713930×10^{-4}	2.189909×10^{-5}
-0.92388	-0.382683	0.270754	.2707490329		4.982108×10^{-6}
-0.92388	0	0.396997	.3969759686	2.630886×10^{-3}	2.054764×10^{-5}
-0.92388	0.382683	0.582060	.5820516436		8.281601×10^{-6}
-0.92388	0.7071	0.805105	.8051129092	2.923752×10^{-3}	7.467376×10^{-6}
-0.92388	0.92388	0.999988	1		1.226586×10^{-5}
-0.92388	1	1.079080	1.079093	1.819082×10^{-3}	1.239371×10^{-5}
0.92388	-1	0.926572	0.926705	1.136674×10^{-2}	1.329385×10^{-4}
0.92388	-0.92388	0.999889	1		1.108533×10^{-4}
0.92388	-0.7071	1.242035	1.242062	4.041806×10^{-3}	2.650721×10^{-5}
0.92388	-0.382683	1.718139	1.718061		7.924835×10^{-5}
0.92388	0	2.519113	2.519044	6.915305×10^{-3}	6.894340×10^{-5}
0.92388	0.382683	3.693430	3.693458		2.709530×10^{-5}
0.92388	0.7071	5.108855	5.108911	9.188658×10^{-3}	5.633941×10^{-5}
0.92388	0.92388	6.345553	6.345584		3.062698×10^{-5}
0.92388	1	6.847455	6.847472	2.385984×10^{-3}	1.660832×10^{-5}
MSE				5.669695×10^{-5}	5.202151×10^{-9}

Figure (2): illustrates comparing the numerical solution with the exact solution using monic polynomial for equation (37) when N=8, $x=-0.92388$ and $x=0.92388$ respectivaly.





Tables (1) and (2), show that two-dimensional monic polynomial is very close to exact solution in solving linear and nonlinear partial differential equations where the mean sequare error is 10^{-5} when N=4 and 10^{-9} when N=8 . The results using monic polynomial approach from the exact solution when N is increasing.

Example 3: let non homogenous and nonlinear burger equations:[17]

$$\frac{\partial u}{\partial t'} + \alpha \frac{\partial^2 u}{\partial x'^2} + \beta u \frac{\partial u}{\partial x'} = f(x', t') \qquad 0 \le t', x' \le 1 \qquad (41)$$

With initial conditions:

$$u(t_0', x') = g(x') \qquad 0 \le t', x' \le 1 \qquad (42)$$

For solving the equation (41) using three method in chapter three, the interval of [0,1] should be transferred to the interval of [-1,1] by suppose [17]

$$t' = 2t - 1 \qquad and \qquad x' = 2x - 1$$

The equation (41) and the boundary conditions (42) become:

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta u \frac{\partial u}{\partial x} = f(x, t) \qquad -1 \le t, x \le 1 \qquad (43)$$

And the initial conditions:

$$u(t_0, x) = g(x) \qquad (44)$$

By application equation (23), (24) and (27), the nonlinear burger's equations (43) into a linear system of equation using the initial conditions (44) in the approximation of non-linear term. Then the equation (43) become:

$$D_t^1 U_{s+1} + \alpha U_{s+1} {D_{xx}^2}^T + \beta U_s U_{s+1} {D_x^1}^T = F \qquad s = 0,1 \qquad (45)$$

Such that U_s when $s = 0$ is the initial conditions. Using Kronecker products, we have:

$$((D_t^1 \otimes I) + (I \otimes (\alpha {D_{xx}^2}^T U_{s+1} + \beta U_s {D_x^1}^T))\, U_{s+1} = F \qquad s = 0,1$$

Then

$$U_{s+1} = ((D_t^1 \otimes I) + (I \otimes (\alpha {D_{xx}^2}^T U_{s+1} + \beta U_s {D_x^1}^T)))^{-1} F \qquad s = 0,1 \qquad (46)$$

Table (3): the comparison between numerical solution for non-linear burger's equations (43) using Monic polynomial and the exact solution When N=4 $\alpha = 1$, $\beta = 5$ and $f(x, y) = 2t + 6x + 15x^2 + 15t^2x^2$ with the exact solution $u = t^2 + x^3$.

t	x	Numerical solution	Exact	Absolute error
-1	-1	-1.097662e-7	0	1.097662×10^{-7}
-1	-0.7071	-0.5	-0.5	8.245888×10^{-8}
-1	0	-1	-1	4.044429×10^{-8}
-1	0.7071	-0.5	-0.5	1.968263×10^{-8}
-1	1	-1.485202e-8	0	1.485202×10^{-8}
0.7071	-1	1.353553	1.353553	1.827112×10^{-7}
0.7071	-0.7071	0.853554	0.853553	1.328207×10^{-7}
0.7071	0	0.353553	0.353553	6.206178×10^{-8}
0.7071	0.7071	0.853553	0.853553	3.108642×10^{-8}
0.7071	1	1.353553	1.353553	2.393065×10^{-8}
MSE				5.884756×10^{-15}

Table (4): the comparison between numerical solution for non-linear burger's equations (43) using Monic polynomial and the exact solution when N=6 $\alpha = 1$, $\beta = {}^{5}/_{x^2}$ and $f(x,y) = 2t + 6x + 15x^2 + 15t^2x^2$ with the exact solution $u = t^2 + x^3$.

t	x	Numerical solution	Exact	Absolute error
-0.5	-1	0.875	0.875	5.536512×10^{-9}
-0.5	-0.866025	0.625	0.625	6.350168×10^{-9}
-0.5	-0.5	0.125	0.125	4.940979×10^{-9}
-0.5	0	-0.124999	-0.125	3.658193×10^{-9}
-0.5	0.5	0.125	0.125	4.629090×10^{-9}
-0.5	0.866025	0.625	0.625	5.330051×10^{-9}
-0.5	1	0.875	0.875	6.433931×10^{-9}
1	-1	2	2	1.491832×10^{-8}
1	-0.866025	1.75	1.75	1.150322×10^{-8}
1	-0.5	1.25	1.25	6.285726×10^{-9}
1	0	1	1	2.494040×10^{-9}
1	0.5	1.25	1.25	2.353412×10^{-9}
1	0.866025	1.75	1.75	3.741889×10^{-9}
1	1	2	2	4.479909×10^{-9}
MSE				3.761572×10^{-17}

6. Conclusion

It is well known that polynomials are used in solving nonlinear ordinary and partial differential equations It requires converting differential equations into nonlinear systems from polynomial transactions, and thus solving these systems with one of the methods of solving nonlinear systems requires additional time and effort. In this paper the differentiation matrix based on the monic Chebyshev polynomials $Q_n(x)$ is presented for solving partial nonlinear ordinary differential equations by relying on operational matrices for derivatives of polynomials and dispensing with the step of finding coefficients and then substituting to find numerical solutions. The main advantage of these polynomials is that the size of the monic polynomial is $\frac{1}{2^{n-1}}$, $n \geq 1$ and this becomes smaller as the degree n increases. The degree n monic polynomial with the smallest maximum on $[-1,1]$ is the modified Chebyshev polynomial $T_n(x)$. This result is used in approximate higher-order differential applications and can be applied to obtain an improvement interpolation scheme. MAPLE 15 has been used in programming and solving examples.

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