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Research Article

Generalised Hvg For Periodic Points Of Period 2^nin One Dimensional Dynamical System

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Abstract
A generalised definition of HVG[4] say m HVG is considered with the vertex sets V_n[4]. To analyse graph theoretic properties of m HVG for any value of m it is very much necessary to find the position of each and every point of the vertex set V_n. For this we have derived some fruitful results which help us to find out the position of each and every vertex set V_n and those results also help us to generate any vertex set V_n of period 2^n without knowing V_{n-1} . Some graph theoretic properties of mHVG is discussed in this paper.

Keywords: Periodic points, Period doubling bifurcation , Horizontal Visibility Graph

Introduction

Horizontal visibility graph algorithm plays an important role for network analysing time series.Zhong et al.[6]have developed a novel multiscale limited penetrable horizontal visibility graph to analyse non linear time series from the perspective of multiscale and complex network analysis.Martin et al.[2] investigated the characteristic of node degree distribution constructed bu using HVG for time series corresponding to 28 chaotic maps, 2 chaotic flows and 3 different stochastic processes.Wang et al.[3] have discussed topological properties of independent and identically distributed random series using horizontal visibility graph.Lacasaet al.[1] have proposed a method to measure a real valued time seies irreversibility by combining two different tools and horizontal visibility algorithm was one of them.

Dutta et al.[4] have considered the period doubling bifurcation points of logistic map and considering the relative positions they have obtained a mathematical modelV_n .The mathematical model have the property that one can say the position of the periodic points of V_n (say) without calculating them. They have taken the elements of V_n and defined horizontal visibility graph.Some properties of the vertices set have been derived by them and by those properties they have derived the degree of some vertices of the Horizontal visibility graph.

To get the degree of all the points of V_n[4] Dutta et al.[5] have partitioned the sets V_n in to n sets and defined level sets.They have derived the important property that each element of a level set is incident with maximum two elements from each partitioned set which have been used to find the total degree of the vertex set.

2. Main Results :

Definition 2.1

$$\boldsymbol{G_{n,m} = (V_n, E_{n,m})}$$

A horizontal visibility graph say $G_{n,m}$ is constructed with the vertices sets of V_n [4] and the edge set is defined as the following:

$E_n = \{((n_1, i), (n_2, j)) \big| N((n_1, i), (n_2, j)) \leq m\}$where$N((n_1, i), (n_2, j))$ represents the number of elements of the form (k, k_1) such that $k_1 > i$ or j and $n_1 < k < n_2$ and m is a non negative integer.

Theorem 2.2:$(2^k , 2^{n-k} - 1)$ exists for a fixed $n$in V_n, $n \geq 2$ and $k = 1,2,3, \dots . n - 1$

Proof : Clearly the result is true for $n = 2$. Let the result be true for $n = p$ i.e in $V_p(2^k, 2^{p-k} - 1)$ exists. To prove in V_{p+1} , $(2^k , 2^{p+1-k} - 1)$ exists for $k = 1,2,3, \dots p$.

In V_{p+1} by the definition of $V_{n,1}$ we have $(2.2^k, 2^{p-k}-1)= (2^{k+1}, 2^{p-k}-1) = (2^{k+1}, 2^{(p+1)-(k+1)}-1) = (2^t, 2^{p+1-t}-1)$ exists for $t = 2,3,\dots,p$ where $k+1 = t$exists.
When $t = 1$ the element $(2, 2^{n-1})$ exists [4].So we can say that the element $(2^k, 2^{n-k})$ exists and hence the theorem.

Theorem 2.3: If $(x, y) \in V_n$ such that $y \le 2^{n-k}-1 \implies (x+2^{k-1}, 2^{n-k+1}-1-y)$ exist in $V_n$where $2 \le k \le (n-1)$.

Proof: We prove the theorem by induction in V_n .Clearly the result is true for $n = 1,2,3$. Let the result be true in V_{n-1} .

Let $(x, y) \in V_n$ such that ≥ 2 . Since $y \le 2^{n-k}-1$ so there exist $(\frac{x}{2}, y)$ in V_{n-1} such that $\left(2.\frac{x}{2}, y\right) \in V_n$.

Now for $k \ge 2$, $\left(\frac{x}{2}, y\right) \in V_{n-1}$ *and* $y \le 2^{n-k}-1$

So for$y \le 2^{n-k}-1$, $(\frac{x}{2}+2^{k-2}, 2^{n-k+1}-1-y)$ exists in V_{n-1}

$\therefore (2.\frac{x}{2}+2.2^{k-2}, 2^{n-k+1}-1-y)$exists in V_n .

Hence the theorem.

Theorem 2.4:$(a, b) \in E_n \Leftrightarrow (a-1, 2^n-1-b) \in O_n$ where $E_n = \{(a, b) \in V_n | \, a \; is \; even\}$ and $O_n = \{(a, b) | a \; is \; odd\}$

Proof: We prove this result by induction. Clearly the result is true in V_1 . Let the result be true in V_{n-1} i.e $(a, b) \in E_{n-1} \Leftrightarrow (a-1, 2^{n-1}-1-b) \in O_{n-1}$. We have to prove the resuly is true in V_n i.e $(a, b) \in E_n \Leftrightarrow (a-1, 2^n-1-b) \in O_n$.

If $(a, b) \in E_n$ then three cases arise

Case 1

$a = 2k$for some integer k, k is even . Since k is even then $(k, b) \in V_{n-1}$i.e $\in E_{n-1}$. So by induction $(k-1, 2^{n-1}-1-b) \in O_{n-1}$. So by the construction of $V_{n,4}$we have$[2(k-1)+1, 2^{n-1}+2^{n-1}-1-b] \in V_n$ i.e $[2k-1, 2^n-1-b) \in V_n = (a-1, 2^n-1-b) \in V_n$.

Case 2

There exist $(k, b) \in V_{n-1}$, k is odd such that $a = 2k+4$. Since k is odd so $(k, b) \in O_{n-1}$. Now $k = k+1-1$ and $k+1$ is even. So by induction $(k+1, 2^{n-1}-1-b) \in E_{n-1}$ .Also by $V_{n,4}^{/}$ we have $[2.(k+1)+1, 2^{n-1}+2^{n-1}-1-b) \in V_n$ i.e $(2k+3, 2^n-1-b) \in V_n$ i.e $(a-1, 2^n-1-b) \in V_n$i.e$(a-1, 2^n-1-b) \in O_n$.

Case 3

If $a = 2$ then $(2, 2^{n-1}-1)$ exist in V_n. Then we know that $(1, 2^{n-1}) \in V_n$ [4]

Also $2^n-1-2^{n-1}+1 = 2^{n-1}$ and hence the theorem.

Theorem 2.5: If $(x, y) \in V_n, 0 \le y \le 2^{n-p}-1$ and $1 < p \le n$ then $(x+1+2+2^2+\cdots+2^{p-2}, 2^{n-1}+2^{n-2}+\cdots+2^{n-p+1}+y)$ exist in V_n .

Proof: Let $(x, y) \in V_n$, $0 \le y \le 2^{n-p}-1$.Then $\left(\frac{x}{2}, y\right) \in V_{n-1}, 0 \le y \le 2^{n-p}-1$ .also by using theorem 2.3 it can be said that the element $(\frac{x}{2}+2^{p-2}, 2^{n-p+1}-1-y)$ exist in V_{n-1} for $2 \le p \le (n-2)$.

Now by theorem 2.4 we have

$(\frac{x}{2}+2^{p-2}-1, \; 2^{n-1}-1-2^{n-p+1}+1+y)$exist in V_{n-1}

$\Rightarrow (\frac{x}{2}+2^{p-2}-1, 2^{n-1}-2^{n-p+1}+y)$exist in V_{n-1} .

Now ,

$2\left(\frac{x}{2}+2^{p-2}-1\right)+1, 2^{n-1}+2^{n-1}-2^{n-p+1}+y)$exist in V_n .

$\Rightarrow (x+2^{p-1}-1, 2^n-2^{n-p+1}+y)$exist in V_n.

$\Rightarrow (x+2^{p-1}-1, 2^{n+1}\left(1-\frac{1}{2^p}\right)-2^n)$exist in V_n.

$\Rightarrow (x+2^{p-1}-1, 2^n-2^{n-p+1})$exist in V_n.

Again,

$$x + 1 + 2 + 2^2 + \cdots + 2^{p-2}$$

$= x + \frac{2^{p-1}-1}{2-1}$
$= x + 2^{p-1} - 1$

Again ,

$$2^{n-1} + 2^{n-2} + \ \ldots + 2^{n-p+1} + y$$

$= 2^n \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-1}}\right) + y$
$= 2^n \left(\frac{1-\left(\frac{1}{2}\right)^p}{1-\frac{1}{2}}\right) + y$
$= 2^{n+1} - \ 2^{n+1-p}$

When $p = 2$
$0 \leq y \leq 2^{n-2} - 1$and to show $(x, y) \in V_n \ \Longrightarrow (x + 1, 2^{n-1} + y) \in V_n$ which follows from the definition of V_n.
And hence the theorem.

Theorem 2.6: If $(a, x) \in V_n$ such that a is even and $a = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_t}$where $x_1 > x_2 > x_3 \ldots > x_t$then $x = ((2^{n-x_t} - 1) - (2^{n-x_{t-1}} - 1) + \cdots \ldots . + (-1)^{t-2}(2^{n-x_2} - 1) + (-1)^{t-1}(2^{n-x_1} - 1))$

Proof: Let $(a, x) \in V_n$ such that a is even
Let $= 2^{x_1} + 2^{x_2} + \cdots + 2^{x_t}$, where $x_1 > x_2 > \cdots > x_t$
First we show that if $= 2^{x_1} + 2^{x_2}$, $x_1 > x_2$ then $(2^{x_1} + 2^{x_2}$, $(2^{n-x_2} - 1) - (2^{n-x_1} - 1))$ exist in V_n.
Using theorem 2.3 we get $(2^{x_1}, 2^{n-x_1} - 1)$, $(2^{x_2}, 2^{n-x_2} - 1)$, $(2^{x_3}, \ 2^{n-x_3} - 1)$, ……..,$(2^{x_t}, 2^{n-x_t} - 1)$ exist in V_n.
Now.

$$x_1 > x_2$$
$$\Rightarrow x_1 \geq x_2 + 1$$
$$\Rightarrow n - x_1 \leq n - x_2 - 1$$
$$\Rightarrow 2^{n-x_1} - 1 \leq 2^{n-x_2-1} - 1$$

Now by using theorem 2.4 we get $(2^{x_1} + 2^{x_2}, \ 2^{n-x_2-1+1} - 1 - (2^{n-x_1} - 1))$ i.e $(2^{x_1} + 2^{x_2}, \ (2^{n-x_2} - 1) - (2^{n-x_1} - 1))$ exist in V_n.
Similarly we can show that if $a = 2^{x_1} + 2^{x_2} + 2^{x_3}$ then $(2^{x_1} + 2^{x_2} + 2^{x_3}, \ (2^{n-x_3} - 1) - (2^{n-x_2} - 1) + (2^{n-x_1} - 1))$ exist in V_n.
Thus if $a = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_t}$ where $x_1 > x_2 > \cdots > x_t$
Then

$$x = ((2^{n-x_t} - 1) - (2^{n-x_{t-1}} - 1) + \cdots \ldots . + (-1)^{t-2}(2^{n-x_2} - 1) + (-1)^{t-1}(2^{n-x_1} - 1))$$

Theorem 2.7: If $(x, y) \in V_n$such that $2^{k-1} \ \leq y \leq 2^k - 1$ then $x = 2^{n-k}m$ for some odd values of m, $k \leq (n - 1)$.

Proof: We know $(2^k, 2^{n-k} - 1)$ exist. So putting $k = n - 1$ it can be said that $(2^{n-1}, 1)$ exist.
Let $A_1 = \{(0,0), (2^{n-1}, 1)\}$ and $A_k = \{(x, y) \in V_n \ such \ that \ 2^{k-1} \ \leq y \leq 2^k - 1\}$
We have to show that$x = 2^{n-k}m$, m is odd.
We show it by induction.
Let $A_{k-1} = \{(x, y) \in V_n \ such \ that \ 2^{k-2} \ \leq y \leq 2^{k-1} - 1\}$ and $x = 2^{n-k+1}m$
We know that if $0 \leq y \leq 2^{n-k} - 1$ then $(x + 2^{k-1}, 2^{n-k+1} - 1 - y)$ exists.
Putting $k = n - k + 1$
$0 \leq y \leq 2^{k-1} - 1$then$(x + 2^{n-k+1-1}, 2^{n-(n-k+1)+1} - 1 - y)$ exists.
$\Rightarrow (x + 2^{n-k}, 2^k - 1 - y)$exists$\forall x$ such that $(x, y) \in \ \bigcup_{i=1}^{k-1} A_{k-1}$
$\Rightarrow (x + 2^{n-k}, 2^k - 1 - y)$exists if $0 \leq y \leq 2^{k-1} - 1$
and$x + 2^{n-k}$ is an odd multiple of 2^{n-k}

and$2^k - 1 - 2^{k-1} + 1 \le 2^k - 1 - y \le 2^k - 1$
Thus $A_k = \{(x,y) \in V_n | 2^{k-1} - 1 \le y \le 2^k - 1\}$ then x is odd multiple of 2^{n-k}
NOTE: By theorem 2.4,2.5,2.6 we can find position of any element of a vertex set V_n.

Corollary 2.8: The elements $\{(a,b)|a \le (x-2-2m)\}$ and $\{(c,d)|\ c \ge (x+2+2m)\}$ can not be incident with the element (x,y) where $(x,y) \in G_{n,m}(V_n, E_{n,m})$ and x is even.

Proof : Let $(x,y) \in G_{n,m}(V_n, E_{n,m})$. The elements $\{(a,b)|a \le (x-2-2m)\}$ can not be incident with (x,y) because there are minimum $m+1$ odd elements between a and x and position of odd elements in V_n are greater than even elements.
Corollary 2.9:

The elements $\{(a,b)|a \le (x-1-2m)\}$ and $\{(c,d)|\ c \ge (x+1+2m)\}$ can not be incident with the element (x,y) where $(x,y) \in G_{n,m}(V_n, E_{n,m})$ and x is odd.

Proof: Since between x and $x-1-2m$ there are $m+1$ odd elements so the result follows.

Corollary 2.10: The elements $\{(a,b)|a \ge (x-1-m)\}$ and $\{(c,d)|\ c \le (x+1+m)\}$are incident with the element (x,y) where $(x,y) \in G_{n,m} = (V_n, E_{n,m})$ and x is odd.

Theorem 2.11: Let (x,j) be in $G_{n,m} = \left(V_n, E_{n,m}\right)$where x is odd. Let $O = \{(p,q)\ |p \text{ is even and } p < x \text{ and } p \ge (x-1-m)\}$. Let$r = 2^k s = maximum\ \{y|(x-1-2m) \le y \le (x-1-m)\}$ such that $(m - n(h) - \sum_{t\le k} r_t) > 0$, where r_t is the number of elements of the form $(2^t u\ , q)$ in $O$and $n(h)$ gives the number of odd elements from $x-1$ to r then (r,v) for some v will be adjacent to (x,j).
Proof: Let (x,j) be in $G_{m,n} = \left(V_n, E_{m,n}\right)$,where x is odd. Let$O = \{(p,q)|p$ is even and $p < x$ and $\ge (x-1-m)\}$. In other words we can say that the elements of O are incident with (x,j) . Clearly if $x > p \ge (x-1-m)$then $(p,q) \in O$. Also whenever $p < (x-1-2m)$ then $(p,q) \notin O$ .Further if We form another set $D = \{(a,b)|(x-1-2m) \le a \le (x-1-m) \text{ and } a \text{ is even.}\}$ then we find the possible condition such that the elements of D will be an element of O.
Let $(r,s) \in D$ such that r is the largest element. Let h be the number of odd elements between $x-1$to r. Let k be the greatest positive integer such that $r = 2^k s$ where s is odd and let r_t be the number of elements of the form $(2^t u\ , q)$ in Owhere $\le k$. Then if $(m - n - \sum_{t\le k} r_t) > 0$implies $(r,s) \in O$.
The condition $(m - n(h) - \sum_{t\le k} r_t) > 0$is taken because between (x,j) and (r,v) if there are $m+1$ elements whose positions are greater than j or v then they will not be adjacent. Since we know that odd elements occupy higher position than even elements so here $n(h)$ is subtracted because $n(h)$ is the number of odd elements from $x-1$ to r and by theorem 2.7 it can be said that $\sum_{t\le k} r_t = \sum_{t\le k} 2^t u$gives the sum of those elements which may occupy higher position than v. Thus if $(m - n(h) - \sum_{t\le k} r_t) > 0$ that means we are sure that there does not exist $m+1$ elements which occupy greater position than v and hence (r,v) will be adjacent with (x,j). Hence the theorem.

The above condition is sufficient but not necessary i.e if $(m - n(h) - \sum_{t\le k} r_t) < 0$ we are not sure whether the element (r,v) will be adjacent with (x,j).At that time we may take help of theorem 2.2, 2.4, 2.6, 2.7
However if $D = \{y|(x-1-2m) \le y \le (x-1-m)\}$ after getting (r,v) we can add the element in the set O and a new set O can be formed and again r will be taken from the set D. Thus we can continue the process.
Theorem 2.12:
Let (x,j) be in $G_{n,m} = \left(V_n, E_{n,m}\right)$, where x is odd. Let $O = \{(p,q)\ |p \text{ is even and } p > x \text{ and } p \le (x+1+m)\}$. .Let $r = 2^k s = minimum\ \{y|(x+1+m) \le y \le (x+1+2m)\}$ such that $(m - n(h) - \sum_{t\le k} r_t) > 0$, where r_t is the number of elements of the form $(2^t u\ , q)$in O. and $n(h)$ gives the number of odd elements from $x+1$ to r then (r,v) for some v will be adjacent to (x,j).

Proof: Let (x,j)be in $G_{m,n} = (V_n, E_{m,n})$, where$x$ is odd.Let
$O = \{(p,q)\ |p\ is\ even\ and\ p > x\ and\ p \le (x+1+m)\}$. In other words we can say that
$O = \{(p,q)\ |p\ is\ even\ and\ p > x\ and\ (p,q) is\ incident\ with\ (x,j)$ }then if $x < p \le (x+1+m)$ then $(p,q) \in O$. Also whenever $p > (x+1+2m)$then $(p,q) \notin O$. Further if We form another set D = $\{(a,b)|(x+1+m) < a \le (x+1+2m)\ and\ a\ is\ even.\}$ then we find the possible conditions such that the elements of D will be an element of O.
Let $(r,s) \in D$ such that r is the smallest integer. Let $n(h)$ be the number of odd elements between $x+1$ to$r$. Let k be the greatest positive integer such that $r = 2^k m$. Let r_t be the number of elements of the form $(2^t m, q)$in . Then if $(m - n - \sum_{t\le k} r_t) > 0$implies $(r,s) \in O$.
If $(m - n - \sum_{t\le k} r_t) < 0$ we are not sure whether the element (r,v) will be adjacent with (x,j). At that time we may take help of theorem 2.2 , 2.4, 2.6, 2.7.

Corollary 2.13: Let $(a,b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level$q$ set of G_n. **let** $= \{(a - 2^{p-1}t, d_t), t = 1,3,5 \dots (m+1), (m+3), \dots (2m+1)\}$. The elements of O belongs to level $p$set[2] where $p < q$.

Corollary 2.14: Let $(a,b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level q set of V_n. Let $= \{(a - 2^{p-1}t, d_t), t = 2,4,6, \dots 2m\}$. The elements of O belongs to level $> p$ set where $p < q$.

Theorem2.15: Let$(a,b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level q set of V_nFor $p < q$, let$O = \{(a - 2^{p-1}t, d_t), t = 1,3,5 \dots (m+1)\}$ ..Let $r = maximum\ \{a - 2^{p-1}t, t = m+3, m+5, \dots, 2m+1\}$such that $(m - n(h) - \sum_{t\le k} r_t) > 0$, where r_t is the number of elements of the form $2^t m + (1 + 2 + 2^2 + \dots + 2^{p-2})$ in O and $t \le k$and if $h = \{(a - 2^{p-1}t, d_t), t = 2,4,6, \dots r\}$ then $n(h)$ gives the cardinality of the set h.Then (r,s) will be adjacent with (a,b) and m is even.

Proof: Let $(a,b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level$q$ set of V_n .let$O =$
$\{(a - 2^{p-1}t, d_t), t = 1,3,5 \dots (m+1)\}$ and m is even.We claim that elements of O are adjacent with (a,b) because if we consider the elements (a $-2^{p-1}t, d_t$) where $t = 1,2, \dots m$ then since m is even so among these $m/2$ elements are in level $> p$ and $\frac{m}{2}$ are in level p .But $(a - 2^{p-1}(m+1), d_{m+1})$ is in level p set. So elements of O are adjacent with (a,b).
Now $(a - 2^{p-1}(m+2), d_{m+2})$ is in level $> p$ set.So the element (a $-2^{p-1}(m+3), b_l$) which is in level p set is to be decided whether it is adjacent with (a,b) or not . The following two sets are obtained when m is even.
$O = \{(a - 2^{p-1}t, d_t), t = 1,3,5 \dots (m+1)\}$and $D = \{(a - 2^{p-1}t, d_t), t = m+3, m+5, \dots, 2m+1\}$
We find the condition under which elements of D will be adjacent with (a,b).

Let $(r,s) \in D$ such that r is the largest integer. Let $n(h)$ be the number of elements belonging to level $> p$ between $a - 2^{p-1}$ to r . Let k be the greatest positive integer such that $r = 2^k m + (1 + 2 + 2^2 + \dots + 2^{p-2})$. Let r_t be the number of elements of the form $(2^t m + (1 + 2 + 2^2 + \dots + 2^{p-2}), d_l)$ in $O$and $\le k$. Then if $(m - n - \sum_{t\le k} r_t) > 0$ implies $(r,s) \in O$ .The condition $(m - n(h) - \sum_{t\le k} r_t) > 0$is taken because between (a,b) and (r,s) if there are $m+1$ elements whose positions are greater than b or s then they will not be adjacent. Since $n(h)$ gives the number of elements belonging to level $> p$ so $n(h)$ is the sum of those elements which occupy higher position than s. $\sum_{t\le k} r_t$gives the sum of those elements which may occupy higher position than s because r_t is the number of elements of the form $(2^t m + 1 + 2 + 2^2 + \dots + 2^{p-2}, d_l)$ in O. Since the elements of O are belonging to level p set and any element of level p set can be expressed as $(2^p q + (1 + 2 + 2^2 + \dots + 2^{p-2})$,d_x[5] so it simplified to $2^t m + (1 + 2 + 2^2 + \dots + 2^{p-2})$. As t increases their position decreases in level p set which can be said with the help of theorem 2.5 and 2.7. Thus if $(m - n(h) - \sum_{t\le k} r_t) > 0$ that means we are sure that there does not exist $m+1$ elements which occupy greater position than s and hence (r,s) will be adjacent with (a,b)
if $(m - n(h) - \sum_{t\le k} r_t) < 0$ we are not sure whether the element (r,s) will be adjacent with (a,b).At that time we may take help of theorem 2.2 , 2.4, 2.5,2.6, 2.7. Hence the theorem.

However if $D = \{(a - 2^{p-1}t, d_t), t = m+3, m+5, ..., 2m+1\}$thenafter getting (r, s) we can add the element in the set O and a new set O can be formed and again r will be taken from the set D. Thus we can continue the process. An algorithm based on this can be generated in the computer .

Theorem 2.16: Let $(a, b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level q[Dutta 1] set of V_n **.** For $< q$, let $O = \{(a + 2^{p-1}t, d_t), t = 1,3,5 ... (m+1)\}$ ..Let $r = minimum \ \{a + 2^{p-1}t, t = m+3, m+5, ..., 2m+1\}$such that $(m - n(h) - \sum_{t \le k} r_t) > 0$, where r_t is the number of elements of the form $2^t m + (1 + 2 + 2^2 + \cdots + 2^{p-2})$ in O and $t \le k$and let $h = \{(a + 2^{p-1}t, d_t), t = 2,4,6, ... r\}$ and $n(h)$ is the cardinality of the set h then (r, s) will be adjacent with (a, b) and m is even.

Proof: Exactly in a similar way like theorem 9we can say that the elements of $O = \{(a + 2^{p-1}t, d_t), t = 1,3,5 ... (m+1)\}$ are adjacent with (r, s) and we form the set $D = \{(a - 2^{p-1}t, d_t), t = m+3, m+5, ..., 2m+1\}$.

Let $(r, s) \in D$ such that r is the smallest integer. Let $n(h)$ be the number of elements belonging to level $> p$ between $a + 2^{p-1}$ to r . Let k be the greatest positive integer such that $r = 2^k m + (1 + 2 + 2^2 + \cdots + 2^{p-2})$. Let r_t be the number of elements of the form $(2^t m + (1 + 2 + 2^2 + \cdots + 2^{p-2}), d_l)$ in $O$and $\le k$. Then if $(m - n - \sum_{t \le k} r_t) > 0$implies $(r, s) \in O$.
If $(m - n - \sum_{t \le k} r_t) < 0$ At that time we may take help of theorem 2.2, 2.4, 2.5, 2.6, 2.7

Theorem 2.17: Let $(a, b) \in G_{n,m} = (V_n, E_{n,m})$ where m is an odd integer. and (a, b)be an element of level q set of V_nFor $p < q$, let $O = \{(a - 2^{p-1}t, d_t), t = 1,3,5 ... m\}$ ..Let $r = maximum \ \{a - 2^{p-1}t, t = m+2, m+4, ..., 2m+1\}$ such that $(m - n(h) - \sum_{t \le k} r_t) > 0$, where r_t is the number of elements of the form $2^t m + (1 + 2 + 2^2 + \cdots + 2^{p-2})$ in O and $t \le k$ and let $h = \{(a - 2^{p-1}t, d_t), t = 2,4,6, ... s\}$and $n(h)$ gives the cardinality of the set $h$where $a - 2^{p-1}s = r$ then (r, s) will be adjacent with (a, b) and m is odd.

Proof: Let $(a, b) \in G_{n,m} = (V_n, E_{n,m})$ and be an element of level$q$ set of V_n **.**let$O = \{(a - 2^{p-1}t, d_t), t = 1,3,5 ... m\}$ and m is odd. We claim that elements of O are adjacent with (a, b) because m is odd implies $\frac{m+1}{2}$ elements are in level p set and $\frac{m-1}{2}$ elements are in level $> p$. But $a - 2^{p-1}(m+1)$ is in level greater than p set. So $a - 2^{p-1}(m+2)$ is in level p set. But $a - 2^{p-1}(m+1)$ is in level $> p$. So when m is odd and $x \le a - 2^{p-1}(m+2)$ then (x, l) may or may not be adjacent with (a, b) .When m is odd the following two sets we are getting
$O = \{(a - 2^{p-1}t, d_t), t = 1,3,5, ... m)\}andD = \{(a - 2^{p-1}t, d_t), t = m+2, m+4, ..., 2m+1\}$
The elements of O are adjacent with (a, b)but the elements of D are to be checked whether they are adjacent with (a, b) or not.
Exactly in a similar way we can say the possible condition is $(m - n - \sum_{t \le k} r_t) > 0$.

Theorem 2.18:
Let $(a, b) \in G_{n,m} = (V_n, E_{n,m})$ where m is an odd integer and(a, b)be an element of level q set of V_n **.** For $p < q$let $= \{(a + 2^{p-1}t, d_t), t = 1,3,5 ... m\}$ ..Let $r = maximum \ \{a - 2^{p-1}t, t = m+2, m+4, ..., 2m+1\}$ such that $(m - n(h) - \sum_{t \le k} r_t) > 0$, where r_t is the number of elements of the form $2^t m + (1 + 2 + 2^2 + \cdots + 2^{p-2})$ in O and $t \le k$ and let $h = \{(a + 2^{p-1}t, d_t), t = 2,4,6, ... r\}$ then (r, s) will be adjacent with (a, b) and m is odd.

Proof: Same as theorem 2.16.

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