Quantum Codes Obtained through Constacyclic Codes Over $\mathbf{Z_p[\nu,\omega,\gamma]/<\nu^2-1,\omega^2-1,\gamma^2-1,\nu\omega-\omega\nu,\omega\gamma-\gamma\omega,\gamma\nu-\nu\gamma>}$

Jagbir Singh[1], Prateek Mor[2]

[1] Department of Mathematics, Maharshi Dayanand University, Rohtak-124001
(E-mail: ahlawatjagbir@gmail.com)

[2] Department of Mathematics, Government College Matanhail, Jhajjar-124103
(E-mail: prateekmor1992@gmail.com)

Article History: Received: 10 November 2020; Revised 12 January 2021 Accepted: 27 January 2021; Published online: 5 April 2021

Abstract

The structural properties and construction of quantum codes over Z_p using Constacyclic codes over the finite commutative non-chain ring $\Re = Z_p[\nu,\omega,\gamma]/<\nu^2-1,\omega^2-1,\gamma^2-1,\nu\omega-\omega\nu,\omega\gamma-\gamma\omega,\gamma\nu-\nu\gamma>$ where $\nu^2 = 1, \omega^2 = 1, \gamma^2 = 1, \nu\omega = \nu\omega, \omega\gamma = \gamma\omega, \gamma\nu = \nu\gamma$ are the focus of htis paperand Z_p is field having p elements with characteristic p where p is an prime such that $p > 2$. A Gray map is defined between $\Re$ and Z_p^8. Decomposing constacyclic codes into cyclic and negacyclic codes over Z_p yields the parameters of quantum codes over Z_p. Some examples of quantum codes of arbitrary length are also obtained as an application.

2010 AMS Classification: 81P68, 94B35, 94B60.

Keywords and phrases: Finite ring, Gray map, Constacyclic codes, Negacyclic, Quantum codes.

1. Introduction

Shor **[10]** demonstrated the presence of a quantum error correcting code in 1995. In 1998, Calderbank et. al **[1]** published a paper in 1998 in which they established a theory for constructing quantum codes using classical error correcting codes. A substantial literature has sprung up around quantum error correcting codes in recent years. Using the Gray image of cyclic codes over some finite rings, some authors created quantum codes. For example, in **[5]**, Qian proposes a new method for constructing quantum codes from cyclic codes over the finite ring $F_2 + vF_2$ where $v^2 = v$. In **[2]** Dertli et. al. derive quantum codes from cyclic codes over

$F_2 + uF_2 + vF_2 + uvF_2$. In **[7]**, Ashraf and Mohammad present a quantum code construction based on cyclic codes over $F_3 + vF_3$ where $v^2 = 1$. Ozen et al. **[9]** investigated several ternary quantum codes derived from the cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$ In 2016. Many researchers have recently obtained new quantum codes over F_p derived from classical cyclic and constacyclic codes, which we refer to as **[3, 4, 6, 8]**.

The rest of the paper is organised as follows: Section 2 is Preliminaries in which some fundamental properties and definitions are given. Section 3 includes Gray Map from the ring $\mathfrak{R}$ to Z_p^4 as well as some gray and ring related details. we discussed the development of quantum codes using constacyclic codes over $\mathfrak{R}$ in section 4, which are exemplified in section 5. Finally, the given paper is concluded in last section.

2. Preliminaries

Let Z_p be a finite filed with p elements for $p > 2$. Now, we first start with a general overview of the ring $\mathfrak{R} = Z_p[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ having characteristic p with restrictions $\nu^2 = 1, \omega^2 = 1, \gamma^2 = 1, \nu\omega = \nu\omega, \omega\gamma = \gamma\omega, \gamma\nu = \nu\gamma$. $\mathfrak{R}$ is a commutative but non-chain finite ring with p^8 elements.

Some of the units of $\mathfrak{R}$ is $\nu, \omega, \nu\omega$ for shake of simplicity we consider ϑ as unit of $\mathfrak{R}$ and also we note that $\vartheta^{-1} = \vartheta$ for each case.

Let us assume $\alpha \in Z_p$ such that $8\alpha \equiv 1 \bmod p$ and

$$\varrho_1 = \alpha(1 + \nu + \omega + \gamma + \nu\omega + \omega\gamma + \gamma\nu + \nu\omega\gamma),$$

$$\varrho_2 = \alpha(1 + \nu + \omega - \gamma + \nu\omega - \omega\gamma - \gamma\nu - \nu\omega\gamma),$$

$$\varrho_3 = \alpha(1 + \nu - \omega + \gamma - \nu\omega - \omega\gamma + \gamma\nu - \nu\omega\gamma),$$

$$\varrho_4 = \alpha(1 - \nu + \omega + \gamma - \nu\omega + \omega\gamma - \gamma\nu - \nu\omega\gamma),$$

$$\varrho_5 = \alpha(1 + \nu - \omega - \gamma - \nu\omega + \omega\gamma - \gamma\nu + \nu\omega\gamma),$$

$$\varrho_6 = \alpha(1 - \nu - \omega + \gamma + \nu\omega - \omega\gamma - \gamma\nu + \nu\omega\gamma),$$

$$\varrho_7 = \alpha(1 - \nu + \omega - \gamma - \nu\omega - \omega\gamma + \gamma\nu + \nu\omega\gamma),$$

$$\varrho_8 = \alpha(1 - \nu - \omega - \gamma + \nu\omega + \omega\gamma + \gamma\nu - \nu\omega\gamma).$$

It is obvious to obtain that $\varrho_i^2 = \varrho_i$, $\varrho_i\varrho_j = 0$ and $\sum_{i=1}^{8} \varrho_i = 1$ for all $i, j = 1,2,\ldots,8$ and $i \neq j$. Now by chinese remainder theorem, the considered ring can be expressed as

$$\Re = \varrho_1 Z_p \oplus \varrho_2 Z_p \oplus \varrho_3 Z_p \oplus \varrho_4 Z_p \oplus \varrho_5 Z_p \oplus \varrho_6 Z_p \oplus \varrho_7 Z_p \oplus \varrho_8 Z_p.$$

Therefore, an arbitrary element $e = e_1 + e_2\nu + e_3\omega + e_4\gamma + e_5\nu\omega + e_6\omega\gamma + e_7\gamma\nu + e_8\nu\omega\gamma$ of $\Re$ where $e_i \in Z_p$ can be uniquely expressed as

$$e = \varrho_1 k_1 + \varrho_2 k_2 + \varrho_3 k_3 + \varrho_4 k_4 + \varrho_5 k_5 + \varrho_6 k_6 + \varrho_7 k_7 + \varrho_8 k_8$$

where $k_i \in Z_p$ for all $i = 1,2,\ldots,8$.

A nonempty subset $\mathcal{K}$ of $\Re^n$ is a linear code over $\Re$ of length n. If $\mathcal{K}$ is an $\Re$-submodule of $\Re^n$ and the elements of $\mathcal{K}$ are codewords. Let $\mathcal{K}$ be a code over $\Re$ of length n and its polynomial representation be $T(\mathcal{K})$, that is,

$$T(\mathcal{K}) = \{\sum_{i=0}^{n-1} \chi_i t^i \mid (\chi_0, \chi_1, \ldots, \chi_{n-1}) \in \mathcal{K}\}$$

Let Υ, Λ and $\mho$ are the maps from $\Re^n$ to $\Re^n$ defined as

$$\Upsilon(\chi_0, \chi_1, \ldots, \chi_{n-1}) = (\chi_{n-1}, \chi_0, \ldots, \chi_{n-2}),$$

$$\Lambda(\chi_0, \chi_1, \ldots, \chi_{n-1}) = (-\chi_{n-1}, \chi_0, \ldots, \chi_{n-2}),$$

$$\mho(\chi_0, \chi_1, \ldots, \chi_{n-1}) = (\vartheta\chi_{n-1}, \chi_0, \ldots, \chi_{n-2}),$$

respectively. Then $\mathcal{K}$ is a cyclic, negacyclic, ϑ-constacyclic if $\Upsilon(\mathcal{K}) = \mathcal{K}$, $\Lambda(\mathcal{K}) = \mathcal{K}$, $\mho(\mathcal{K}) = \mathcal{K}$ respectively. A code $\mathcal{K}$ over $\Re$ of length n is cyclic, negacyclic and ϑ-constacyclic if and only if $T(\mathcal{K})$ is an ideal of $\Re[t]/< t^n - 1 >$, $\Re[t]/< t^n + 1 >$ and $\Re[t]/< t^n - \vartheta >$ respectively.

For the arbitrary elements $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1})$ and $\psi = (\psi_0, \psi_1, \ldots, \psi_{n-1})$ of $\Re$, the inner product is defined as

$$\chi.\psi = \sum_{i=0}^{n-1} \chi_i \psi_i.$$

If $\chi.\psi = 0$, then χ and ψ are orthogonal. If $\mathcal{K}$ is a linear code over $\Re$ of length n, then the dual code of $\mathcal{K}$ is defined as

$$\mathcal{K}^\perp = \{\chi \in \Re^n : \chi.\psi = 0 \text{ for all } \psi \in \mathcal{K}\}.$$

which is also a linear code over the ring $\Re$ of length n. A code $\mathcal{K}$ is said to be self orthogonal if $\mathcal{K} \subseteq \mathcal{K}^\perp$ and said to be self dual if $\mathcal{K} = \mathcal{K}^\perp$.

3. Gray Map over $\mathfrak{R}$

The hamming weight $w_H(\chi)$ for any codeword $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1}) \in \mathfrak{R}^n$ is defined as the number of all non-zero components in $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1})$. The minimum weight of a code $\mathcal{K}$, that is, $w_H(\mathcal{K})$ is the least weight among all of its non zero codewords. The Hamming distance between two codes $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1})$ and $\hat{\chi} = (\hat{\chi}_0, \hat{\chi}_1, \ldots, \hat{\chi}_{n-1})$ of $\mathfrak{R}^n$, denoted by $d_H(\chi, \hat{\chi}) = w_H(\chi - \hat{\chi})$ and is defined as

$$d_H(\chi, \psi) = |\{i \mid \chi_i \neq \psi_i\}|.$$

Minimum distance of $\mathcal{K}$, denoted by d_H and is given by minimum distance between the different pairs of codewords of the linear code $\mathcal{K}$. For any codeword $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1}) \in \mathfrak{R}^n$, the lee weight is defined as $w_L(\chi) = \sum_{i=0}^{n-1} w_L(\chi_i)$ and lee distance of $(\chi, \hat{\chi})$ is given by $d_L(\chi, \hat{\chi}) = w_L(\chi - \hat{\chi}) = \sum_{i=0}^{n-1} w_L(\chi_i - \hat{\chi}_i)$.

Minimum lee distance of $\mathcal{K}$ is denoted by d_L and is given by minimum lee distance of different pairs of codewords of the linear code $\mathcal{K}$.

The Gray map φ from $\mathfrak{R}$ to Z_p^8, that is, $\varphi\colon \mathfrak{R} \to Z_p^8$ is defined as

$$\varphi(k = \sum_{i=1}^{8} \varrho_i k_i) = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8).$$

Where

$$\beta_1 = (k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8),$$
$$\beta_2 = (k_1 + k_2 + k_3 - k_4 + k_5 - k_6 - k_7 - k_8),$$
$$\beta_3 = (k_1 + k_2 - k_3 + k_4 - k_5 - k_6 + k_7 - k_8),$$
$$\beta_4 = (k_1 - k_2 + k_3 + k_4 - k_5 + k_6 - k_7 - k_8),$$
$$\beta_5 = (k_1 + k_2 - k_3 - k_4 - k_5 + k_6 - k_7 + k_8),$$
$$\beta_6 = (k_1 - k_2 - k_3 + k_4 + k_5 - k_6 - k_7 + k_8),$$
$$\beta_7 = (k_1 - k_2 + k_3 - k_4 - k_5 - k_6 + k_7 + k_8),$$
$$\beta_8 = (k_1 - k_2 - k_3 - k_4 + k_5 + k_6 + k_7 - k_8).$$

Theorem 3.1 The Gray map φ is linear and distance preserving isometry map from $(\mathfrak{R}^n, d_L)$ to (Z_p^{8n}, d_H), where d_L and d_H are the lee distance and hamming distance in $\mathfrak{R}^n$ and Z_p^{8n} respectively.

Proof. Let $k_1, k_2 \in \mathfrak{R}$ and $\alpha \in Z_p$ then

$$\varphi(\kappa k_1 + k_2) = \kappa\varphi(k_1) + \varphi(k_1)$$

So, φ is linear map.

Now we show that φ is distance preserving map.

By the above definitions, $d_L(\chi,\hat{\chi}) = w_L(\chi-\hat{\chi}) = w_H(\varphi(\chi-\hat{\chi})) = w_H(\varphi(\chi)-\phi(\hat{\chi})) = d_H(\varphi(\chi),\varphi(\hat{\chi}))$.

Hence φ is distance preserving map from $(\mathfrak{R}^n, d_L)$ to (Z_p^{8n}, d_H).

Theorem 3.2 If $\mathcal{K}$ is a linear code over the ring $\mathfrak{R}$ of length n with $|\mathcal{K}| = p^k$, $d_L(\mathcal{K}) = d$, then $\varphi(\mathcal{K})$ is a linear code having parameters $[8n,k,d]$.

Theorem 3.3 Let $\mathcal{K}$ be a linear code over the ring $\mathfrak{R}$ of length n. If $\mathcal{K}$ is self orthogonal, then $\varphi(\mathcal{K})$ is also self orthogonal.

Proof. Let $\mathcal{K}$ be a self orthogonal code and $\eta_1,\eta_2 \in \mathcal{K}$ such that $\eta_1 = \sum_{i=1}^{8} \varrho_i k_i$ and $\eta_2 = \sum_{i=1}^{8} \varrho_i k_i'$ where $k_i, k_i' \in Z_p$ for $i = 1,2,\ldots,8$ from the definition of self orthogonality, $\eta_1.\eta_2 = 0$, that is, $\sum_{i=1}^{8} \varrho_i k_i k_i' = 0$, it follows that $k_i k_i' = 0$ for $i = 1,2,\ldots,8$. Now, applying φ on η_1,η_2 we have $\varphi(\eta_1).\varphi(\eta_2) = \sum_{i=1}^{8} 8k_i k_i' = 0$ that implies $\varphi(\mathcal{K})$ is self orthogonal.

Theorem 3.4 [4] Let $\mathcal{K}$ be a linear code over the ring $\mathfrak{R}$ of length n. Then $\varphi(\mathcal{K}^{\perp}) = (\varphi(\mathcal{K}))^{\perp}$. Further, $\mathcal{K}$ is self dual if and only if $\varphi(\mathcal{K})$ is self dual.

4. Quantum codes obtained through ϑ-constacyclic codes

Let S_i's be the linear codes for $i = 1,2,\ldots,8$. we denote

$$S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus S_5 \oplus S_6 \oplus S_7 \oplus S_8 = \{s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8 | s_i \in S_i \text{ for } i = 1,2,\ldots,8\}$$

and

$$S_1 \otimes S_2 \otimes S_3 \otimes S_4 \otimes S_5 \otimes S_6 \otimes S_7 \otimes S_8 = \{(s_1,s_2,s_3,s_4\ s_5,s_6,s_7,s_8) | s_i \in S_i \text{ for } i = 1,2,\ldots,8\}$$

For a linear code $\mathcal{K}$ of length n over $\mathfrak{R}$, we define

$$K_1 = \{s_1 \in Z_p^n \text{ such that } \textstyle\sum_{i=1}^{8} s_i\varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 1 \text{ and } 1 \leq j \leq 8\},$$

$$K_2 = \{s_2 \in Z_p^n \text{ such that } \textstyle\sum_{i=1}^{8} s_i\varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 2 \text{ and } 1 \leq j \leq 8\},$$

$$K_3 = \{s_3 \in Z_p^n \text{ such that } \textstyle\sum_{i=1}^{8} s_i\varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 3 \text{ and } 1 \leq j \leq 8\},$$

$K_4 = \{s_4 \in Z_p^n \text{ such that } \sum_{i=1}^{8} s_i \varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 4 \text{ and } 1 \leq j \leq 8\},$

$K_5 = \{s_5 \in Z_p^n \text{ such that } \sum_{i=1}^{8} s_i \varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 5 \text{ and } 1 \leq j \leq 8\},$

$K_6 = \{s_6 \in Z_p^n \text{ such that } \sum_{i=1}^{8} s_i \varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 6 \text{ and } 1 \leq j \leq 8\},$

$K_7 = \{s_7 \in Z_p^n \text{ such that } \sum_{i=1}^{8} s_i \varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 7 \text{ and } 1 \leq j \leq 8\},$

$K_8 = \{s_8 \in Z_p^n \text{ such that } \sum_{i=1}^{8} s_i \varrho_i \in \mathcal{K}, \text{for some } k_j \in Z_p^n, j \neq 8 \text{ and } 1 \leq j \leq 8\}.$

Clearly, $K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8$ are the linear codes over Z_p of length n.

Theorem 4.1 [4] Let $\mathcal{K}$ be a linear code over the ring $\mathfrak{R}$ of length n. Then $\varphi(\mathcal{K}) = K_1 \otimes K_2 \otimes K_3 \otimes K_4 \otimes K_5 \otimes K_6 \otimes K_7 \otimes K_8$ and $|\mathcal{K}| = |K_1||K_2||K_3|K_4||K_5||K_6||K_7|K_8|$.

Corollary 4.2 [4] If $\varphi(\mathcal{K}) = K_1 \otimes K_2 \otimes K_3 \otimes K_4 \otimes K_5 \otimes K_6 \otimes K_7 \otimes K_8$ then $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$.

By the help of Theorem 4.1 and Corollary 4.2, we say that the linear code $\mathcal{K}$ can be uniquely expressed as the linear code $\mathcal{K}$ can be uniquely expressed as

$$\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$$

and also

$$|\mathcal{K}| = |K_1||K_2||K_3|K_4||K_5||K_6||K_7|K_8|.$$

If $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ and G_8 are the generator matrices of the linear codes $K_1, K_2, K_3, K_4, K_5, K_6, K_7$ and K_8 respectively. Then, the generator matrix of $\mathcal{K}$ is

$$G = \begin{bmatrix} \varrho_1 G_1 & \varrho_2 G_2 & \varrho_3 G_3 & \varrho_4 G_4 & \varrho_5 G_5 & \varrho_6 G_6 & \varrho_7 G_7 & \varrho_8 G_8 \end{bmatrix}^T,$$

and that of $\varphi(\mathcal{K})$ is

$$\varphi(G) = \begin{bmatrix} \varphi(\varrho_1 G_1) & \varphi(\varrho_2 G_2) & \varphi(\varrho_3 G_3) & \varphi(\varrho_4 G_4) & \varphi(\varrho_5 G_5) & \varphi(\varrho_6 G_6) & \varphi(\varrho_7 G_7) & \varphi(\varrho_8 G_8) \end{bmatrix}^T$$

Note: Now, we consider different cases of ϑ.

Case 1. $\vartheta = \nu\omega$

Theorem 4.3 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ be a linear code over the ring $\mathfrak{R}$ of length n where K_i for $i = 1,2,\ldots 8$ are the linear code over Z_p. Then, $\mathcal{K}$ is a $\nu\omega$-constacyclic codes over the ring $\mathfrak{R}$ of length n if and only if K_i for $i = 1,2,6,8$ are cyclic and K_j for $j = 3,4,5,7$ are negacyclic codes over Z_p of length n.

Proof. Let,

$$\theta^i = (\theta_0^i, \theta_1^i, \ldots, \theta_{n-1}^i) \in K_i, \text{ for } i = 1,2,\ldots,8.$$

For an arbitrary element $\zeta_i \in \mathcal{K}$, uniquely expressed as

$$\zeta_j = \varrho_1\theta_j^1 + \varrho_2\theta_j^2 + \varrho_3\theta_j^3 + \varrho_4\theta_j^4 + \varrho_5\theta_j^5 + \varrho_6\theta_j^6 + \varrho_7\theta_j^7 + \varrho_8\theta_j^8 = \sum_{i=1}^{8} \varrho_i\theta_j^i,$$

where $\theta_j^i \in Z_p$ for $j = 0,1,\ldots,n-1$.

Let,

$$\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \mathfrak{R}^n.$$

First we assume that $\mathcal{K}$ is $\nu\omega$-constacyclic codes over the ring $\mathfrak{R}$ of length n, then

$$\begin{aligned}
\mho(\zeta) &= ((\nu\omega)\zeta_{n-1}, \zeta_0, \ldots, \zeta_{n-2}) \\
&= (\varrho_1\theta_{n-1}^1 + \varrho_2\theta_{n-1}^2 - \varrho_3\theta_{n-1}^3 - \varrho_4\theta_{n-1}^4 - \varrho_5\theta_{n-1}^5 + \varrho_6\theta_{n-1}^6 - \varrho_7\theta_{n-1}^7 + \varrho_8\theta_{n-1}^8, \\
&\quad \varrho_1\theta_1^1 + \varrho_2\theta_1^2 + \varrho_3\theta_1^3 + \varrho_4\theta_1^4 + \varrho_5\theta_1^5 + \varrho_6\theta_1^6 + \varrho_7\theta_7^1 + \varrho_8\theta_1^8, \ldots, \\
&\quad \varrho_1\theta_{n-2}^1 + \varrho_2\theta_{n-2}^2 + \varrho_3\theta_{n-2}^3 + \varrho_4\theta_{n-2}^4 + \varrho_5\theta_{n-2}^5 + \varrho_6\theta_{n-2}^6 + \varrho_1\theta_{n-2}^1 + \varrho_8\theta_{n-2}^8) \\
&= \varrho_1\Upsilon(\theta^1) + \varrho_2\Upsilon(\theta^2) + \varrho_3\Lambda(\theta^3) + \varrho_4\Lambda(\theta^4) + \varrho_5\Lambda(\theta^5) + \varrho_6\Upsilon(\theta^6) + \varrho_7\Lambda(\theta^7) + \varrho_8\Upsilon(\theta^8)
\end{aligned}$$

which is an element of the linear code $\mathcal{K}$. Therefore, K_i for $i = 1,2,6,8$ are cyclic and K_j for $j = 3,4,5,7$ are negacyclic codes over the ring Z_p of length n respectively.

Conversely, for any $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \mathcal{K}$, where $\zeta_j = \sum_{i=1}^{8} \varrho_i\theta_j^i$, and where $\theta_j^i \in Z_p$ for $j = 0,1,\ldots,n-1$. If K_i for $i = 1,2,6,8$ are cyclic and K_j for $j = 3,4,5,7$ are negacyclic codes over the ring Z_p of length n respectively, then $\Upsilon(\theta^1) \in K_1, \Upsilon(\theta^2) \in K_2, \Lambda(\theta^3) \in K_3, \Lambda(\theta^4) \in K_4, \Lambda(\theta^5) \in K_5, \Upsilon(\theta^6) \in K_6, \Lambda(\theta^7) \in K_7, \Upsilon(\theta^8) \in K_8$.

Hence, we have

$$\varrho_1\Upsilon(\theta^1)+\varrho_2\Upsilon(\theta^2)+\varrho_3\Lambda(\theta^3)+\varrho_4\Lambda(\theta^4)+\varrho_5\Lambda(\theta^5)+\varrho_6\Upsilon(\theta^6)+\varrho_7\Lambda(\theta^7)+\varrho_8\Upsilon(\theta^8) \in \mathcal{K}$$

where it given that

$$\mho(\zeta) = \varrho_1\Upsilon(\theta^1)+\varrho_2\Upsilon(\theta^2)+\varrho_3\Lambda(\theta^3)+\varrho_4\Lambda(\theta^4)+\varrho_5\Lambda(\theta^5)+\varrho_6\Upsilon(\theta^6)+\varrho_7\Lambda(\theta^7) +\varrho_8\Upsilon(\theta^8),$$

which implies that $\mho(\zeta) \in \mathcal{K}$.

Therefore, $\mathcal{K}$ is a $\nu\omega$-constacyclic codes over the ring $\mathfrak{R}$ of length n .

Case 2. $\vartheta = \nu$

Theorem 4.4 Let $\mathcal{K} = \varrho_1K_1 \oplus \varrho_2K_2 \oplus \varrho_3K_3 \oplus \varrho_4K_4 \oplus \varrho_5K_5 \oplus \varrho_6K_6 \oplus \varrho_7K_7 \oplus \varrho_8K_8$ be a linear code over the ring $\mathfrak{R}$ of length n where K_i for $i = 1,2,\ldots 8$ are the linear code over Z_p. Then, $\mathcal{K}$ is a ν-constacyclic codes over the ring $\mathfrak{R}$ of length n if and only if K_i for $i = 1,2,3,5$ are cyclic and K_j for $j = 4,6,7,8$ are negacyclic codes over Z_p of length n.

Proof. Let,

$$\theta^i = (\theta^i_0,\theta^i_1,\ldots,\theta^i_{n-1}) \in K_i,\ \text{for } i = 1,2,\ldots,8.$$

For an arbitrary element $\zeta_i \in \mathcal{K}$, uniquely expressed as

$$\zeta_j = \varrho_1\theta^1_j+\varrho_2\theta^2_j+\varrho_3\theta^3_j+\varrho_4\theta^4_j+\varrho_5\theta^5_j+\varrho_6\theta^6_j+\varrho_7\theta^7_j+\varrho_8\theta^8_j = \textstyle\sum_{i=1}^{8}\varrho_i\theta^i_j,$$

where $\theta^i_j \in Z_p$ for $j = 0,1,\ldots,n-1$.

Let,

$$\zeta = (\zeta_0,\zeta_1,\ldots,\zeta_{n-1}) \in \mathfrak{R}^n.$$

First we assume that $\mathcal{K}$ is ν-constacyclic codes over the ring $\mathfrak{R}$ of length n, then

$$\begin{aligned}\mho(\zeta) &= ((\nu\omega)\zeta_{n-1},\zeta_0,\ldots,\zeta_{n-2})\\ &= (\varrho_1\theta^1_{n-1}+\varrho_2\theta^2_{n-1}+\varrho_3\theta^3_{n-1}-\varrho_4\theta^4_{n-1}+\varrho_5\theta^5_{n-1}-\varrho_6\theta^6_{n-1}-\varrho_7\theta^7_{n-1}-\\ &\quad \varrho_8\theta^8_{n-1},\ \varrho_1\theta^1_1+\varrho_2\theta^2_1+\varrho_3\theta^3_1+\varrho_4\theta^4_1+\varrho_5\theta^5_1+\varrho_6\theta^6_1+\varrho_7\theta^1_7+\varrho_8\theta^8_1,\ldots,\\ &\quad \varrho_1\theta^1_{n-2}+\varrho_2\theta^2_{n-2}+\varrho_3\theta^3_{n-2}+\varrho_4\theta^4_{n-2}+\varrho_5\theta^5_{n-2}+\varrho_6\theta^6_{n-2}+\varrho_1\theta^1_{n-2}+\\ &\quad \varrho_8\theta^8_{n-2})\\ &= \varrho_1\Upsilon(\theta^1)+\varrho_2\Upsilon(\theta^2)+\varrho_3\Upsilon(\theta^3)+\varrho_4\Lambda(\theta^4)+\varrho_5\Upsilon(\theta^5)+\varrho_6\Lambda(\theta^6)+\varrho_7\Lambda(\theta^7)\\ &\quad +\varrho_8\Lambda(\theta^8)\end{aligned}$$

which is an element of the linear code $\mathcal{K}$. Therefore, K_i for $i = 1,2,3,5$ are cyclic and K_j for $j = 4,6,7,8$ are negacyclic codes over the ring Z_p of length n respectively.

Conversely, for any $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \mathcal{K}$, where $\zeta_j = \sum_{i=1}^{8} \varrho_i \theta_j^i$, and where $\theta_j^i \in Z_p$ for $j = 0,1,\ldots,n-1$. If K_i for $i = 1,2,3,5$ are cyclic and K_j for $j = 4,6,7,8$ are negacyclic codes over the ring Z_p of length n respectively, then $\Upsilon(\theta^1) \in K_1, \Upsilon(\theta^2) \in K_2, \Upsilon(\theta^3) \in K_3$, $\Lambda(\theta^4) \in K_4, \Upsilon(\theta^5) \in K_5, \Lambda(\theta^6) \in K_6, \Lambda(\theta^7) \in K_7, \Lambda(\theta^8) \in K_8$.
Hence, we have

$$\varrho_1\Upsilon(\theta^1) + \varrho_2\Upsilon(\theta^2) + \varrho_3\Upsilon(\theta^3) + \varrho_4\Lambda(\theta^4) + \varrho_5\Upsilon(\theta^5) + \varrho_6\Lambda(\theta^6) + \varrho_7\Lambda(\theta^7) + \varrho_8\Lambda(\theta^8) \in \mathcal{K}$$

where it given that

$$\mho(\zeta) = \varrho_1\Upsilon(\theta^1) + \varrho_2\Upsilon(\theta^2) + \varrho_3\Upsilon(\theta^3) + \varrho_4\Lambda(\theta^4) + \varrho_5\Upsilon(\theta^5) + \varrho_6\Lambda(\theta^6) + \varrho_7\Lambda(\theta^7) + \varrho_8\Lambda(\theta^8),$$

which implies that $\mho(\zeta) \in \mathcal{K}$.
Therefore, $\mathcal{K}$ is a ν-constacyclic codes over the ring $\mathfrak{R}$ of length n .

Case 3. $\vartheta = \omega$

Theorem 4.5 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ be a linear code over the ring $\mathfrak{R}$ of length n where K_i for $i = 1,2,\ldots 8$ are the linear code over Z_p. Then, $\mathcal{K}$ is a ω-constacyclic codes over the ring $\mathfrak{R}$ of length n if and only if K_i for $i = 1,2,4,7$ are cyclic and K_j for $j = 3,5,6,8$ are negacyclic codes over Z_p of length n.

Proof. The proof of this theorem is similar to proof of Theorem 4.4.

The following Theorem is Similar to Theorem 7 **[4]**.

Theorem 4.6 Let $\mathcal{K}$ be a ϑ-constacyclic codes over the ring $\mathfrak{R}$ of length n. Then

$$\mathcal{K} = < \varrho_1 g_1(t), \varrho_2 g_2(t), \varrho_3 g_3(t), \varrho_4 g_4(t), \varrho_5 g_5(t), \varrho_6 g_6(t), \varrho_7 g_7(t), \varrho_8 g_8(t) >$$
$$= < \varrho_1 g_1(t) + \varrho_2 g_2(t) + \varrho_3 g_3(t) + \varrho_4 g_4(t) + \varrho_5 g_5(t) + \varrho_6 g_6(t) + \varrho_7 g_7(t) + \varrho_8 g_8(t) >$$

where $g_i(t)$ are the generator polynomials of K_i for $i = 1,2,\ldots,8$ respectively.
Moreover, $|\mathcal{K}| = p^{8n - \sum_{i=1}^{8} \deg(g_i(t))}$

Theorem 4.7 Let $\mathcal{K}$ be a ϑ-constacyclic codes over the ring $\mathfrak{R}$ of length n. Then $\mathcal{K}^\perp$ is also a ϑ-constacyclic codes over the ring $\mathfrak{R}$ of length n. Moreover,

1. $\mathcal{K}^{\perp} = \varrho_1 K_1^{\perp} \oplus \varrho_2 K_2^{\perp} \oplus \varrho_3 K_3^{\perp} \oplus \varrho_4 K_4^{\perp} \oplus \varrho_5 K_5^{\perp} \oplus \varrho_6 K_6^{\perp} \oplus \varrho_7 K_7^{\perp} \oplus \varrho_8 K_8^{\perp}$
2. $\mathcal{K}^{\perp} = < \varrho_1 g_1^{\star}(t), \varrho_2 g_2^{\star}(t), \varrho_3 g_3^{\star}(t), \varrho_4 g_4^{\star}(t), \varrho_5 g_5^{\star}(t), \varrho_6 g_6^{\star}(t), \varrho_7 g_7^{\star}(t), \varrho_8 g_8^{\star}(t) >$
$= < \varrho_1 g_1^{\star}(t) + \varrho_2 g_2^{\star}(t) + \varrho_3 g_3^{\star}(t) + \varrho_4 g_4^{\star}(t) + \varrho_5 g_5^{\star}(t) + \varrho_6 g_6^{\star}(t) + \varrho_7 g_7^{\star}(t) + \varrho_8 g_8^{\star}(t) >$
3. $|\mathcal{K}^{\perp}| = p^{\sum_{i=1}^{8} \deg(g_i(t))}$

where $g_i^{\star}(t)$ are the reciprocal polynomial of $\frac{x^n-1}{g_i(t)}$ $\frac{x^n+1}{g_j(t)}$ for different i and j for different case of ϑ respectively.

Lemma 4.8 [1] If $\mathcal{K}$ is a cyclic or negacyclic code over the ring Z_p with a generator polynomial g(t). Then, $\mathcal{K}$ contains its dual code if and only if

$$x^n - \iota \equiv 0 \bmod (g(t)g^{\star}(t))$$

where $\iota = \pm 1$.

Case 1. $\vartheta = \nu\omega$

Theorem 4.9 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ is a ϑ-constacyclic code over the ring $\mathfrak{R}$ of length n. Then, $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ if and only if

$$x^n - 1 \equiv 0 \bmod (g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t) g_j^{\star}(t)).$$

for $i = 1,2,6,8$ and $j = 3,4,5,7$.

Proof. Let $\mathcal{K} = < g(t) > = < \sum_{i=1}^{8} \varrho_i g_i(t) >$ be a ϑ-constacyclic code over $\mathfrak{R}$ of length n. Then, $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ where $g_i(t)$ are the generator polynomial of K_i for $i = 1,2,\ldots,8$ respectively.
First we consider

$$x^n - 1 \equiv 0 \bmod (g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t) g_j^{\star}(t)).$$

for $i = 1,2,6,8$ and $j = 3,4,5,7$. Then by above lemma, we have

$$K_1^{\perp} \subseteq K_1, K_2^{\perp} \subseteq K_2, K_3^{\perp} \subseteq K_3, K_4^{\perp} \subseteq K_4, K_5^{\perp} \subseteq K_5, K_6^{\perp} \subseteq K_6, K_7^{\perp} \subseteq K_7, K_8^{\perp} \subseteq K_8,$$

and therefore

$$\varrho_1 K_1^{\perp} \subseteq \varrho_1 K_1, \varrho_2 K_2^{\perp} \subseteq \varrho_2 K_2, \varrho_3 K_3^{\perp} \subseteq \varrho_3 K_3, \varrho_4 K_4^{\perp} \subseteq \varrho_4 K_4,$$
$$\varrho_5 K_5^{\perp} \subseteq \varrho_5 K_5, \varrho_6 K_6^{\perp} \subseteq \varrho_6 K_6, \varrho_7 K_7^{\perp} \subseteq \varrho_7 K_7, \varrho_8 K_8^{\perp} \subseteq \varrho_8 K_8$$

which implies that

$$\varrho_1 K_1^{\perp} \oplus \varrho_2 K_2^{\perp} \oplus \varrho_3 K_3^{\perp} \oplus \varrho_4 K_4^{\perp} \oplus \varrho_5 K_5^{\perp} \oplus \varrho_6 K_6^{\perp} \oplus \varrho_7 K_7^{\perp} \oplus \varrho_8 K_8^{\perp} \subseteq \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$$

Thus, we have

$$\mathcal{K}^{\perp} \subseteq \mathcal{K}.$$

Conversely let us consider

$$\mathcal{K}^{\perp} \subseteq \mathcal{K},$$

then

$$\varrho_1 K_1^{\perp} \oplus \varrho_2 K_2^{\perp} \oplus \varrho_3 K_3^{\perp} \oplus \varrho_4 K_4^{\perp} \oplus \varrho_5 K_5^{\perp} \oplus \varrho_6 K_6^{\perp} \oplus \varrho_7 K_7^{\perp} \oplus \varrho_8 K_8^{\perp} \subseteq \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$$

which implies that

$$\varrho_1 K_1^{\perp} \subseteq \varrho_1 K_1, \varrho_2 K_2^{\perp} \subseteq \varrho_2 K_2, \varrho_3 K_3^{\perp} \subseteq \varrho_3 K_3, \varrho_4 K_4^{\perp} \subseteq \varrho_4 K_4,$$
$$\varrho_5 K_5^{\perp} \subseteq \varrho_5 K_5, \varrho_6 K_6^{\perp} \subseteq \varrho_6 K_6, \varrho_7 K_7^{\perp} \subseteq \varrho_7 K_7, \varrho_8 K_8^{\perp} \subseteq \varrho_8 K_8$$

that implies

$$K_1^{\perp} \subseteq K_1, K_2^{\perp} \subseteq K_2, K_3^{\perp} \subseteq K_3, K_4^{\perp} \subseteq K_4, K_5^{\perp} \subseteq K_5, K_6^{\perp} \subseteq K_6, K_7^{\perp} \subseteq K_7, K_8^{\perp} \subseteq K_8,$$

Then by above lemma,

$$x^n - 1 \equiv 0 \bmod (g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t) g_j^{\star}(t)).$$

for $i = 1,2,6,8$ and $j = 3,4,5,7$.

Case 2. $\vartheta = \nu$

Theorem 4.10 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ is a ϑ-constacyclic code over the ring $\mathfrak{R}$ of length n. Then, $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ if and only if

$$x^n - 1 \equiv 0 \bmod (g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t)g_j^{\star}(t)).$$

for $i = 1,2,3,5$ and $j = 4,6,7,8$.

Proof. Let $\mathcal{K} = <g(t)> = <\sum_{i=1}^{8} \varrho_i g_i(t)>$ be a ϑ-constacyclic code over $\mathfrak{R}$ of length n. Then, $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ where $g_i(t)$ are the generator polynomial of K_i for $i = 1,2,\ldots,8$ respectively.
First we consider

$$x^n - 1 \equiv 0 \bmod (g_i(t)g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t)g_j^{\star}(t)).$$

for $i = 1,2,3,5$ and $j = 4,6,7,8$. Then by above lemma, we have

$$K_1^{\perp} \subseteq K_1, K_2^{\perp} \subseteq K_2, K_3^{\perp} \subseteq K_3, K_4^{\perp} \subseteq K_4, K_5^{\perp} \subseteq K_5, K_6^{\perp} \subseteq K_6, K_7^{\perp} \subseteq K_7, K_8^{\perp} \subseteq K_8,$$

and therefore

$$\varrho_1 K_1^{\perp} \subseteq \varrho_1 K_1, \varrho_2 K_2^{\perp} \subseteq \varrho_2 K_2, \varrho_3 K_3^{\perp} \subseteq \varrho_3 K_3, \varrho_4 K_4^{\perp} \subseteq \varrho_4 K_4,$$
$$\varrho_5 K_5^{\perp} \subseteq \varrho_5 K_5, \varrho_6 K_6^{\perp} \subseteq \varrho_6 K_6, \varrho_7 K_7^{\perp} \subseteq \varrho_7 K_7, \varrho_8 K_8^{\perp} \subseteq \varrho_8 K_8$$

which implies that

$$\varrho_1 K_1^{\perp} \oplus \varrho_2 K_2^{\perp} \oplus \varrho_3 K_3^{\perp} \oplus \varrho_4 K_4^{\perp} \oplus \varrho_5 K_5^{\perp} \oplus \varrho_6 K_6^{\perp} \oplus \varrho_7 K_7^{\perp} \oplus \varrho_8 K_8^{\perp} \subseteq \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$$

Thus, we have

$$\mathcal{K}^{\perp} \subseteq \mathcal{K}.$$

Conversely let us consider

$$\mathcal{K}^{\perp} \subseteq \mathcal{K},$$

then

$$\varrho_1 K_1^{\perp} \oplus \varrho_2 K_2^{\perp} \oplus \varrho_3 K_3^{\perp} \oplus \varrho_4 K_4^{\perp} \oplus \varrho_5 K_5^{\perp} \oplus \varrho_6 K_6^{\perp} \oplus \varrho_7 K_7^{\perp} \oplus \varrho_8 K_8^{\perp} \subseteq \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$$

which implies that

$$\varrho_1 K_1^{\perp} \subseteq \varrho_1 K_1, \varrho_2 K_2^{\perp} \subseteq \varrho_2 K_2, \varrho_3 K_3^{\perp} \subseteq \varrho_3 K_3, \varrho_4 K_4^{\perp} \subseteq \varrho_4 K_4,$$

$$\varrho_5 K_5^{\perp} \subseteq \varrho_5 K_5, \varrho_6 K_6^{\perp} \subseteq \varrho_6 K_6, \varrho_7 K_7^{\perp} \subseteq \varrho_7 K_7, \varrho_8 K_8^{\perp} \subseteq \varrho_8 K_8$$

that implies

$K_1^{\perp} \subseteq K_1, K_2^{\perp} \subseteq K_2, K_3^{\perp} \subseteq K_3, K_4^{\perp} \subseteq K_4, K_5^{\perp} \subseteq K_5, K_6^{\perp} \subseteq K_6, K_7^{\perp} \subseteq K_7, K_8^{\perp} \subseteq K_8$,

Then by above lemma,

$$x^n - 1 \equiv 0 \bmod (g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t) g_j^{\star}(t)).$$

for $i = 1,2,3,5$ and $j = 4,6,7,8$.

Case 3. $\vartheta = \omega$

Theorem 4.11 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ is a ϑ-constacyclic code over the ring $\mathfrak{R}$ of length n. Then, $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ if and only if

$$x^n - 1 \equiv 0 \bmod (g_i(t) g_i^{\star}(t))$$

and

$$x^n + 1 \equiv 0 \bmod (g_j(t) g_j^{\star}(t)).$$

for $i = 1,2,4,7$ and $j = 3,5,6,8$.

Proof. The proof of this theorem is similar to proof of Theorem 4.10.

By above Theorems, we have the following Corollary.

Corollary 4.12 Let $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8$ be a ϑ-constacyclic code over $\mathfrak{R}$ of length n where K_i for $i = 1,2,\ldots 8$ are the linear code over Z_p. Then, $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ if and only if $K_i^{\perp} \subseteq K_i$ for $i = 1,2,\ldots,8$.

Lemma 4.13 [1](CSS Construction) Let $\mathcal{K}$ be a linear code over the ring Z_p having parameters $[n, k, d]$. Then a quantum code having parameters $[n, 2k - n, \geq d]_3$ can be obtained if $\mathcal{K}^{\perp} \subseteq \mathcal{K}$.

The following theorem defines the construction of quantum codes by the use of Corollary 4.12 and Lemma 4.13.

Theorem 4.14 If $\mathcal{K} = \varrho_1 K_1 \oplus \varrho_2 K_2 \oplus \varrho_3 K_3 \oplus \varrho_4 K_4 \oplus \varrho_5 K_5 \oplus \varrho_6 K_6 \oplus \varrho_7 K_7 \oplus \varrho_8 K_8 = < \varrho_1 g_1(t), \varrho_2 g_2(t), \varrho_3 g_3(t), \varrho_4 g_4(t), \varrho_5 g_5(t), \varrho_6 g_6(t), \varrho_7 g_7(t), \varrho_8 g_8(t) >$ is a ϑ-constacyclic code over the ring $\mathfrak{R}$ of length n where $g_i(t)$ are the generator polynomials of K_i for $i = 1,2,\ldots,8$ respectively. If $K_i^{\perp} \subseteq K_i$ for $i = 1,2,\ldots,8$, then $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ and there exists a quantum code having parameters $[8n, 2k - 8n, \geq d_L]_p$ where k is the dimension of linear code $\varphi(\mathcal{K})$ and d_L is minimum lee distance of a linear code $\mathcal{K}$.

5. Examples

In this section some examples are provided to illustrate the main result. Here, the quantum codes through ϑ-constacyclic codes over the ring $\mathfrak{R} = Z_p[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ where $\nu^2 = 1, \omega^2 = 1, \gamma^2 = 1, \nu\omega = \nu\omega, \omega\gamma = \gamma\omega, \gamma\nu = \nu\gamma$ are obtains.

Example 5.1 In $Z_3(t)$, $t^{15} - 1 = (t+2)^3(t^4 + t^3 + t^2 + t + 1)^3$ and $t^{15} + 1 = (t - 2)^3(t^4 + 2t^3 + t^2 + 2t + 1)^3$. Now, let $\mathcal{K}$ be a $\nu\omega$-constacyclic code over $\mathfrak{R} = Z_3[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ of length 15. Let $g_1(t) = g_2(t) = g_6(t) = g_8(t) = t + 2$ and $g_3(t) = g_4(t) = g_5(t) = g_7(t) = (t - 2)^2$ then $g(t) = \varrho_1(t+2) + \varrho_2(t+2) + \varrho_3(t-2)^2 + \varrho_4(t-2)^2 + \varrho_5(t-2)^2 + \varrho_6(t+2) + \varrho_7(t-2)^2 + \varrho_8(t+2)$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{15} - 1$ for $i = 1,2,6,8$ respectively and $g_j(t)g_j^*(t)|t^{15} + 1$ for $j = 3,4,5,7$ respectively, then by the use of Theorem 4.9, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over Z_3 having parameters $[120, 108, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[120, 96, \geq 3]_3$.

Example 5.2 In $Z_3(t)$, $t^{21} - 1 = (t-1)^3(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)^3$ and $t^{21} + 1 = (t+1)^3(t^6 + 2t^5 + t^4 + 2t^3 + t^2 + 2t + 1)^3$. Now, let $\mathcal{K}$ be a ν-constacyclic codes over the ring $\mathfrak{R} = Z_3[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ of length 21. Let $g_1(t) = g_2(t) = g_3(t) = g_5(t) = t - 1$ and $g_4(t) = g_6(t) = g_7(t) = g_8(t) = (t+1)^2$ then $g(t) = \varrho_1(t-1) + \varrho_2(t-1) + \varrho_3(t-1) + \varrho_4(t+1)^2 + \varrho_5(t-1) + \varrho_6(t+1)^2 + \varrho_7(t+1)^2 + \varrho_8(t+1)^2$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{21} - 1$ for $i = 1,2,3,5$ respectively and $g_j(t)g_j^*(t)|t^{21} + 1$ for $j = 4,6,7,8$ respectively, then by the use of Theorem 4.10, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_3 having parameters $[168, 156, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[168, 144, \geq 3]_3$.

Example 5.3 In $Z_5(t)$, $t^{15} - 1 = (t+4)^5(t^2+t+1)^5$ and $t^{15} + 1 = (t-4)^5(t^2+4t+1)^5$. Now, let $\mathcal{K}$ be a ω-constacyclic codes over the ring $\mathfrak{R} = Z_5[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ of length 15. Let $g_1(t) = g_2(t) = g_4(t) = g_7(t) = (t+4)^2$ and $g_3(t) = g_5(t) = g_6(t) = g_8(t) = (t-4)^2$ then $g(t) = \varrho_1(t+4)^2 + \varrho_2(t+4)^2 + \varrho_3(t-4)^2 + \varrho_4(t+4)^2 + \varrho_5(t-4)^2 + \varrho_6(t-4)^2 + \varrho_7(t+4)^2 + \varrho_8(t-4)^2$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{15} - 1$ for $i = 1,2,4,7$ respectively and $g_j(t)g_j^*(t)|t^{15} + 1$ for $j = 3,5,6,8$ respectively, then by the use of Theorem 4.11, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_5 having parameters $[120, 104, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[120, 88, \geq 3]_5$.

Example 5.4 In $Z_5(t)$, $t^{20} - 1 = (t+1)^5(t+2)^5(t+3)^5(t+4)^5$ and $t^{20} + 1 = (t^2+2)^5(t^2+3)^5$. Now, let $\mathcal{K}$ be a $\nu\omega$-constacyclic codes over the ring $\mathfrak{R} = Z_5[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ of length 20. Let $g_1(t) = g_2(t) = g_6(t) = g_8(t) = (t+4)^2$ and $g_3(t) = g_4(t) = g_5(t) = g_7(t) = (t^2+2)$ then $g(t) = \varrho_1(t+4)^2 + \varrho_2(t+4)^2 + \varrho_3(t^2+2) + \varrho_4(t^2+2) + \varrho_5(t^2+2) + \varrho_6(t+4)^2 + \varrho_7(t^2+2) + \varrho_8(t+4)^2$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{20} - 1$ for $i = 1,2,6,8$ respectively and $g_j(t)g_j^*(t)|t^{20} + 1$ for $j = 3,4,5,7$ respectively, then by the use of Theorem 4.9, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_5 having parameters $[160, 144, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[160, 128, \geq 3]_5$.

Example 5.5 In $Z_5(t)$, $t^{30} - 1 = (t+1)^5(t+4)^5(t^2+t+1)^5(t^2+4t+1)^5$ and $t^{30} + 1 = (t+2)^5(t+3)^5(t^2+2t+4)^5(t^2+3t+4)^5$. Now, let $\mathcal{K}$ be a ω-constacyclic codes over the ring $\mathfrak{R} = Z_5[\nu, \omega, \gamma]/< \nu^2 - 1, \omega^2 - 1, \gamma^2 - 1, \nu\omega - \omega\nu, \omega\gamma - \gamma\omega, \gamma\nu - \nu\gamma >$ of length 30. Let $g_1(t) = g_2(t) = g_4(t) = g_7(t) = t^2+t+1$ and $g_3(t) = g_5(t) = g_6(t) = g_8(t) = t^2+3t+4$ then $g(t) = \varrho_1(t^2+t+1) + \varrho_2(t^2+t+1) + \varrho_3(t^2+3t+4) + \varrho_4(t^2+t+1) + \varrho_5(t^2+3t+4) + \varrho_6(t^2+3t+4) + \varrho_7(t^2+t+1) + \varrho_8(t^2+3t+4)$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{30} - 1$ for $i = 1,2,4,7$

respectively and $g_j(t)g_j^*(t)|t^{30}+1$ for $j = 3,5,6,8$ respectively, then by the use of Theorem 4.11, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_5 having parameters $[240, 224, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[120, 208, \geq 3]_5$.

Example 5.6 In $Z_7(t)$, $t^{20}-1 = (t+1)(t+6)(t^2+1)(t^4+t^3+t^2+t+1)(t^4+3t^3+4t^2+4t+1)(t^4+4t^3+4t^2+3t+1)(t^4+6t^3+t^2+6t+1)$ and $t^{20}+1 = (t^2+3t+1)(t^2+4t+1)(t^4+t^3+6t^2+3t+1)(t^4+3t^3+6t^2+t+1)(t^4+4t^3+6t^2+6t+1)(t^4+6t^3+6t^2+4t+1)$. Now, let $\mathcal{K}$ be a ν-constacyclic codes over the ring $\mathfrak{R} = Z_7[\nu,\omega,\gamma]/<\nu^2-1,\omega^2-1,\gamma^2-1,\nu\omega-\omega\nu,\omega\gamma-\gamma\omega,\gamma\nu-\nu\gamma>$ of length 20. Let $g_1(t) = g_2(t) = g_3(t) = g_5(t) = t+6$ and $g_4(t) = g_6(t) = g_7(t) = g_8(t) = t^2+3t+1$ then $g(t) = \varrho_1(t+6)+\varrho_2(t+6)+\varrho_3(t+6)+\varrho_4(t^2+3t+1)+\varrho_5(t+6)+\varrho_6(t^2+3t+1)+\varrho_7(t^2+3t+1)+\varrho_8(t^2+3t+1)$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{20}-1$ for $i = 1,2,3,5$ respectively and $g_j(t)g_j^*(t)|t^{20}+1$ for $j = 4,6,7,8$ respectively, then by the use of Theorem 4.10, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_7 having parameters $[160, 148, 2]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[160, 136, \geq 2]_7$.

Example 5.7 In $Z_7(t)$, $t^{21}-1 = (t+3)^7(t+5)^7(t+6)^7$ and $t^{20}+1 = (t+1)^7(t+2)^7(t+4)^7$. Now, let $\mathcal{K}$ be a $\nu\omega$-constacyclic codes over the ring $\mathfrak{R} = Z_7[\nu,\omega,\gamma]/<\nu^2-1,\omega^2-1,\gamma^2-1,\nu\omega-\omega\nu,\omega\gamma-\gamma\omega,\gamma\nu-\nu\gamma>$ of length 15. Let $g_1(t) = g_2(t) = g_6(t) = g_8(t) = (t+3)^2$ and $g_3(t) = g_4(t) = g_5(t) = g_7(t) = (t+4)^2$ then $g(t) = \varrho_1(t+3)^2+\varrho_2(t+3)^2+\varrho_3(t+4)^2+\varrho_4(t+4)^2+\varrho_5(t+4)^2+\varrho_6(t+3)^2+\varrho_7(t+4)^2+\varrho_8(t+3)^2$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{21}-1$ for $i = 1,2,6,8$ respectively and $g_j(t)g_j^*(t)|t^{21}+1$ for $j = 3,4,5,7$ respectively, then by the use of Theorem 4.9, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over Z_7 having parameters $[168, 152, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[168, 136, \geq 3]_7$.

Example 5.8 In $Z_{11}(t)$, $t^{18}-1 = (t+1)(t+10)(t^2+t+1)(t^2+10t+1)(t^6+t^3+1)(t^6+10t^3+1)$ and $t^{18}+1 = (t^2+1)(t^2+5t+1)(t^2+6t+1)(t^6+5t^3+1)(t^6+6t^3+1)$. Now, let $\mathcal{K}$ be a $\nu\omega$-constacyclic codes over the ring $\mathfrak{R} = Z_{11}[\nu,\omega,\gamma]/<\nu^2-$

$1, \omega^2-1, \gamma^2-1, \nu\omega-\omega\nu, \omega\gamma-\gamma\omega, \gamma\nu-\nu\gamma >$ of length 18. Let $g_1(t) = g_2(t) = g_6(t) = g_8(t) = t+10$ and $g_3(t) = g_4(t) = g_5(t) = g_7(t) = t^2+5t+1$ then $g(t) = \varrho_1(t+10)+\varrho_2(t+10)+\varrho_3(t^2+5t+1)+\varrho_4(t^2+5t+1)+\varrho_5(t^2+5t+1)+\varrho_6(t+10)+\varrho_7(t^2+5t+1)+\varrho_8(t+10)$ be the generator polynomials of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{18}-1$ for $i = 1,2,6,8$ respectively and $g_j(t)g_j^*(t)|t^{18}+1$ for $j = 3,4,5,7$ respectively, then by the use of Theorem 4.9, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$. Further $\varphi(\mathcal{K})$ is a linear code over Z_{11} having parameters $[144, 132, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[144, 120, \geq 3]_{11}$.

Example 5.9 In $Z_{11}(t)$, $t^{33}-1 = (t+10)^{11}(t^2+t+1)^{11}$ and $t^{33}+1 = (t+1)^{11}(t^2+10t+1)^{11}$. Now, let $\mathcal{K}$ be a ν-constacyclic codes over the ring $\mathfrak{R} = Z_{11}[\nu,\omega,\gamma]/< \nu^2-1, \omega^2-1, \gamma^2-1, \nu\omega-\omega\nu, \omega\gamma-\gamma\omega, \gamma\nu-\nu\gamma >$ of length 33. Let $g_1(t) = g_2(t) = g_3(t) = g_5(t) = (t+10)^2$ and $g_4(t) = g_6(t) = g_7(t) = g_8(t) = t^2+10t+1$ then $g(t) = \varrho_1(t+10)^2+\varrho_2(t+10)^2+\varrho_3(t+10)^2+\varrho_4(t^2+10t+1)+\varrho_5(t+10)^2+\varrho_6(t^2+10t+1)+\varrho_7(t^2+10t+1)+\varrho_8(t^2+10t+1)$ be the generator polynomial of $\mathcal{K}$. Since $g_i(t)g_i^*(t)|t^{33}-1$ for $i = 1,2,3,5$ respectively and $g_j(t)g_j^*(t)|t^{33}+1$ for $j = 4,6,7,8$ respectively, then by the use of Theorem 4.10, we get $\mathcal{K}^{\perp} \subseteq \mathcal{K}$ Further $\varphi(\mathcal{K})$ is a linear code over the ring Z_{11} having parameters $[264, 248, 3]$. Then, by the application of Theorem 4.14, we obtain the quantum codes having parameters $[264, 232, \geq 3]_{11}$.

6. Conclusion

In this work, we have given a construction for quantum codes through ϑ-constacyclic codes over the finite non-chain ring $\mathfrak{R} = Z_p[\nu,\omega,\gamma]/< \nu^2-1, \omega^2-1, \gamma^2-1, \nu\omega-\omega\nu, \omega\gamma-\gamma\omega, \gamma\nu-\nu\gamma >$ where $\nu^2 = 1, \omega^2 = 1, \gamma^2 = 1, \nu\omega = \nu\omega, \omega\gamma = \gamma\omega, \gamma\nu = \nu\gamma$ for different case of ϑ. We have derived self-orthogonal codes over the ring Z_p as Gray images of linear codes over the ring $Z_p[\nu,\omega,\gamma]/< \nu^2-1, \omega^2-1, \gamma^2-1, \nu\omega-\omega\nu, \omega\gamma-\gamma\omega, \gamma\nu-\nu\gamma >$. In particular, the parameters of quantum codes over the ring Z_p are obtained by decomposing constacyclic codes into cyclic and negacyclic codes over the ring Z_p. Also, it can be interesting to look at other classes of constacyclic codes over $Z_p[\nu,\omega,\gamma]/< \nu^2-1, \omega^2-1, \gamma^2-1, \nu\omega-\omega\nu, \omega\gamma-\gamma\omega, \gamma\nu-\nu\gamma >$.

References

[1] A. Calderbank, E. Rains, P. Shor, N. J. A. Sloane, Quantum Error Correction via Codes over GF(4), IEEE Trans. Inf. Theory 44(4) (1998), 1369-1387.

[2] A. Dertli, Y. Cengellenmis, S. Eren, On quantum codes obtained from cyclic codes over A_2, Int. J. Quantum Inform. 13(2) (2015), 1550031.

[3] F. Ma, J. Gao, F.W. Fu, Constacyclic codes over the ring $F_p + vF_p + v^2F_p$ and their applications of constructing new non-binary quantum codes, Quantum Inf. Process. 17 (2018), 122.

[4] H. Islam, O. Prakash, Quantum codes from the cyclic codes over $F_p[u, v, w]/(u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw)$, J. Appl. Math. Comput. 60 (2019), 625–635.

[5] J. Qian, Quantum codes from cyclic codes over $F_2 + vF_2$, Journal of Inform. and computational Science 10 (2013), 1715-1722.

[6] J. Li, J. Gao, Y. Wang, Quantum codes from $(1 - 2v)$-constacyclic codes over the ring $F_q + uF_q + vF_q + uvF_q$, Discrete Math. Algorithms Appl. 10(4) (2018), 1850046.

[7] M. Ashraf, G. Mohammad, Quantum codes from cyclic codes $F_3 + vF_3$, Int. J. Quantum Inform 12 (2014), 1450042 .

[8] M. Ashraf, G. Mohammad, Quantum codes over F_p from cyclic codes over $F_p[u, v]/(u^2 - 1, v^3 - v, uv - vu)$, Cryptogr. Commun. (2018)

[9] M. Ozen, N.T. Ozzaim, H. Ince, Quantum codes from cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$. Int. Conf. Quantum Sci. Appl. J. Phys. Conf. Ser. 766 (2016).

[10] P. W. Shor, Scheme for reducing decoherence in quantum memory, Phys. Rev. A 52 (1995), 2493-2496.