(Λ,Μ)-Multi Fuzzy Subgroup Of A Group

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ABSTRACT

In this paper, we have defined (λ,μ)- multi fuzzy subgroups of a group G and discussed some of its properties by using $(\alpha,\beta)-$ cuts . Also We have defined (λ,μ)-multi fuzzy cosets of a group and proved some related theorems with examples.

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1.INTRODUCTION

After the presentation of the fluffy set by way of L.A.Zadeh[23] some professionals investigated the speculation of concept of fluffy set. The concept of intuitionistic fluffy set(IFS) became provided by way of Krassimir.T.Atanassov [1] as a hypothesis of Zadeh's fluffy set.

In recent years, some variants and extensions of fuzzy groups emerged. In 1996, Bhakat and Das proposed the concept of an $(\in, \in \vee q)$-fuzzy subgroup in [6] and investigated their fundamental properties. They showed that A is an $(\in, \in \vee q)$-fuzzy subgroup if and only if $A\alpha$ is a crisp group for any $\alpha \in (0, 0.5]$ provided $A\alpha$ $\neq 0$. A question arises naturally: can we define a type of fuzzy subgroups such that all of their nonempty α-level sets are crisp subgroups for any α in an interval $(\lambda, \mu]$? In 2003, Yuan et al. [22,23] answered this question by defining a so-called (λ, μ)-fuzzy subgroups, which is an extension of $(\in, \in \vee q)$-fuzzy subgroup. As in the case of fuzzy group, some counterparts of classic concepts can be found for (λ, μ)-fuzzy subgroups. For instance, (λ, μ)-fuzzy normal subgroups and (λ, μ)-fuzzy quotient groups are defined and their elementary properties are investigated, and an equivalent characterization of (λ, μ)-fuzzy normal subgroups was presented in [22,23]. However, there is much more research on (λ, μ)-fuzzy subgroups if we consider rich results both in the classic group theory and the fuzzy group theory in the sense of Rosenfeld.

S. Sabu and T.V. Ramakrishnan [17] proposed the theory of multi fuzzy sets in terms of multi dimentional membership functions and investigated some properties of multi level fuzziness. An element of a multi fuzzy set can occur more than once with possibly [same or different membership values]. R.Muthuraj and S.Balamurugan[15] proposed the intuitionistic multi fuzzy subgroup and its level subgroups. The notion of t-intuitionistic fuzzy set, t-intuitionistic fuzzy group, t-intuitionistic fuzzy coset was introduced by P.K.Sharma[18,19]. And KR.Balasubramanian et al[3]. introduced the notion of t-intuitionistic multi fuzzy set and t-intuitionistic multi fuzzy subgroup of a group. In this paper we conduct a detailed investigation on (λ, μ)-multi fuzzy subgroups of a group."

2. PRILIMINARIES

Definition 2.1[23]

Let X be a non-empty set .A fuzzy subset A of X is defined by a function A:X→[0,1].

Definition 2.3[15,16]

Let X be a non-empty set. A multi fuzzy set A in X is defined as the set of ordered sequences as follows.

$A = \{(x, A_1(x), A_2(x), \dots, A_k(x), \dots) : x \in X\}$.Where $A_i : X \longrightarrow [0,1]$ for all i.

Definition 2.5[16]

Let X be a non-empty set. A k-dimensional multi fuzzy set A in X is defined by the set

$A = \{(x, (A_1(x), A_2(x), \dots A_k(x))), : x \in X\}$. Where $A_i : X \longrightarrow [0,1]$ for $i = 1,2,3, \dots, k$

Definition 2.6 [16]:For any three MFSs A, B and C,we have:

1.Commutative Law : $A \cap B = B \cap A$ and $A \cup B = B \cup A$

2.Idempotent Law : $A \cap A = A$ and $A \cup A = A$

3.De Morgan's Law :$\neg(A \cup B) = (\neg A \cap \neg B)$and $(\neg A \cap B) = (\neg A \cup \neg B)$

4.Associative Law : $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

5.Distributive Law : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

$$A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$$

Definition 2 [21,22] Let A be a fuzzy subset of G. A is called a (λ, μ)-fuzzy subgroup of G if, for all $x, y \in G$,

$$\text{(i)}A(xy) \vee \lambda \geqslant A(x) \wedge A(y) \wedge \mu$$

$$\text{(ii)}\ (x^{-1}) \vee \lambda \geqslant A(x) \wedge \mu$$

Clearly, a (0, 1)-fuzzy subgroup is just a fuzzy subgroup, and thus a (λ, μ)-fuzzy subgroup is a generalization of fuzzy subgroup .

3. Main Results

Definition .3.1

Let A be a fuzzy subset of G. Then a (λ, μ)- fuzzy subset $A^{(\lambda,\mu)}$ of a fuzzy set A of G is defined as $A^{(\lambda,\mu)} = (x, A \vee \lambda \wedge \mu : x \in G)$.

Definition .3.2

Let A be a multi fuzzy subset of G. Then a (λ, μ)- multi fuzzy subset $A^{(\lambda,\mu)}$ of a fuzzy set A of G is defined as $A^{(\lambda,\mu)} = (x, A \vee \lambda \wedge \mu : x \in G)$. That is, $A_i^{(\lambda_i,\mu_i)} = (x, A_i \vee \lambda_i \wedge \mu_i : x \in G)$

Clearly, a (0, 1)-multi fuzzy subset is just a multi fuzzy subset of G, and thus a (λ, μ)- multi fuzzy subgroup is a generalization of fuzzy subgroup. Where (0, 1)-multi fuzzy subset A is defined as $A^{(0,1)} = \left(A_i^{(0_i,1_i)}\right)$

Definition .3.3

Let A be a multi fuzzy subset of G. $A = (A_i)$ is called a (λ, μ)-multi fuzzy subgroup of G if, for all $x \in G$,

$$A(xy) \vee \lambda \geqslant \min\{A(x), A(y)\} \wedge \mu,$$

That is,

$A_i(xy) \vee \lambda_i \geqslant \min\{A_i(x), \{A_i(y)\} \wedge \mu_i$

Clearly, a (0, 1)-multi fuzzy subgroup is just a multifuzzy subgroup of G, and thus a (λ, μ)- multi fuzzy subgroup is a generalization of multi fuzzy subgroup.

Definition 3.3

Let $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ be any two (λ, μ)- multi fuzzy sets having the same dimension k of X. Then

(i). $A^{(\lambda,\mu)} \subseteq B^{(\lambda,\mu)}$, iff $A^{(\lambda,\mu)}(x) \leq B^{(\lambda,\mu)}(x)$ forall $x \in X$

(ii). $A^{(\lambda,\mu)} = B^{(\lambda,\mu)}$,iff $A^{(\lambda,\mu)}(x) = B^{(\lambda,\mu)}(x)$ forall $x \in X$

(iii). $\hat{}A^{(\lambda,\mu)} = \{(x, 1 - A^{(\lambda,\mu)}): x \in X\}$

(iv). $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)} = \{\left(x, \left(A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}\right)(x): x \epsilon X\right\}$,

where $\left(A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}\right)(x) = \min\left\{A^{(\lambda,\mu)}(x), B^{(\lambda,\mu)}(x)\right\} = \min\left\{A_i^{(\lambda_i,\mu_i)}(x), B_i^{(\lambda_i,\mu_i)}(x)\right\}$ for $i = 1,2, \dots, k$

(v). $A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)} = \left\{\left(x, A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}(x)\ \right): x \in X\right\}$,

where $\left(A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}\right)(x) = \max\left\{A^{(\lambda,\mu)}(x), B^{(\lambda,\mu)}(x)\right\} = \max\left\{A_i^{(\lambda_i,\mu_i)}(x), B_i^{(\lambda_i,\mu_i)}(x)\right\}$ for $i = 1,2, \dots, k$

Here, $\left\{A_i^{(\lambda_i,\mu_i)}(x)\right\}$ and $\left\{B_i^{(\lambda_i,\mu_i)}(x)\right\}$ represents the corresponding i^{th} position membership values of $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ respectively (see the definition 4.6,in ref.[17]).

Definition 3.4 :

For any three (MFSs $A^{(\lambda,\mu)}$, $B^{(\lambda,\mu)}$ and $C^{(\lambda,\mu)}$, we have:

1. Commutative Law : $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)} = B^{(\lambda,\mu)} \cap A^{(\lambda,\mu)}$ and

$$A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)} = B^{(\lambda,\mu)} \cup A^{(\lambda,\mu)}$$

2. Idempotent Law : $A^{(\lambda,\mu)} \cap A^{(\lambda,\mu)} = A^{(\lambda,\mu)}$ and $A^{(\lambda,\mu)} \cup A^{(\lambda,\mu)} = A^{(\lambda,\mu)}$

3. De Morgan's Law :$\neg(A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}) = \neg(A^{(\lambda,\mu)} \cap \neg B^{(\lambda,\mu)})$ and $\neg(A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}) = \neg(A^{(\lambda,\mu)} \cup \neg B^{(\lambda,\mu)})$

4. Associative Law : $A^{(\lambda,\mu)} \cup \left(B^{(\lambda,\mu)} \cup C^{(\lambda,\mu)}\right) = \left(A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}\right) \cup C^{(\lambda,\mu)}$ and $A^{(\lambda,\mu)} \cap \left(B^{(\lambda,\mu)} \cap C^{(\lambda,\mu)}\right) = (A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}) \cap A^{(\lambda,\mu)}$

5. Distributive Law : $A^{(\lambda,\mu)} \cup \left(B^{(\lambda,\mu)} \cap C^{(\lambda,\mu)}\right) = (A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}) \cap (A^{(\lambda,\mu)} \cap C^{(\lambda,\mu)})$ and $A^{(\lambda,\mu)} \cap \left(B^{(\lambda,\mu)} \cup C^{(\lambda,\mu)}\right) = (A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}) \cup (A^{(\lambda,\mu)} \cap C^{(\lambda,\mu)})$

Definition 3.5 :

Let $A^{(\lambda,\mu)} = \{(x, A^{(\lambda,\mu)}(x)) : x \epsilon X\}$ be a (λ, μ) −MFS of dimension k and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in [0,1]^k$, where each $\alpha_i \in [0,1]$ for all i. Then the $\alpha -$ cut of $A^{(\lambda,\mu)}$ is the set of all x such that ${A_i}^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$, ∀i and is denoted by $[A^{(\lambda,\mu)}]_{(\alpha)}$. Clearly it is a crisp set.

Definition 3.6 :

Let $A^{(\lambda,\mu)} = \{(x, A^{(\lambda,\mu)}(x)) : x \epsilon X\}$ be a (λ, μ) −MFS of dimension k and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in [0,1]^k$,where each $\alpha_i \in [0,1$ for all i. Then the strong $\alpha -$ cut of $A^{(\lambda,\mu)}$ is the set of all x such that ${A_i}^{(\lambda,\mu)}(x) > \alpha_i$,∀i and is denoted by $[A^{(\lambda,\mu)}]_{\alpha^*}$. Clearly it is also a crisp set.

Theorem 3.7 (ref.[19]):

Let A and B are any two (λ, μ) −MFSs of dimension k taken from a non –empty set X. Then $A \subseteq B$ if and only if $[A^{(\lambda,\mu)}]_{(\alpha)} \subseteq [B^{(\lambda,\mu)}]_{(\alpha)}$for every $\in [0,1]^k$.

Definition 3.8 :

A MFS $A = \{(x, A(x)) : x \epsilon X\}$ of a group G is said to be a (λ, μ)-multifuzzy sub group of G (MFSG), if it satisfies the following: For $\lambda, \mu \in [0,1]^k, 0 \leq \lambda_i \leq \mu_i \leq 1, 0 \leq \lambda_i + \mu_i \leq 1$

(i) $A(xy) \vee \lambda \geq \min\{A(x), A(y)\} \wedge \mu$

(ii) $A(x^{-1}) \vee \lambda \geq A(x) \wedge \mu$ for all $x, y \in G$. That is,

(i) $A_i(xy) \vee \lambda_i \geq \min\{A_i(x), A_i(y)\} \wedge \mu_i$

(ii) $A_i(x^{-1}) \vee \lambda_i \geq A_i(x) \wedge \mu_i$ for all $x, y \in G$.

Clearly, a (0, 1)-multi fuzzy subgroup is just a multi fuzzy subgroup of G,and thus a (λ, μ)- multi fuzzy subgroup is a generalization of multi fuzzy subgroup.

An alternative definition for (λ, μ)-MFG is as follows:

Definition 3.9 :

A MFS A of a group G is said to be a (λ, μ)-multi-fuzzy sub group of G ((λ, μ)-MFSG), if it satisfies.

$A(xy^{-1}) \vee \lambda \geq \min\{A(x), A(y)\} \wedge \mu$ for all $x,y \in G$

Where, $A(xy^{-1}) \vee \lambda = (A_1(xy^{-1}) \vee \lambda_1, A_2(xy^{-1}) \vee \lambda_2, \ldots, A_k(xy^{-1}) \vee \lambda_k)$ and $\min\{A(x), A(y)\} \wedge \mu = (\min\{A_1(x), A_1(y)\} \wedge \mu_1, \min\{A_2(x), A_2(y)\} \wedge \mu_2, \ldots, \min\{A_k(x), A_k(y)\} \wedge \mu_k)$ for all x, y and xy^{-1} in G.

Remark 3.10 :

(i) If A is a (λ, μ) −MFSG of G,then the complement of A need not be an $(\lambda, \mu) -$ MFSG of G

(ii) A is a MFSG of a group ⟺ each (λ, μ) −FS $A^{(\lambda_i,\mu_i)}$ $(A^{(\lambda_i,\mu_i)}) : x \epsilon G\}_{i=1,2,\ldots,k}$ is a (λ, μ) −FSG of G.

Definition 3.11(ref.[6,9,12]) :

A $(\lambda,\mu)-$ MFSG $A^{(\lambda,\mu)}$ of a group G is said to be an (λ,μ) −multi fuzzy normal subgroup $((\lambda,\mu)-\text{MFNSG})$ of G, it satisfies

$A^{(\lambda,\mu)}(xy)=A^{(\lambda,\mu)}(yx)$ for all $x,y\in G$

Theorem 3.12 :

A (λ,μ) −MFSG $A^{(\lambda,\mu)}$ of a group G is normal, it satisfies

$A^{(\lambda,\mu)}(g^{-1}xg)=A^{(\lambda,\mu)}(x)$ for all $x,y\in G$ and $g\in G$

Proof :Let $x\in A^{(\lambda,\mu)}$ and $g\in G$.

Then $A^{(\lambda,\mu)}(g^{-1}xg)=A^{(\lambda,\mu)}(g^{-1}(xg))=A^{(\lambda,\mu)}((xg)g^{-1})$,since $A^{(\lambda,\mu)}$ is normal.

$A^{(\lambda,\mu)}((xg)g^{-1})=A^{(\lambda,\mu)}(x(gg^{-1}))=A^{(\lambda,\mu)}(xe)=A^{(\lambda,\mu)}(x)$, Hence (i) is true.

Definition 3.13 :

Let $(G,.)$ be a Groupoid and $A^{(\lambda,\mu)}$,$B^{(\lambda,\mu)}$ be any two (λ,μ) −MFSs having same dimension k of G.Then the product of $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$, denoted by $A^{(\lambda,\mu)}\circ B^{(\lambda,\mu)}$ and is defined as:

$$A^{(\lambda,\mu)}\circ B^{(\lambda,\mu)}(x)=\begin{cases}\max[\min\{A^{(\lambda,\mu)}(y),B^{(\lambda,\mu)}(z)\}:yz=x,\forall\, y,z\in G]\\ \lambda_k=(\lambda,\lambda,\dots,\lambda_{k\text{ times}}),\text{if x is not expressible sa }x=yz\end{cases},\forall\, x\in G$$

That is ,$\forall\, x\in G$,

$$A^{(\lambda,\mu)}\circ B^{(\lambda,\mu)}(x)=\begin{cases}(\max\big[\min\{A^{(\lambda,\mu)}(y),B^{(\lambda,\mu)}(z)\}:yz=x,\forall\, y,z\in G\big]\\ (\lambda_k) \qquad ,\text{if x is not expressible as }x=yz\end{cases}$$

Definition 3.14 :

Let X and Y be any two non-empty sets and $f:X\longrightarrow Y$ be a mapping. Let $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ be any two (λ,μ) −MFSs of X and Y respectively having the same dimention k.Then the image of $A^{(\lambda,\mu)}(\subseteq X)$ under the map f is denoted by $f(A^{(\lambda,\mu)})$, is defined as :$\forall y\in Y$,

$$f\big(A^{(\lambda,\mu)}\big)(y)=\begin{cases}\max\{A^{(\lambda,\mu)}(x):x\in f^{-1}(y)\\ \lambda_k, \qquad \text{otherwise}\end{cases}$$

Also, the pre − image of $B^{(\lambda,\mu)}(\subseteq Y)$ under the map f is denoted by $f^{-1}\big(B^{(\lambda,\mu)}\big)$ and it is defined as: $f^{-1}\big(B^{(\lambda,\mu)}\big)(x)=(B^{(\lambda,\mu)}(f(x)),\forall x\in X$.

4. Properties of α −cuts of the (λ,μ) −MFSGs of a group

In this section, we have proved some theorems on (λ,μ) −MFSGs of a group G by using some of their $\alpha-$ cuts.

Proposition 4.1 :

If $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are any two (λ,μ) −MFSs of a universal set X

Then the following are hold good :

(i) $[A^{(\lambda,\mu)}]_\alpha\subseteq[A^{(\lambda,\mu)}]_\delta$ if $\alpha\geq\delta$

(ii) $A^{(\lambda,\mu)}\subseteq B^{(\lambda,\mu)}$ implies $[A^{(\lambda,\mu)}]_\alpha\subseteq[B^{(\lambda,\mu)}]_\delta$

(iii)$\left[A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}\right]_\alpha = [A^{(\lambda,\mu)}]_\alpha \cap [B^{(\lambda,\mu)}]_\alpha$

(iv)$\left[A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}\right]_\alpha \supseteq [A^{(\lambda,\mu)}]_\alpha \cup [B^{(\lambda,\mu)}]_\alpha$ (here equality holds if $\alpha_i = 1, \forall$ i)

(v) $[\cap A_i^{(\lambda,\mu)}]_\alpha = \cap [A_i^{(\lambda,\mu)}]_\alpha$, where $\alpha \in [0,1]^k$

Proposition 4.2 : Let (G,.) be a groupoid and $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are any two (λ,μ) −MFSs of G. Then we have $\left[A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)}\right]_\alpha = \left[A^{(\lambda,\mu)}\right]_\alpha \left[B^{(\lambda,\mu)}\right]_\alpha, where\ \ \alpha \in [0,1]^k$.

Theorem 4.3 :

If $A^{(\lambda,\mu)}$ is a(λ,μ) −multi fuzzy subgroup of G and $\alpha \in [0,1]^k$, then $\left[A^{(\lambda,\mu)}\right]_\alpha$ is a subgroup of G, where $A^{(\lambda,\mu)}(e) \geq \alpha$, and '$e$' is the identity element of G.

Proof :

Since $A^{(\lambda,\mu)}(e) \geq \alpha, e \in [A^{(\lambda,\mu)}]_\alpha$. There fore $[A^{(\lambda,\mu)}]_\alpha \neq \emptyset$.

Let $x, y \in [A^{(\lambda,\mu)}]_\alpha$.Then $A^{(\lambda,\mu)}(x) \geq \alpha\ and\ A^{(\lambda,\mu)}(y) \geq \alpha$.

Then for all $i, A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$ and $A_i^{(\lambda_i,\mu_i)}(y) \geq \alpha_i$,

$\Rightarrow min\left\{A_i^{(\lambda_i,\mu_i)}(x), A_i^{(\lambda_i,\mu_i)}(y)\right\} \geq \alpha_i$, $\forall\ i$ … … … . . . . (1)

$\Rightarrow A_i^{(\lambda_i,\mu_i)}(xy^{-1}) \geq min\left\{A_i^{(\lambda_i,\mu_i)}(x), A_i^{(\lambda_i,\mu_i)}(y)\right\} \geq \alpha_i$, $\forall\ i$, since $A^{(\lambda,\mu)}$ is a (λ,μ) −multi fuzzy subgroup of a group G and by (1).

$\Rightarrow A_i^{(\lambda_i,\mu_i)}(xy^{-1}) \geq \alpha_i$, $\forall\ i$.

$\Rightarrow A^{(\lambda,\mu)}(xy^{-1}) \geq \alpha$

$\Rightarrow xy^{-1} \in [A^{(\lambda,\mu)}]_\alpha$

$\Rightarrow [A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G.

Theorem 4.4 :

If $A^{(\lambda,\mu)}$ is a (λ,μ) −multi fuzzy subset of a group G, then $A^{(\lambda,\mu)}$ is a (λ,μ) −multi fuzzy subgroup of $G \Leftrightarrow$ each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G, for all $\alpha \in [0,1]^k$ for all i .

Proof : ($\Rightarrow$)Let $A^{(\lambda,\mu)}$ be a $(\lambda,\mu)-$ multi-fuzzy subgroup of a group G.Then by the theorem 3.4, each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G for all $\alpha \in [0,1]^k$.

($\Leftarrow$) Conversely, let $A^{(\lambda,\mu)}$ be a (λ,μ)–multifuzzy subset of a group G such that each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G for all $\alpha \in [0,1]^k$, $\forall\ i$.

To prove that $A^{(\lambda,\mu)}$ is a (λ,μ) −multi fuzzy subgroup of G, we must prove that : (i) $A^{(\lambda,\mu)}(xy) \geq min\{A^{(\lambda,\mu)}(x), A^{(\lambda,\mu)}(y)\}$, $\forall\ x, y \in G$

(ii) $A^{(\lambda,\mu)}(x^{-1}) = A^{(\lambda,\mu)}(x)$

Let $x, y \in G$ and for all i, let $\alpha_i = min\{A_i^{(\lambda_i,\mu_i)}(x), A_i^{(\lambda_i,\mu_i)}(y)\}$. Then $\forall\ i$,

We have $A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$, $A_i^{(\lambda_i,\mu_i)}(y) \geq \alpha_i$

That is , $\forall\ i$,we have $A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$, and $A_i^{(\lambda_i,\mu_i)}(y) \geq \alpha_i$

Then we have $A^{(\lambda,\mu)}(x) \geq \alpha$ and $A^{(\lambda,\mu)}(y) \geq \alpha$. That is, $x \in [A^{(\lambda,\mu)}]_\alpha$ and $y \in [A^{(\lambda,\mu)}]_\alpha$ therefore, $xy \in [A^{(\lambda,\mu)}]_\alpha$, since each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup by hypothesis .

Therefore , $\forall\ i$, we have $A_i^{(\lambda_i,\mu_i)}(xy) \geq \alpha_i = min\{A_i^{(\lambda_i,\mu_i)}(x), A_i^{(\lambda_i,\mu_i)}(y)$.

That is, $A^{(\lambda,\mu)}(xy) \geq min\ \{A^{(\lambda,\mu)}(x), A^{(\lambda,\mu)}(y)\}$ hence (i) is true.

Now, let $x \in G$ and $\forall\ i$, let $A_i^{(\lambda_i,\mu_i)}(x) = \alpha_i$. Then $A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$ is true for all i. Therefore $A^{(\lambda,\mu)}(x) \geq \alpha$. Thus , $x \in [A^{(\lambda,\mu)}]_\alpha$.

Since each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G forall $\alpha,\beta \in [0,1]^k$ and $x \in [A^{(\lambda,\mu)}]_{(\alpha,\beta)}$, we have $x^{-1} \in [A^{(\lambda,\mu)}]_\alpha$ which implies that $A_i^{(\lambda_i,\mu_i)}(x^{-1}) \geq \alpha_i$ is true $\forall\ i$. Which implies that $A_i^{(\lambda_i,\mu_i)}(x^{-1}) \geq A_i^{(\lambda_i,\mu_i)}(x)$ is true $\forall\ i$. Thus , $\forall\ i, A_i^{(\lambda_i,\mu_i)}(x) = A_i^{(\lambda_i,\mu_i)}((x^{-1})^{-1}) \geq A_i^{(\lambda_i,\mu_i)}(x^{-1}) \geq A_i^{(\lambda_i,\mu_i)}(x)$ which implies that $A_i^{(\lambda_i,\mu_i)}(x^{-1}) = A_i^{(\lambda_i,\mu_i)}(x)$. Hence $A^{(\lambda,\mu)}$ is a $(\lambda,\mu)-$ multifuzzy subgroup of G.

Theorem 4.5 :

If $A^{(\lambda,\mu)}$ is a (λ,μ) -multi fuzzy normal subgroup of a group G and for every $\alpha \in [0,1]^k$, then $[A^{(\lambda,\mu)}]_\alpha$ is a normal subgroup of G, where $A^{(\lambda,\mu)}(e) \geq \alpha$ and 'e' is the identity element of G.

Proof :

Let $x \in [A^{(\lambda,\mu)}\ ]_\alpha$ and $g \in G$.Then , $A^{(\lambda,\mu)}(e) \geq \alpha$.

That is , $A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$, $\forall\ i$ …………(1)

Since $A^{(\lambda,\mu)}$ is a (λ,μ) −MFNSG of G,

$A_i^{(\lambda_i,\mu_i)}(g^{-1}xg) = A_i^{(\lambda_i,\mu_i)}(x)$, $\forall\ i$.

$\Longrightarrow A_i^{(\lambda_i,\mu_i)}(g^{-1}xg) = A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$,$\forall\ i$,by using (1).

$\Longrightarrow A_i^{(\lambda_i,\mu_i)}(g^{-1}xg) \geq \alpha_i$,$\forall\ i$

$\Longrightarrow A^{(\lambda,\mu)}(g^{-1}xg) \geq \alpha$

$\Longrightarrow g^{-1}xg \in [A^{(\lambda,\mu)}]_\alpha$

$\Longrightarrow [A^{(\lambda,\mu)}]_\alpha$ is normal subgroup of G

Theorem 4.6 :

If $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are any two (λ,μ) −multi fuzzy subgroups ((λ,μ) −MFSGs) of a group G, then $(A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)})$ is also a (λ,μ) −multi fuzzy subgroup of G.

Proof:

By the above theorem 4.6, $A^{(\lambda,\mu)}$ is a $(\lambda,\mu)-$ multi fuzzy subgroup of $G \Leftrightarrow$ each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G for all $\alpha \in [0,1]^k$ with $\leq \alpha_i \leq 1, \forall\ i$. But, since $[A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}]_\alpha = [A^{(\lambda,\mu)}]_\alpha \cap [B^{(\lambda,\mu)}]_\alpha$ and both $[A^{(\lambda,\mu)}]_\alpha$ and $[B^{(\lambda,\mu)}]_\alpha$ are subgroups of G (as $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are $(\lambda,\mu)-$ multi fuzzy subgroups) and the intersection of any two subgroups is also a subgroup of G , which implies that $[A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}]_\alpha$ is a subgroup of G and hence $(A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)})$ is a $(\lambda,\mu)-$ multi fuzzy subgroup of G.

Remark 4.7 :

The union of two $(\lambda,\mu)-$ multi fuzzy subgroups of a group G need not be a $(\lambda,\mu)-$ MFSG of the group G.

Proof: Consider the Klein's four group G={e, a, b, ab },where $a^2 = e = b^2$ and $ba = ab$. For $0 \leq i \leq 5$, let $t_i, s_i \in [0,1]^k$ such that $r_0 > r_1 > \ldots \ldots > r_5$ and $s_0 < s_1 < \ldots \ldots < s_5$. Define $(\lambda,\mu) - MFSs\ A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ of dimension k as follows : $A^{(\lambda,\mu)} = \{(x, A^{(\lambda,\mu)}(x)): x\epsilon G\}$ and $B^{(\lambda,\mu)} = \{(x, B^{(\lambda,\mu)}(x)): x\epsilon G\}$, where $A_i^{(\lambda_i,\mu_i)}(e) = r_1 \vee \lambda_i \wedge \mu_i, A_i^{(\lambda_i,\mu_i)}(a) = r_3 \vee \lambda_i \wedge \mu_i, A_i^{(\lambda_i,\mu_i)}(b) = r_4 \vee \lambda_i \wedge \mu_i = A_i^{(\lambda_i,\mu_i)}(ab)$ and $B_i^{(\lambda_i,\mu_i)}(e) = r_0 \vee \lambda_i \wedge \mu_i, B_i^{(\lambda_i,\mu_i)}(a) = r_5 \vee \lambda_i \wedge \mu_i = B_i^{(\lambda_i,\mu_i)}(ab), B_i^{(\lambda_i,\mu_i)}(b) = r_2 \vee \lambda_i \wedge \mu_i$.

Clearly $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are $(\lambda,\mu) -$ multi fuzzy subgroups of G.

Now $\left(A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}\right)(x) = max\{A^{(\lambda,\mu)}(x), A^{(\lambda,\mu)}(x)\}$= $(max\{A_i{}^{(\lambda_i,\mu_i)}(x), B_i{}^{(\lambda_i,\mu_i)}(x)\})_{i=1}^{k}$

$(A_i{}^{(\lambda_i,\mu_i)} \cup B_i{}^{(\lambda_i,\mu_i)})(e) = r_0 \vee \lambda_i \wedge \mu_i, (A_i{}^{(\lambda_i,\mu_i)} \cup B_i{}^{(\lambda_i,\mu_i)})(a) = r_3 \vee \lambda_i \wedge \mu_i, (A_i{}^{(\lambda_i,\mu_i)} \cup B_i{}^{(\lambda_i,\mu_i)})(b) = r_2 \vee \lambda_i \wedge \mu_i$; $A_i{}^{(\lambda_i,\mu_i)}(ab) = r_4 \vee \lambda_i \wedge \mu_i$.

$$[A_i{}^{(\lambda_i,\mu_i)}]_{r_3} = \{x: x \in G \text{ such that } A_i{}^{(\lambda_i,\mu_i)}(x) \geq r_3\} = \{e,a\}$$

$$[B_i{}^{(\lambda_i,\mu_i)}]_{r_3} = \{x: x \in G \text{ such that } B_i{}^{(\lambda_i,\mu_i)}(x) \geq r_3\} = \{e\}$$

$$[A_i{}^{(\lambda_i,\mu_i)} \cup B_i{}^{(\lambda_i,\mu_i)}]_{r_3} = \{x: x \in G \text{ such that } A_i{}^{(\lambda_i,\mu_i)}(x) \geq r_3\}$$
$$= \{x: x \in G \text{ such that} max\{A_i{}^{(\lambda_i,\mu_i)}(x), B_i{}^{(\lambda_i,\mu_i)}(x)\} \geq r_3\} = \{e,a,b\}$$

Since {e,a,b } is not a subgroup of G, $[A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}]_{r_3}$ is not a subgroup of G.Hence $[A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}]$ is not a subgroup of G and there fore $[A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}]$ is not a (λ,μ) −MFSG of the group G.

Example 4.8 : There are two cases needed to clarify the previous theorem 3.7 and remark.

$Case\ (i)$: Consider the abelian group $G = \{e,a,b,ab\}$ with usual multiplication such that $a^2 = e = b^2$ and $ab = ba$. Let $A^{(\lambda,\mu)} = \{< e, (0.6 \vee \lambda_1 \wedge \mu_1, 0.8 \vee \lambda_2 \wedge \mu_2) >, < a, (0.4 \vee \lambda_1 \wedge \mu_1, 0.4 \vee \lambda_2 \wedge \mu_2) >, < b, (0.3 \vee \lambda_1 \wedge \mu_1, 0.3 \vee \lambda_2 \wedge \mu_2) >, < ab, (0.3 \vee \lambda_1 \wedge \mu_1, 0.3 \vee \lambda_2 \wedge \mu_2) >\}$ and $B^{(\lambda,\mu)} = \{< e, (0.7 \vee \lambda_1 \wedge \mu_1, 0.7 \vee \lambda_2 \wedge \mu_2) >, < a, (0.2 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) >, < b, (0.4 \vee \lambda_1 \wedge \mu_1, 0.4 \vee \lambda_2 \wedge \mu_2) >, < ab, (0.2 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) >\}$ be two (λ,μ) −MFSs having dimension two of G. Clearly $A^{(\lambda,\mu)}$and $A^{(\lambda,\mu)}$ are $(\lambda,\mu) - MFSGs$ of G.

Then $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)} = \left\{< e, (0.6 \vee \lambda_1 \wedge \mu_1, 0.7 \vee \lambda_2 \wedge \mu_2) >, < a, (0.2 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2), < b, (0.3 \vee \lambda_1 \wedge \mu_1, 0.3 \vee \lambda_2 \wedge \mu_2) >, < ab, \begin{pmatrix} 0.2 \vee \lambda_1 \wedge \mu_1, \\ 0.2 \vee \lambda_2 \wedge \mu_2 \end{pmatrix} >\right\}$

$$and\ A^t \cup B^t = \{< e, (0.7 \vee \lambda_1 \wedge \mu_1, 0.8 \vee \lambda_2 \wedge \mu_2) >, < a, \begin{pmatrix} 0.4 \vee \lambda_1 \wedge \mu_1, \\ 0.4 \vee \lambda_2 \wedge \mu_2 \end{pmatrix} >,$$

$< b, (0.4 \vee \lambda_1 \wedge \mu_1, 0.4 \vee \lambda_2 \wedge \mu_2) >, < ab, (0.3 \vee \lambda_1 \wedge \mu_1, 0.3 \vee \lambda_2 \wedge \mu_2) > \}$

Therefore it is easily verified that in this case $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}$ is a $(\lambda,\mu) - MFSG$ of G and $A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}$ is not a $(\lambda,\mu) - MFSG$ of G. Hence $case(i)$."

$Case(ii)$: Consider the abelian group $G = \{e,a,b,ab\}$ "with usual multiplication such that $a^2 = e = b^2$ and $ab = ba$. Let $A^{(\lambda,\mu)} = \{< e, (0.5 \vee \lambda_1 \wedge \mu_1, 0.9 \vee \lambda_2 \wedge \mu_2) >, < a, (0.4 \vee \lambda_1 \wedge \mu_1, 0.6 \vee \lambda_2 \wedge \mu_2) >, < b, (0.1 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) >, < ab, (0.1 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) >\}$ and $A^{(\lambda,\mu)} = \{< e, (0 \vee \lambda_1 \wedge \mu_1, 0.7 \vee \lambda_2 \wedge \mu_2) >, < a, (0 \vee \lambda_1 \wedge \mu_1, 0.4 \vee \lambda_2 \wedge \mu_2) >, < b, (0 \vee \lambda_1 \wedge \mu_1, 0.1 \vee \lambda_2 \wedge \mu_2) >, < ab, (0 \vee \lambda_1 \wedge \mu_1, 0.1 \vee \lambda_2 \wedge \mu_2) >\}$ be two (λ,μ) −MFSs having dimension two of G. Clearly $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are $(\lambda,\mu) - MFSGs$ of G.

Then $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)} = \{< e, (0 \vee \lambda_1 \wedge \mu_1, 0.7 \vee \lambda_2 \wedge \mu_2) >, < a, (0 \vee \lambda_1 \wedge \mu_1, 0.4 \vee \lambda_2 \wedge \mu_2) >, < b, (0 \vee \lambda_1 \wedge \mu_1, 0.1 \vee \lambda_2 \wedge \mu_2) >, < ab, (0 \vee \lambda_1 \wedge \mu_1, 0.1 \vee \lambda_2 \wedge \mu_2) >\}$ and $A^t \cup B^t = \{< e, (0.5 \vee \lambda_1 \wedge \mu_1, 0.9 \vee \lambda_2 \wedge \mu_2) >, < a, (0.4 \vee \lambda_1 \wedge \mu_1, 0.6 \vee \lambda_2 \wedge \mu_2) >, < b, (0.1 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) >, < ab, (0.1 \vee \lambda_1 \wedge \mu_1, 0.2 \vee \lambda_2 \wedge \mu_2) > \}$.

Here, it can be easily verified that both $A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}$ and $A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}$ are $(\lambda,\mu) - MFSGs$ of G. Hence $case\ (ii)$.

From the conclusion of the above example, we come to the point that there is an uncertainty in verifying whether or not $A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}$ is a $(\lambda,\mu) - MFSG$ of G.

Theorem 4.9 :

If $A^{(\lambda,\mu)}$and $B^{(\lambda,\mu)}$ be any two (λ,μ) −MFSGs of a group G. Then $A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)}$ is a (λ,μ) −MFSG of G ⇔ $A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)} = B^{(\lambda,\mu)} \circ A^{(\lambda,\mu)}$

$Proof$: Since $A^{(\lambda,\mu)}$ and $B^{(\lambda,\mu)}$ are $(\lambda,\mu) - MFSGs$ of G , each $[A^{(\lambda,\mu)}]_\alpha$ and $[B^{(\lambda,\mu)}]_\alpha$ are subgroups of G,$\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i$ … … … … … … … … … … … … … … (1)

Suppose $A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)}$ is a (λ,μ) −MFSG of G ⇔ each $[A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)}]_\alpha$ are subgroups of G,$\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i$.

Now, from (1), $[A^{(\lambda,\mu)}]_\alpha \circ [B^{(\lambda,\mu)}]_\alpha$ is a subgroup of G ⇔ $[A^{(\lambda,\mu)}]_\alpha \circ [B^{(\lambda,\mu)}]_\alpha = [B^{(\lambda,\mu)}]_\alpha \circ [A^{(\lambda,\mu)}]_\alpha$, since if H and K are any two subgroups of G, then HK is a subgroup of G ⇔ HK=KH ⇔ $[A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)}]_\alpha = [B^{(\lambda,\mu)} \circ A^{(\lambda,\mu)}]_\alpha$, $\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i.$ ⇔ $A^{(\lambda,\mu)} \circ B^{(\lambda,\mu)} = B^{(\lambda,\mu)} \circ A^{(\lambda,\mu)}$.

Theorem 4.10 :

If $A^{(\lambda,\mu)}$ is any (λ,μ) −MFSG of a group G, then $A^{(\lambda,\mu)} \circ A^{(\lambda,\mu)} = A^{(\lambda,\mu)}$.

Proof: Since $A^{(\lambda,\mu)}$ is a (λ,μ) −MFSG of a group G, each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G, $\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i.$

$\Rightarrow [A^{(\lambda,\mu)}]_\alpha \circ [A^{(\lambda,\mu)}]_\alpha = [A^{(\lambda,\mu)}]_\alpha$, since H is a subgroup of G ⇒ HH=H.

$\Rightarrow [A^{(\lambda,\mu)} \circ A^{(\lambda,\mu)}]_\alpha = [A^{(\lambda,\mu)}]_\alpha$, $\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i.$

$\Rightarrow A^{(\lambda,\mu)} \circ A^{(\lambda,\mu)} = A^{(\lambda,\mu)}$.

5. (λ,μ) −multi fuzzy cosets of a group

Definition 5.1 :

Let G be a group and $A^{(\lambda,\mu)}$ be a $(\lambda,\mu) - MFSG$ of G. Let $x \in G$ be a fixed element. Then the set $xA^{(\lambda,\mu)} = \{\left(g, xA^{(\lambda,\mu)}(g)\right): g \in G\}$ where $xA^{(\lambda,\mu)}(g) = A^{(\lambda,\mu)}(x^{-1}g)$,$\forall g \in G$ is called the $(\lambda,\mu) -$ multi fuzzy left coset of G determined by $A^{(\lambda,\mu)}$ and x.

Similarly, the set $A^{(\lambda,\mu)}x = \{\left(g, A^{(\lambda,\mu)}x(g)\right): g \in G\}$ where $A^{(\lambda,\mu)}x(g) = A^{(\lambda,\mu)}(gx^{-1})$,$\forall g \in G$ is called the (λ,μ) −multifuzzy right coset of G determined by $A^{(\lambda,\mu)}$ and x.

Remark 5.2 :

It is clear that if $A^{(\lambda,\mu)}$ is a (λ,μ) −multi fuzzy normal subgroup of G, then the $(\lambda,\mu) -$ multi fuzzy left coset and the (λ,μ) −multi fuzzy right coset of $A^{(\lambda,\mu)}$ on G coincides and in this case, we simply call it as (λ,μ) −multi fuzzy coset.

Example 5.3 :

Let G be a group. Then $A^{(\lambda,\mu)} = \{\left(x, A^{(\lambda,\mu)}(x)\right) : x \in G / A^{(\lambda,\mu)}(x) = A^{(\lambda,\mu)}(e)\}$ is a (λ,μ) −multi fuzzy normal subgroup of G.

Theorem 5.4 :

Let $A^{(\lambda,\mu)}$ be a $(\lambda,\mu) -$ multifuzzy subgroup of G and x be any fixed element of G. Then the following hold :

(i) $x[A^{(\lambda,\mu)}]_\alpha = [x\,A^{(\lambda,\mu)}]_\alpha$

(ii) $[A^{(\lambda,\mu)}]_\alpha x = [\,A^{(\lambda,\mu)}x]_\alpha$, $\forall\ \alpha \in [0,1]^k$ with $0 \le \alpha_i \le 1, \forall i.$

Proof :

(i) $[x\,A^{(\lambda,\mu)}]_\alpha = \left\{g \in G : x\,A^{(\lambda,\mu)}(g) \ge \alpha\right\}$ $with\ 0 \le \alpha_i \le 1, \forall i.$

Also $x[\,A^{(\lambda,\mu)}]_\alpha = x\{y \in G:\ A^{(\lambda,\mu)}(y) \geq \alpha\}$

$$= \{xy\ \in G:\ A^{(\lambda,\mu)}(y) \geq \alpha\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Put $xy = g\ \Rightarrow y = x^{-1}g$. Then (1) can be written as ,

$$x[\,A^{(\lambda,\mu)}]_\alpha = \{g \in G:\ A^{(\lambda,\mu)}(x^{-1}g) \geq \alpha\,\} = \{g \in G:\ xA^{(\lambda,\mu)}(g) \geq \alpha\,\} = [x\,A^{(\lambda,\mu)}]_\alpha$$

Therefore, $x[A^{(\lambda,\mu)}]_\alpha = [x\,A^{(\lambda,\mu)}]_\alpha, \forall\, \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i.$

(ii) Now $[A^{(\lambda,\mu)}x]_\alpha = \{g \in G: A_x{}^{(\lambda,\mu)}(g) \geq \alpha\,\}\, with\ 0 \leq \alpha_i \leq 1, \forall i\}$. Also

$[A^{(\lambda,\mu)}]_\alpha x = \{y \in G: A^{(\lambda,\mu)}(y) \geq \alpha\}x$

$$= \{yx\ \in G: A^{(\lambda,\mu)}(y) \geq \alpha\,\} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Set $yx = g\ \Rightarrow y = gx^{-1}$. Then (2) can be written as $[A^{(\lambda,\mu)}]_\alpha x = \{g \in G: A^{(\lambda,\mu)}(gx^{-1}) \geq \alpha\,\}$

$$= \{g \in G: A_x{}^{(\lambda,\mu)}(g) \geq \alpha\,\} = [A_x{}^{(\lambda,\mu)}]_\alpha$$

Therefore, $[A^{(\lambda,\mu)}]_\alpha x = [A_x{}^{(\lambda,\mu)}]_\alpha,\ \forall\, \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i.$

Theorem 5.5 :

Let $A^{(\lambda,\mu)}$ be a $(\lambda,\mu) - MFSG$ of a group G. Let x, y be any two elements of G such that $= min\{A^{(\lambda,\mu)}(x), A^{(\lambda,\mu)}(y)\}$. Then the following hold :

$(i)\ xA^{(\lambda,\mu)} = yA^{(\lambda,\mu)} \Leftrightarrow x^{-1}y \in [A^{(\lambda,\mu)}]_\alpha$

$(ii)\ A^{(\lambda,\mu)}x = A^{(\lambda,\mu)}y \Leftrightarrow yx^{-1} \in [A^{(\lambda,\mu)}]_\alpha$

Proof :

$(i) xA^{(\lambda,\mu)} = yA^{(\lambda,\mu)} \Leftrightarrow [xA^{(\lambda,\mu)}]_\alpha = [yA^{(\lambda,\mu)}]_\alpha, \forall\, \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i.$

$\Leftrightarrow x[A^{(\lambda,\mu)}]_\alpha = y[A^{(\lambda,\mu)}]_\alpha$, by Theorem 4.5 (i).

$\Leftrightarrow x^{-1}y \in [A^{(\lambda,\mu)}]_\alpha$, since each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G.

$(ii)\ A^{(\lambda,\mu)}x =\ A^{(\lambda,\mu)}y \Leftrightarrow [A^{(\lambda,\mu)}x]_\alpha = [A^{(\lambda,\mu)}y]_\alpha, \forall\, \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i.$

$\Leftrightarrow [A^{(\lambda,\mu)}]_\alpha x = [A^{(\lambda,\mu)}]_\alpha y$, by Theorem 4.5 (ii).

$\Leftrightarrow xy^{-1} \in [A^{(\lambda,\mu)}]_\alpha$, since each $[A^{(\lambda,\mu)}]_\alpha$ is a subgroup of G.

6. Homomorphisms of $(\boldsymbol{\lambda}, \boldsymbol{\mu}) -$ Multi fuzzy subgroup

In this section we shall prove some theorems on (λ,μ) −MFSGs of a group by homomorphism.

Preposition 6.1 :

Let $f: X \to Y$ be an onto map. If $A\ and\ B$ are multi-fuzzy sets with dimension k of X and Y respecively , then the following hold :

$(i)\ f\left(\left[A^{(\lambda,\mu)}\right]_\alpha\right) \subseteq [f(A^{(\lambda,\mu)})]_\alpha)$

$(ii) f^{-1}\left([B^{(\lambda,\mu)}]_\alpha\right) = [f^{-1}\left(B^{(\lambda,\mu)}\right)]_\alpha\big], \forall\, \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i.$

Proof :

(i) Let $y \in f\left(\left[A^{(\lambda,\mu)}\right]_\alpha\right)$. Then there exist an element $x \in \left[A^{(\lambda,\mu)}\right]_\alpha$ such that $f(x) = y$. Then we have $A^{(\lambda,\mu)}(x) \geq \alpha$,

Since $x \in \left[A^{(\lambda,\mu)}\right]_\alpha$

$\Rightarrow A_i^{(\lambda_i,\mu_i)}(x) \geq \alpha_i$

$\Rightarrow max\{A_i^{(\lambda_i,\mu_i)}(x): x \in f^{-1}(y)\} \geq \alpha_i, \forall i.$

$\Rightarrow max\{A^{(\lambda,\mu)}(x): x \in f^{-1}(y)\} \geq \alpha$

$\Longrightarrow f\left(A^{(\lambda,\mu)}\right)(y) \geq \alpha \Longrightarrow y \in [f\left(f\left(A^{(\lambda,\mu)}\right)\right)]_\alpha$

Therefore, $\left(\left[A^{(\lambda,\mu)}\right]_\alpha\right) \subseteq [f\left(A^{(\lambda,\mu)}\right)]_\alpha, \forall A^{(\lambda,\mu)} \in (\lambda,\mu) - MFS(X)$.

(ii) Let $x \in [f^{-1}\left(B^{(\lambda,\mu)}\right)]_\alpha \Leftrightarrow \{x \in X : f^{-1}\left(B^{(\lambda,\mu)}\right)(x) \geq \alpha\}$

$\Leftrightarrow \left\{x \in X : f^{-1}\left(B_i^{(\lambda_i,\mu_i)}\right)(x) \geq \alpha_i\right\}, \forall i.$

$\Leftrightarrow \left\{x \in X : B_i^{(\lambda_i,\mu_i)}(f(x)) \geq \alpha_i\right\}, \forall i.$

$\Leftrightarrow \left\{x \in X : B^{(\lambda,\mu)}(f(x)) \geq \alpha\right\}, \forall i.$

$\Leftrightarrow \{x \in X : f(x) \in [B^{(\lambda,\mu)}]_\alpha \Leftrightarrow \{x \in X : x \in f^{-1}(\left[B^{(\lambda,\mu)}\right]_\alpha)\}$

$\Leftrightarrow f^{-1}(\left[B^{(\lambda,\mu)}\right]_\alpha)$

Theorem 6.2

Let $f: G_1 \to G_2$ be an onto homomorphism and if $A^{(\lambda,\mu)}$ is a (λ,μ) −MFSG of G_1, then $f(B^{(\lambda,\mu)})$ is a (λ,μ) −MFSG of group G_2.

Proof :

By theorem 4.4 , it is enough to prove that each $[f(A^{(\lambda,\mu)})]_\alpha$ is a subgroup of G_2. $\forall \alpha \in [0,1]^k$ with $0 \leq \alpha_i \leq 1, \forall i$. Let $y_1, y_2 \in [f(A^{(\lambda,\mu)})]_\alpha$.

Then $f\left(A^{(\lambda,\mu)}\right)(y_1) \geq \alpha$ and $f\left(A^{(\lambda,\mu)}\right)(y_2) \geq \alpha$

$\Rightarrow f\left(A_i^{(\lambda_i,\mu_i)}\right)(y_1) \geq \alpha_i$

$\Rightarrow f\left(A_i^{(\lambda_i,\mu_i)}\right)(y_2) \geq \alpha_i, \forall i \ldots\ldots\ldots (1)$

By the proposition 6.1(i),we have $f\left(\left[f\left(A^{(\lambda,\mu)}\right)\right]_\alpha\right) \subseteq \left[f(f\left(A^{(\lambda,\mu)}\right))\right]_\alpha), \forall f\left(A^{(\lambda,\mu)}\right) \in (\lambda,\mu) - MFS(G_1)$.

Since f is onto, there exists some x_1 and x_2 in G_1 such that $f(x_1)=y_1$ and $f(x_2)=y_2$.Therefore, (1) can be written as $f\left(A_i^{(\lambda_i,\mu_i)}\right)(f(x_1)) \geq \alpha_i \; and \; f\left(A_i^{(\lambda_i,\mu_i)}\right)(f(x_2)) \geq \alpha_i, \forall i.$

$\Rightarrow f\left(A_i^{(\lambda_i,\mu_i)}\right)(x_1) \geq f\left(A_i^{(\lambda_i,\mu_i)}\right)(f(x_1)) \geq \alpha_i$ and $A_i^{(\lambda_i,\mu_i)}(x_2) \geq f\left(A_i^{(\lambda_i,\mu_i)}\right)(f(x_2)) \geq \alpha_i, \forall i.$

$\Rightarrow A_i^{(\lambda_i,\mu_i)}(x_1) \geq \alpha_i$ and $A_i^{(\lambda_i,\mu_i)}(x_2) \geq \alpha_i, \forall i.$

$\Rightarrow A^{(\lambda,\mu)}(x_1) \geq \alpha$ and $A^{(\lambda,\mu)}(x_2) \geq \alpha$.

$\Rightarrow min\{A^{(\lambda,\mu)}(x_1), A^{(\lambda,\mu)}(x_2)\} \geq \alpha$.

$\Rightarrow A^{(\lambda,\mu)}(x_1 {x_2}^{-1}) \geq min\{A^{(\lambda,\mu)}(x_1), A^{(\lambda,\mu)}(x_2)\} \geq \alpha$, since $A^{(\lambda,\mu)} \in (\lambda,\mu) - MFSG(G_1)$.

$\Rightarrow A^{(\lambda,\mu)}(x_1 x_2{}^{-1}) \geq \alpha$

$\Rightarrow x_1 x_2{}^{-1} \in [A^{(\lambda,\mu)}]_\alpha \Longrightarrow f(x_1 x_2{}^{-1}) \in f([A^{(\lambda,\mu)}]_\alpha) \subseteq [f(A^{(\lambda,\mu)})]_\alpha$

$\Longrightarrow f(x_1)f(x_2{}^{-1}) \in [f(A^{(\lambda,\mu)})]_\alpha \Longrightarrow f(x_1)f(x_2)^{-1} \in [f(A^{(\lambda,\mu)})]_\alpha \Longrightarrow y_1 y_2{}^{-1} \in [f(A^{(\lambda,\mu)})]_\alpha \Longrightarrow [f(A^{(\lambda,\mu)})]_\alpha$ is a subgroup of $G_2, \forall \alpha \in [0,1]^k \Rightarrow f(A^{(\lambda,\mu)}) \in (\lambda,\mu) - MFSG(G_2)$

Corollary 6.3 :

If $f: G_1 \to G_2$ be a homomorphism of a group G_1 onto a group G_2 and $\{ A_i{}^{(\lambda_i,\mu_i)} : i \in I \}$ be a family of $(\lambda,\mu) - MFSG$s of G_1, then $f\big(\cap A_i{}^{(\lambda_i,\mu_i)}\big)$ is an $(\lambda,\mu) - MFSG$ of G_2.

Theorem 6.4 :

Let $f: G_1 \to G_2$ be a homomorphism of a group G_1 into a group G_2. If $B^{(\lambda,\mu)}$ is an $(\lambda,\mu) - MFSG$ of G_2, then $f^{-1}(B^{(\lambda,\mu)})$ is also a $(\lambda,\mu) - MFSG$ of G_1.

Proof :

By theorem 4.4, it is enough to prove that $[f^{-1}(B^{(\lambda,\mu)})]_\alpha$ is a subgroup of G_1, with $0 \leq \alpha_i \leq 1, \forall i$.

Let $x_1, x_2 \in [f^{-1}(B^{(\lambda,\mu)})]_\alpha$. Then $f^{-1}\big(B^{(\lambda,\mu)}\big)(x_1) \geq \alpha$ and $f^{-1}\big(B^{(\lambda,\mu)}\big)(x_2) \geq \alpha \Longrightarrow B^{(\lambda,\mu)}\big(f(x_1)\big) \geq \alpha \; and \; B^{(\lambda,\mu)}(f(x_2)) \geq \alpha$

$\Longrightarrow min\{B^{(\lambda,\mu)}\big(f(x_1)\big), B^{(\lambda,\mu)}(f(x_2))\} \geq \alpha$

$\Longrightarrow B^{(\lambda,\mu)}(f(x_1)f(x_2)^{-1} \geq min\{B^{(\lambda,\mu)}\big(f(x_1)\big), B^{(\lambda,\mu)}(f(x_2))\} \geq \alpha$, since $B^{(\lambda,\mu)} \in (\lambda,\mu) - MFSG(G_2)$.

$\Longrightarrow (f(x_1)f(x_2)^{-1} \in [B^{(\lambda,\mu)}]_\alpha \Longrightarrow f(x_1 x_2{}^{-1}) \in [B^{(\lambda,\mu)}]_\alpha$,since f is a homomorphism.

$\Longrightarrow x_1 x_2{}^{-1} \in f^{-1}([B^{(\lambda,\mu)}]_\alpha)= \big[f^{-1}(B^{(\lambda,\mu)})\big]_\alpha$, by the preposition 6.1(ii).

$\Longrightarrow x_1 x_2{}^{-1} \in \big[f^{-1}(B^{(\lambda,\mu)})\big]_\alpha \Longrightarrow \big[f^{-1}(B^{(\lambda,\mu)})\big]_\alpha$ is a subgroup of G_1.

$\Longrightarrow f^{-1}(B^{(\lambda,\mu)})$ is a $(\lambda,\mu) - MFSG$ of G_1.

Theorem 6.5 :

Let $f: G_1 \to G_2$ be a surjective homomorphism and if $A^{(\lambda,\mu)}$is a $(\lambda,\mu) - MFSG$ of a group G_1, then $f\big(A^{(\lambda,\mu)}\big)$ is also a $(\lambda,\mu) - MFNSG$ of a group G_2.

Proof :

Let $g_2 \in G_2$ and $y \in f\big(A^{(\lambda,\mu)}\big)$. Since f is surjective, there exists $g_1 \in G_1$ and $x \in A^{(\lambda,\mu)}$, such that $f(x) = y$ and $f(g_1) = g_2$.

Also, since $A^{(\lambda,\mu)}$ is a $(\lambda,\mu) - MFNSG$ of G_1, $A^{(\lambda,\mu)}(g_1{}^{-1}xg_1) = A^{(\lambda,\mu)}(x), \forall \; x \in A^{(\lambda,\mu)} \; and \; g_1 \in G_1$.

Now consider, $f\big(A^{(\lambda,\mu)}\big)(g_2{}^{-1}xg_2) = f\big(A^{(\lambda,\mu)}\big)\big(f(g_1{}^{-1}xg_1)\big) = f\big(A^{(\lambda,\mu)}\big)(y')$, since f is a homomorphism, where $y' = f(g_1{}^{-1}xg_1) = g_2{}^{-1}yg_2 = max \, \big\{A^{(\lambda,\mu)}(x') : f(x') = y' for \; x' \in G_1\big\} = max \, \big\{A^{(\lambda,\mu)}(x') : f(g_1{}^{-1}xg_1) \; for \; x' \in G_1\big\}$ $=max\{ \; A^{(\lambda,\mu)}(g_1{}^{-1}xg_1) : \; f(g_1{}^{-1}xg_1) = y'\} = \; g_2{}^{-1}yg_2 \; for \; x \in \; A^{(\lambda,\mu)} \; , g_1 \in G_1\} = max\{ \; A^{(\lambda,\mu)}(x) : \; f(g_1{}^{-1}xg_1) = y'\} = \; g_2{}^{-1}yg_2 \; for \; x \in A^{(\lambda,\mu)} \; , g_1 \in G_1\} = \; max\{ \; A^{(\lambda,\mu)}(x) : f(g_1)^{-1}f(x)f(g_1) = \; g_2{}^{-1}yg_2 \; for \; x \in \; A^{(\lambda,\mu)} \; , g_1 \in G_1\} = \; max\{ \; A^{(\lambda,\mu)}(x) : \; g_2{}^{-1}f(x)g_2 = g_2{}^{-1}yg_2 \; for \; x \in G_1\} = max\big\{A^{(\lambda,\mu)}(x) : f(x) = y \; for \; x \in G_1\big\} = \; f(A^{(\lambda,\mu)})(y)$. Hence $f(A^{(\lambda,\mu)})$ is a $(\lambda,\mu) - MFNSG$ of G_2.

Theorem 6.6 :

If $A^{(\lambda,\mu)}$ is a $(\lambda,\mu) - MFNSG$ of a group G, then there exists a natural homomorphism $f: G \to G/A^{(\lambda,\mu)}$ defined by $f(x) = xA^{(\lambda,\mu)}$,$\forall x \in G$.

Proof :

Let $f: G \rightarrow G/A^{(\lambda,\mu)}$ defined by $(x) = xA^{(\lambda,\mu)}$,$\forall x \in G$.

Claim 1: f is a homomorphism

That is, to prove that : $f(xy) = f(x)f(y), \forall\, x, y \in G, or\ \ (xy)A^{(\lambda,\mu)} = \left(xA^{(\lambda,\mu)}\right)\left(yA^{(\lambda,\mu)}\right), \forall\, x, y \in G$

Since $A^{(\lambda,\mu)}$ is a $(\lambda, \mu) - MFNSG$ of G, we have$A^{(\lambda,\mu)}(g^{-1}xg) = A^{(\lambda,\mu)}(x)$, $\forall\, x \in A^{(\lambda,\mu)}\ and\ g \in G$.

Equivalently, $A^{(\lambda,\mu)}(xy) = A^{(\lambda,\mu)}(yx)$, $\forall\, x, y \in G$.

Also,$\forall\, g \in G$, we have $\left(xA^{(\lambda,\mu)}\right)(g) = (A^{(\lambda,\mu)}(x^{-1}g))$

$\left(yA^{(\lambda,\mu)}\right)(g) = (A^{(\lambda,\mu)}(y^{-1}g))$

$[(xy)A^{(\lambda,\mu)}](g) = \left(A^{(\lambda,\mu)}((xy)^{-1}g)\right)$, $\forall\, g \in G$.

By definition 3.13, we have $\left[\left(xA^{(\lambda,\mu)}\right)\left(yA^{(\lambda,\mu)}\right)\right](g) = (min\{xA^{(\lambda,\mu)}(r), yA^{(\lambda,\mu)}(s)\}: g = rs\ )$

$=[min\{A^{(\lambda,\mu)}(x^{-1}r), A^{(\lambda,\mu)}(y^{-1}s)\}: g = rs]$

Claim 2 : $A^{(\lambda,\mu)}[(xy)^{-1}g\ ] = max[min\ \{\ A^{(\lambda,\mu)}(x^{-1}r), A^{(\lambda,\mu)}(y^{-1}s)\ \} : g = rs\ ]$, $\forall\, g \in G$.

Consider $A^{(\lambda,\mu)}[(xy)^{-1}g] = A^{(\lambda,\mu)}[y^{-1}x^{-1}g] = A^{(\lambda,\mu)}[y^{-1}x^{-1}rs]$, since $g = rs$.

$= A^{(\lambda,\mu)}[y^{-1}(x^{-1}rsy^{-1})y] = A^{(\lambda,\mu)}[x^{-1}rsy^{-1}]$, $since$ $A^{(\lambda,\mu)}$ is normal.

$\geq \min\{\ A^{(\lambda,\mu)}(x^{-1}r), A^{(\lambda,\mu)}(sy^{-1})\ \}$, since $A^{(\lambda,\mu)}$ is $(\lambda, \mu) - MFSG$.

$= \min\{\ A^{(\lambda,\mu)}(x^{-1}r), A^{(\lambda,\mu)}(y^{-1}s)\ \}$, $\forall\, g = rs\ \in G$, since $A^{(\lambda,\mu)}$ is normal.

Therefore, $A^{(\lambda,\mu)}[(xy)^{-1}g\ ] = \max[\min\ \{\ A^{(\lambda,\mu)}(x^{-1}r), A^{(\lambda,\mu)}(y^{-1}s)\ \} : g = rs\ ]$, $\forall\, g \in G$.

Which proves the Claim 2.

Thus, $\left[(xy)A^{(\lambda,\mu)}\right](g) = \left[\left(xA^{(\lambda,\mu)}\right)\left(yA^{(\lambda,\mu)}\right)\right](g), \forall\, g\ \in G \Longrightarrow (xy)A^{(\lambda,\mu)} = \left(xA^{(\lambda,\mu)}\right)\left(yA^{(\lambda,\mu)}\right)$

$\Longrightarrow f(xy) = f(x)f(y) \Longrightarrow$ f is a homomorphism. This proves the Claim 1 and hence the Theorem .

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