Negative Binomial Logistic Distribution

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Abstract:
A new family of distributions via, Binomial Logistic Distribution (NBLD) is introduced. The various characteristics of the distribution are derived. The structural analysis of the distribution includes moments, mode, skewness, kurtosis, hazard rate. Also describes the quantile method of estimation, likelihood method of estimation, order statistics and stochastic orders. The goodness of the distribution is tested with a real data.

Keywords: Discrete Binomial distribution, logistic distribution, hazard rate, estimation, Marshall-Olkin family

1. Introduction

The simplicity of the logistic distribution and its importance as a growth curve has made it one of the many important statistical distributions. The shape of the logistic distribution is similar to that of the normal distribution makes it simpler and also profitable on suitable occasions to replace the normal distribution by the logistic distribution with negligible errors in the respective theories. Pear and Reed (1920) Pear et al. (1940), and Schultz (1930) applied the logistic model as a growth model in human populations as well as in some biological organisms. Oliver (1964) used the logistic function in terms of modelling data related to agricultural population. A few more interesting uses of the logistic function are in the analysis of survival data, Plackett (1959). Gupta and Kundu (2010) discussed various properties of the two generalizations of the logistic distributions, namely the skew logistic and the second type which they termed as proportional reversed hazard family with the baseline distribution as the logistic distribution. The second one is alternatively known as Type I generalized logistic distribution. However, the skew logistic distribution (SLD) was first proposed by Wahed and Ali (2001). Nadarajah (2009) extended this SLD by introducing a scale parameter, and he studied its distributional properties. Chakraborty et al. (2012) has proposed a new SLD by considering a new skew function where the skew function is not a cumulative distribution function (c.d.f.). The importance of the logistic distribution has already been felt in many areas of human endeavour. Verhulst (1845) used it in economics and demographic studies. Berkson (1944 1951) used the distribution extensively in analyzing bioassay and quantal response data. The works Berkson (1953), George-et-al (1980), Ojo (1989), Ojo (2002) are a few of many publications on logistic distribution.

Recently there has been increasing interest in defining new generated families of univariate continuous distributions by introducing additional shape parameters to the baseline model. The generated distributions have attracted several statisticians to develop new models.

Marshall and Olkin(1997) introduced a new family of distributions by adding a parameter to a family of distributions. They started with a survival function $\overline{F}(x)$ and considered a family of survival functions given by

$$\overline{G}\left(x,\alpha\right)=\frac{\alpha\overline{F}(x)}{1-(1-\alpha)\overline{F}(x)} \qquad For -\infty < x < \infty \text{ and } 0<\alpha<\infty \tag{1.1}$$

An interesting property of this family of distributions is the following: Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with survival function $\overline{F}(x)$. Let N be a geometric random variable with probability mass function $P\left(N=n\right)=\alpha(1-\alpha)^{n-1}$; n = 1, 2 ... and $0<\alpha<1$. Then $U_N=\min\left(X_1,X_2\ldots X_N\right)$ has a survival function given by equation (1). If $\alpha>1$, N is geometric random variable with probability mass function $P\left(N=n\right)=\alpha^{-1}(1-\alpha^{-1})^{n-1}$ n= 1,2,... Then $V_N=\max\left(X_1,X_2\ldots X_N\right)$ also has the survival function (1.1).

Many authors have proposed various univariate distributions belonging to the family of Marshall-Olkin distributions. A few among them are Marshall-Olkin Pareto by Alice and Jose (2003), Marshall-Olkin Weibull by Ghitany et al.(2005), Marshall-Olkin semi Weibull by Alice and Jose(2005), Marshall–Olkin Extended Lomax Distribution and Its Application to Censored Data(2007), Marshall –Olkin q Weibull by Jose et al.(2010)), etc:. Also Jayakumar and Thomas (2008), explained a generalization to Marshall–Olkin scheme and its application to Burr type XII distribution. They proposed a generalization to the family of distributions as (1.1) as

$$\bar{G}\left(x,\alpha,\gamma\right)=\left(\frac{\alpha\bar{F}(x)}{1-(1-\alpha)\bar{F}(x)}\right)^{\gamma},\quad \alpha>0,\gamma>0 \tag{1.2}$$

Nataraja et-al.(2013) proposed a generalization to the Marshal-Olkin form by replacing the geometric distribution with truncated negative binomial distribution having p.m.f.

$$P\left(N=n\right)=\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\binom{\theta+n-1}{\theta-1}\left(1-\alpha\right)^{n},\quad \text{for n}=1,2,...$$

and arrived in a form

$$\bar{G}\left(x,\alpha,\theta\right)=\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\frac{1}{\left(F\left(x\right)+\alpha\bar{F}(x)\right)^{\theta}}-1\right],\quad \alpha>0,\theta>0\,;-\infty<x<\infty \tag{1.3}$$

and when $\alpha\to1$, $\bar{G}\left(x\right)\to\bar{F}(x)$ When $\theta=1$, the introduced family of distributions in (1.3) becomes the family of Marshall-Olkin distributions.

This family of distributions can be interpreted as follows: Suppose the failure times of a device are observed. Every time a failure occur, the device is repaired to resume function. Suppose also that the device is seemed no longer usable when a failure occurs that exceeds a certain level of severity. Let $X_1,X_2,\ldots$ denote the failure times and let N denote the number of failures, then U_N will represent the time to first failure of device. Hence the new model could be used to represent the time to first failure and life time.

Third generalized family of distribution is introduced by Sankaran and Jayakumar (2016), by replacing the distribution N by discrete Mittag-Leffler distribution. They derived a family of distributions with parameters α and c having survival function

$$\bar{G}(x)=\frac{1-F^{\alpha}(x)}{1-cF^{\alpha}(x)},c>0,0<\alpha<\infty. \tag{1.4}$$

Note that the Marshall-Olkin method applied to F^{α} the exponential form of a parent distribution function F. will also give rise to $\bar{G}(x)$ in (1.4). The family of distribution generated by truncated discrete Mittag-leffler distribution can also be considered as a generalization to Marshall-Olkin family of distributions since it reduces to Marshall-Olkin family when $\alpha=1$ and $c=\frac{1-p}{p}$.

2. Negative binomial family of logistic distribution

The logistic survival function is defined as

$$\bar{F}\left(x,\beta\right)=\frac{1}{1+e^{\beta x}},\quad \beta>0,-\infty<x<\infty \tag{2.1}$$

Substituting this in (1.3), we get

$$\overline{G}(x,\alpha,\theta,\beta)=\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\frac{\left(e^{\beta x}+1\right)^{\theta}}{\left(\alpha+e^{\beta x}\right)^{\theta}}-1\right] \tag{2.2}$$

The density function is

$$g(x,\alpha,\theta,\beta)=\frac{\alpha^{\theta}}{\left(1-\alpha^{\theta}\right)}\frac{\theta\beta(1-\alpha)e^{\beta x}\left(e^{\beta x}+1\right)^{\theta-1}}{\left[\alpha+e^{\beta x}\right]^{\theta+1}} \tag{2.3}$$

The expression for r^th order moment is

$$E\left(X^{r}\right) =\int_{-\infty}^{\infty} x^{r}\frac{\alpha^{\theta}}{\left(1-\alpha^{\theta}\right)}\frac{\theta\beta(1-\alpha)e^{\beta x}\left(e^{\beta x}+1\right)^{\theta-1}}{\left[\alpha+e^{\beta x}\right]^{\theta+1}}dx, \quad r=1,2,...$$

Measure of skewness $\gamma_1=\sqrt{\beta_1}=\sqrt{\left(\frac{\mu_3^2}{\mu_2^3}\right)}$.

Moment measure of kurtosis $\gamma_2=\beta_2-3$.

Since the expression is not easy to calculate, we find the mean, variance, measure of skewness and kurtosis by numerical methods for $\beta=1$. Table 1 provides the mean for various values of α and θ. Table 2 for measure of skewness and table 3 for kurtosis for various values of α and θ.

Table 1. : Table of mean for various values of α and θ, for β =1

α/θ	1	2	3	4	5	6	7	8	9	10	11
0.1	-2.303	-3.121	-3.640	-3.992	-4.252	-4.458	-4.628	-4.774	-4.901	-5.014	-5.115
0.3	-1.204	-1.742	-2.186	-2.537	-2.814	-3.036	-3.219	-3.374	-3.508	-3.626	-3.731
0.5	-0.693	-1.026	-1.336	-1.615	-1.863	-2.079	-2.267	-2.431	-2.575	-2.702	-2.816
0.7	-0.357	-0.533	-0.706	-0.874	-1.035	-1.189	-1.334	-1.471	-1.600	-1.720	-1.832
0.9	-0.105	-0.158	-0.211	-0.263	-0.315	-0.367	-0.419	-0.470	-0.521	-0.572	-0.622
1.1	0.095	0.143	0.190	0.238	0.285	0.332	0.379	0.426	0.472	0.518	0.564
1.3	0.262	0.393	0.522	0.649	0.773	0.893	1.010	1.123	1.232	1.336	1.435
1.5	0.405	0.605	0.800	0.988	1.166	1.334	1.491	1.637	1.772	1.897	2.012
1.7	0.531	0.790	1.038	1.270	1.485	1.681	1.858	2.018	2.161	2.290	2.407
1.9	0.642	0.952	1.243	1.509	1.748	1.959	2.145	2.309	2.454	2.582	2.697
2.1	0.742	1.097	1.423	1.714	1.968	2.187	2.377	2.541	2.684	2.810	2.922
2.3	0.833	1.227	1.582	1.891	2.155	2.378	2.568	2.731	2.872	2.996	3.106
2.5	0.916	1.345	1.724	2.047	2.317	2.541	2.730	2.891	3.031	3.153	3.262
2.7	0.993	1.453	1.852	2.185	2.458	2.683	2.871	3.030	3.168	3.288	3.396
2.9	1.065	1.552	1.968	2.308	2.584	2.808	2.994	3.152	3.288	3.408	3.515
3.1	1.131	1.644	2.073	2.420	2.696	2.919	3.104	3.260	3.395	3.514	3.620

Table 2: Table of variance for various values of α and θ, for β =1

α/θ	1	2	3	4	5	6	7	8	9	10	11
0.1	3.290	2.620	2.230	2.041	1.942	1.883	1.844	1.816	1.795	1.778	1.765
0.3	3.290	3.000	2.678	2.412	2.222	2.093	2.006	1.946	1.903	1.870	1.846
0.5	3.290	3.179	3.019	2.839	2.662	2.501	2.365	2.253	2.163	2.092	2.035
0.7	3.290	3.259	3.209	3.143	3.066	2.979	2.889	2.796	2.705	2.618	2.536
0.9	3.290	3.287	3.282	3.276	3.268	3.258	3.246	3.233	3.219	3.202	3.185
1.1	3.290	3.288	3.284	3.279	3.272	3.264	3.254	3.243	3.231	3.218	3.203
1.3	3.290	3.273	3.245	3.208	3.161	3.108	3.049	2.986	2.920	2.852	2.784
1.5	3.290	3.250	3.187	3.105	3.010	2.908	2.803	2.699	2.600	2.508	2.425
1.7	3.290	3.223	3.120	2.995	2.858	2.721	2.591	2.474	2.370	2.281	2.206
1.9	3.290	3.194	3.053	2.889	2.722	2.566	2.429	2.314	2.218	2.141	2.079
2.1	3.290	3.164	2.987	2.793	2.606	2.444	2.309	2.202	2.118	2.053	2.001
2.3	3.290	3.135	2.925	2.707	2.510	2.348	2.221	2.124	2.051	1.995	1.951
2.5	3.290	3.106	2.868	2.632	2.431	2.273	2.155	2.067	2.003	1.954	1.917
2.7	3.290	3.079	2.816	2.567	2.365	2.214	2.104	2.026	1.968	1.925	1.892
2.9	3.290	3.053	2.768	2.511	2.311	2.167	2.065	1.994	1.942	1.903	1.874
3.1	3.290	3.028	2.724	2.462	2.266	2.129	2.034	1.969	1.921	1.886	1.859

Table 3: Table of γ_1 for various values of α and θ, β =1

α/θ	1	2	3	4	5	6	7	8	9	10	11
0.1	-1.387	-1.289	-1.222	-1.184	-1.160	-1.144	-1.133	-1.124	-1.117	-1.111	-1.106
0.3	-1.321	-1.435	-1.418	-1.370	-1.326	-1.290	-1.262	-1.240	-1.223	-1.209	-1.197
0.5	-0.980	-1.257	-1.400	-1.455	-1.461	-1.445	-1.419	-1.392	-1.367	-1.343	-1.323
0.7	-0.564	-0.806	-1.007	-1.166	-1.285	-1.369	-1.425	-1.458	-1.476	-1.482	-1.480
0.9	-0.174	-0.259	-0.343	-0.426	-0.506	-0.584	-0.658	-0.730	-0.798	-0.863	-0.925
1.1	0.157	0.235	0.311	0.387	0.460	0.532	0.602	0.669	0.733	0.795	0.854
1.3	0.424	0.618	0.794	0.949	1.080	1.189	1.276	1.344	1.396	1.433	1.459
1.5	0.634	0.893	1.098	1.250	1.354	1.421	1.458	1.475	1.479	1.473	1.462
1.7	0.798	1.082	1.275	1.389	1.448	1.470	1.469	1.457	1.439	1.419	1.398
1.9	0.927	1.210	1.371	1.444	1.465	1.458	1.438	1.414	1.389	1.365	1.344
2.1	1.027	1.295	1.420	1.458	1.453	1.430	1.401	1.373	1.347	1.324	1.305
2.3	1.107	1.351	1.442	1.453	1.431	1.399	1.367	1.339	1.314	1.293	1.275
2.5	1.169	1.388	1.449	1.440	1.408	1.372	1.339	1.311	1.288	1.269	1.253
2.7	1.219	1.411	1.447	1.423	1.385	1.347	1.315	1.289	1.267	1.249	1.235
2.9	1.259	1.425	1.440	1.406	1.364	1.326	1.295	1.270	1.250	1.234	1.220
3.1	1.291	1.432	1.430	1.389	1.345	1.308	1.278	1.255	1.236	1.221	1.209

Table 4: Table of γ_2 for various values of α and θ, for β =1

α/θ	1	2	3	4	5	6	7	8	9	10	11
0.1	2.414	1.915	1.680	1.557	1.483	1.433	1.397	1.370	1.348	1.330	1.316
0.3	3.391	2.896	2.513	2.242	2.054	1.921	1.823	1.750	1.692	1.646	1.608
0.5	3.881	3.614	3.330	3.061	2.824	2.625	2.460	2.324	2.213	2.120	2.043
0.7	4.110	4.023	3.915	3.790	3.656	3.516	3.377	3.242	3.113	2.992	2.880
0.9	4.192	4.184	4.173	4.160	4.144	4.126	4.105	4.083	4.058	4.031	4.002
1.1	4.193	4.187	4.178	4.167	4.154	4.139	4.122	4.103	4.083	4.060	4.036
1.3	4.150	4.102	4.040	3.965	3.881	3.789	3.691	3.591	3.489	3.389	3.290
1.5	4.084	3.975	3.840	3.689	3.530	3.371	3.216	3.070	2.936	2.813	2.702
1.7	4.006	3.831	3.627	3.414	3.206	3.014	2.841	2.689	2.558	2.444	2.345
1.9	3.923	3.685	3.424	3.169	2.938	2.737	2.568	2.425	2.307	2.207	2.124
2.1	3.840	3.545	3.240	2.961	2.723	2.527	2.368	2.239	2.134	2.048	1.976
2.3	3.758	3.414	3.078	2.788	2.552	2.365	2.219	2.103	2.010	1.934	1.872
2.5	3.679	3.294	2.936	2.642	2.414	2.239	2.104	1.999	1.916	1.849	1.794
2.7	3.604	3.183	2.812	2.521	2.301	2.137	2.014	1.919	1.844	1.783	1.734
2.9	3.532	3.083	2.704	2.418	2.208	2.055	1.941	1.854	1.785	1.731	1.685
3.1	3.465	2.992	2.610	2.330	2.130	1.987	1.881	1.801	1.738	1.687	1.646

Close observation on the four tables we can have an approximate idea of parameters of a given data set.

Random variable generation $X = \frac{1}{\beta}\ln\left(\frac{\alpha\left(1-U\alpha^{\theta}+\alpha^{\theta}\right)^{1/\theta}-\alpha}{\alpha-\left(1-U\alpha^{\theta}+\alpha^{\theta}\right)^{1/\theta}}\right)$ where $U \sim U(0,1)$.

$$\operatorname{Median}(X) = \frac{1}{\beta}\ln\left(\frac{\alpha\left(\frac{1-\alpha^{\theta}}{2}\right)^{\frac{1}{\theta}}-\alpha}{\alpha-\left(\frac{1-\alpha^{\theta}}{2}\right)^{\frac{1}{\theta}}}\right)$$

The first derivative of $\log(g)$ is

$$\left(\ln(g)\right)^{/} = \beta + \frac{(\theta-1)\beta e^{\beta x}}{\left(1+e^{\beta x}\right)} - \frac{(\theta+1)\beta e^{\beta x}}{\left(\alpha+e^{\beta x}\right)}$$

$$\operatorname{Mode}(x) = \frac{1}{\beta}\ln\left(\frac{\theta(\alpha-1)\pm\sqrt{\theta^2(\alpha-1)^2+4\alpha}}{2}\right)$$

$$\text{Mode} = \begin{cases} 0 & for\ \alpha = 1 \\ < 0 & for\ \alpha < 1 \\ > 0 & for\ \alpha > 1 \end{cases}$$

p.d.f is decreasing for $x > \frac{1}{\beta}\ln\left(\frac{\theta(\alpha-1)\pm\sqrt{\theta^2(\alpha-1)^2+4\alpha}}{2}\right)$

increasing for $x < \frac{1}{\beta}\ln\left(\frac{\theta(\alpha-1)\pm\sqrt{\theta^2(\alpha-1)^2+4\alpha}}{2}\right)$

Second derivative of log G is

$$\left(\ln(g)\right)^{//} = \frac{(\theta-1)\beta^2 e^{\beta x}}{\left(1+e^{\beta x}\right)^2} - \frac{(\theta+1)\alpha\beta^2 e^{\beta x}}{\left(\alpha+e^{\beta x}\right)^2}$$

Point of inflexion is $\frac{1}{\beta}\ln\left(\frac{2\alpha\pm\sqrt{4\alpha^2(1+\theta)+\alpha\left(\theta^2-1\right)\left(\alpha^2-1\right)}}{(\theta-\alpha\theta-\alpha-1)}\right)$

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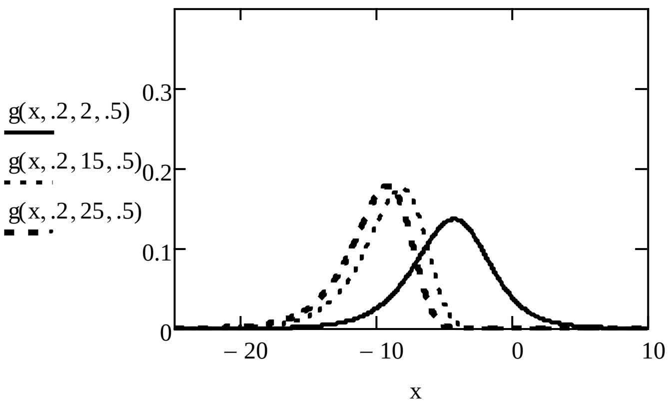
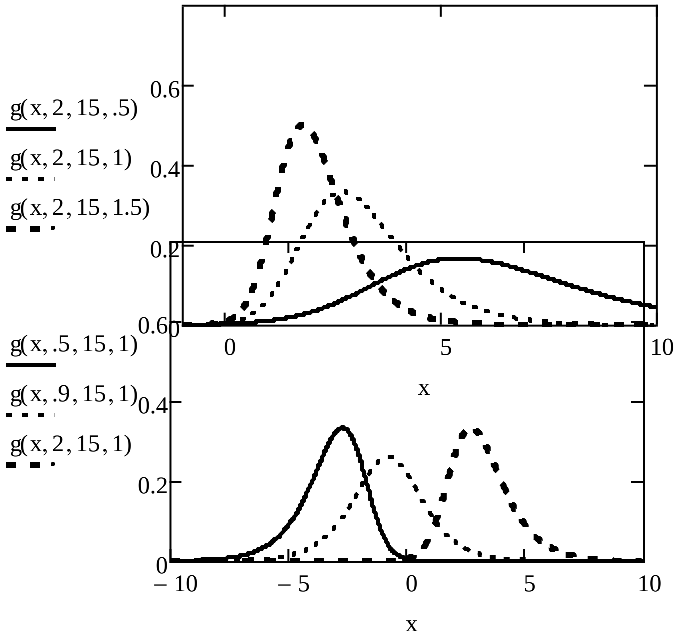




The hazard rate function is given by

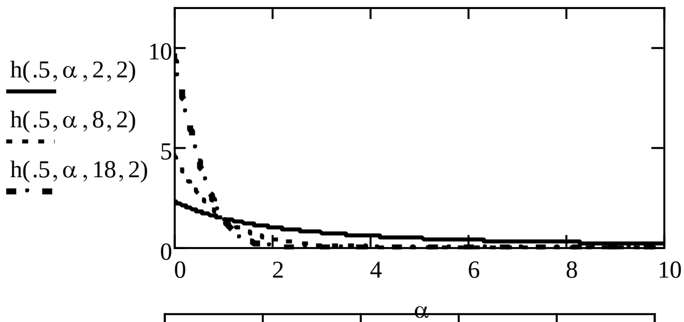
$$h(x)=\frac{\beta\theta(\alpha-1)e^{\beta x}}{\left[\left(\alpha+e^{\beta x}\right)^{\theta}-\left(1+\right.\right.}$$

$$h(0)=\frac{\beta\theta(\alpha-1)2^{\theta-1}}{\left((\alpha+1)^{\theta}-2^{\theta}\right)(\alpha+1)}$$

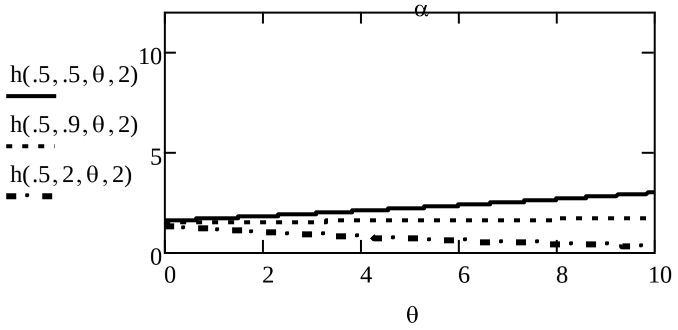
As β increases hazard rate increases for fixed value of θ *and* α.

$$h(0)=\frac{\beta\theta\left(1-\frac{1}{\alpha}\right)2^{\theta-1}}{\left((\alpha+1)^{\theta}-2^{\theta}\right)\left(1+\frac{1}{\alpha}\right)}$$

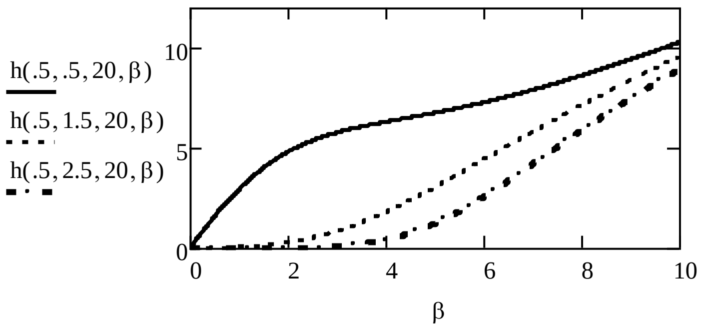
As α increases hazard rate decreases exponentially for fixed value of β *and* θ.



As α increases hazard rate decreases and the rate of decreases increases as θ increases.



As θ increases hazard rate decreases for $\alpha<1$, and increases for $\alpha>1$.



As β increases hazard rate increases fast for $\alpha < 1$, Increases fast as $\alpha > 1$.

3. Quantile method of estimation

$$X_{Q_1} = \frac{1}{\beta}\ln\left(\frac{\alpha\left(\frac{3+\alpha^{\theta}}{4}\right)^{1/\theta} - \alpha}{\alpha - \left(\frac{3+\alpha^{\theta}}{4}\right)^{1/\theta}}\right)$$

$$X_{Q_3} = \frac{1}{\beta}\ln\left(\frac{\alpha\left(\frac{1+3\alpha^{\theta}}{4}\right)^{1/\theta} - \alpha}{\alpha - \left(\frac{1+3\alpha^{\theta}}{4}\right)^{1/\theta}}\right)$$

$$\beta = \frac{1}{X_{Q_2}}\ln\left(\frac{\alpha\left(\frac{1+\alpha^{\theta}}{2}\right)^{1/\theta} - \alpha}{\alpha - \left(\frac{1+\alpha^{\theta}}{2}\right)^{1/\theta}}\right) \tag{3.1}$$

$$\alpha = \alpha\left(\frac{3+\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}} + \left(\left(\frac{3+\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}} - \alpha\right)\left[\frac{\alpha - \alpha\left(\frac{1+3\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}}}{\left(\frac{1+3\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}} - \alpha}\right]^{\frac{X_{Q_3}}{X_{Q_1}}} \tag{3.2}$$

$$\theta = \frac{\ln\left(\frac{1+\alpha^{\theta}}{2}\right)}{\ln\left[\alpha + \frac{\left(\alpha - \alpha\left(\frac{1+\alpha^{\theta}}{2}\right)^{\frac{1}{\theta}}\right)\left(\left(\frac{3+\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}} - \alpha\right)^{\frac{X_{Q_2}}{X_{Q_3}}}}{\left(\alpha - \alpha\left(\frac{3+\alpha^{\theta}}{4}\right)^{\frac{1}{\theta}}\right)^{\frac{X_{Q_2}}{X_{Q_3}}}}\right]} \tag{3.3}$$

Solving (3.1), (3.2) and (3.3) iteratively we get the values of parameters. For getting initial values of parameters one may use the tables of mean variance skewness and Kurtosis.

4. Likelihood method of estimation

The log likelihood equations are

$$\ln\ln(L) = n\theta ln(\alpha) + n.ln(\theta) + n.ln(\beta) + n.ln(1-\alpha) - n.ln(1-\alpha^{\theta}) + n\beta\bar{x}$$
$$+(\theta-1)\sum\ln\left(1+e^{\beta x_i}\right) - (\theta+1)\sum\ln\left(\alpha+e^{\beta x_i}\right) \quad (4.1)$$

$$\frac{\partial \ln(L)}{\partial\theta} = n.ln(\alpha) + \frac{n}{\theta} + \frac{n\alpha^{\theta}\ln(\alpha)}{1-\alpha^{\theta}} + \sum\ln\left(1+e^{\beta x_i}\right) - \sum\ln\left(\alpha+e^{\beta x_i}\right) \quad (4.2)$$

$$\frac{\partial \ln(L)}{\partial\alpha} = \frac{n\theta}{\alpha} - \frac{n}{1-\alpha} + \frac{n\theta\alpha^{\theta-1}}{1-\alpha^{\theta}} - \sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)} \quad (4.3)$$

$$\frac{\partial \ln(L)}{\partial\beta} = \frac{n}{\beta} + n\bar{x} + (\theta-1)\sum\frac{x_i e^{\beta x_i}}{\left(1+e^{\beta x_i}\right)} - (\theta+1)\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)} \quad (4.4)$$

The maximum likelihood estimates of α, θ *and* β are the solutions of simultaneous equations (4.2), (4.3) and (4.4). The solution of the four equations is not having a closed form. So numerical technique such as Newton Rapson method can be employed to get a solution. For getting initial values the tables for mean, variance, measure of skewness and Kurtosis can be used.

Now as in Bozidar et-al (2016) we study the existence and uniqueness of MLE when the other parameters are known or given.

Theorem 4.1

Let g1 = $\frac{\partial \ln(L)}{\partial\theta}$ where α *and* β are known. Then their exist a unique solution for g1 = 0 for $\theta \in (0,\infty)$.

Proof :

We have , $$g1 = n.ln(\alpha) + \frac{n}{\theta} + \frac{n\alpha^{\theta}\ln(\alpha)}{\alpha^{\theta}-1} + \sum\ln\left(1+e^{\beta x_i}\right) - \sum\ln\left(\alpha+e^{\beta x_i}\right)$$

Now

$$\underset{\theta\to 0}{Lim}\; g1 = n.ln(\alpha) + \frac{n}{\theta} + \frac{n\ln(\alpha)}{1-\alpha^{-\theta}} + \sum\ln\left(1+e^{\beta x_i}\right) - \sum\ln\left(\alpha+e^{\beta x_i}\right) = \infty$$

On the other hand

$$\underset{\theta\to\infty}{Lim}\; g1 = n.ln(\alpha) + \frac{n}{\theta} - \frac{n\alpha^{\theta}\ln(\alpha)}{\alpha^{\theta}-1} + \sum\ln\left(1+e^{\beta x_i}\right) - \sum\ln\left(\alpha+e^{\beta x_i}\right) < 0$$

Therefore their exist at least one root, say $\hat{\theta} \in (0,\infty)$ such that g1 = 0. To show uniqueness the first derivative of g1 is

$$\frac{\partial g1}{\partial\theta} = -\frac{n}{\theta^2} - \frac{n\ln(\alpha)\alpha^{-\theta}}{\left(1-\alpha^{-\theta}\right)^2} < 0$$

Hence their exist a solution for g1=0 and the root of $\hat{\theta}$ is unique.

Theorem 4.2

Let g2 = $\frac{\partial \ln(L)}{\partial\alpha}$ where θ *and* β are known. Then their exist a unique solution for g1 = 0 for $\alpha \in (0,\infty)$.

$$g2 = \frac{n\theta}{\alpha} - \frac{n}{1-\alpha} + \frac{n\theta\alpha^{\theta-1}}{1-\alpha^{\theta}} - \sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}$$

$$\lim_{\alpha\to 0} g2 = \lim_{\alpha\to 0}\left(\frac{n\theta}{\alpha}-\frac{n}{1-\alpha}+\frac{n\theta\alpha^{\theta-1}}{1-\alpha^{\theta}}-\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}\right)=\infty$$

Also

$$\lim_{\alpha\to \infty} g2 = \lim_{\alpha\to \infty}\left(\frac{n\theta}{\alpha}-\frac{n}{1-\alpha}+\frac{n\theta\alpha^{\theta-1}}{1-\alpha^{\theta}}-\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}\right)<0$$

$$\frac{\partial g2}{\partial\alpha}=\frac{-n\theta}{\alpha^2}-\frac{n}{(1-\alpha)^2}-\frac{n\theta^2\alpha^{-\theta}}{\left(\alpha^{-\theta+1}-\alpha\right)^2}-\frac{n\theta}{\left(\alpha^{-\theta+1}-\alpha\right)^2}+\frac{n\theta\alpha^{-\theta}}{\left(\alpha^{-\theta+1}-\alpha\right)^2}+\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)^2}$$

$$\frac{\partial g2}{\partial\alpha}<0 \text{ If } \frac{n\theta\alpha^{-\theta}}{\left(\alpha^{-\theta+1}-\alpha\right)^2}+\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)^2}<\frac{n\theta}{\alpha^2}+\frac{n}{(1-\alpha)^2}+\frac{n\theta^2\alpha^{-\theta}}{\left(\alpha^{-\theta+1}-\alpha\right)^2}+\frac{n\theta}{\left(\alpha^{-\theta+1}-\alpha\right)^2}$$

There for their exist at least one root say $\alpha\in(0,\infty)$ such that g2=0 provided condition is satisfied.

Theorem 4.3

Let $g3=\frac{\partial LogL}{\partial\beta}$ Where α *and* θ are known then their exist a unique solution for $g3=0$ for $\beta\in(0,\infty)$

$$g3=\frac{n}{\beta}+n\bar{x}+(\theta-1)\sum\frac{x_i e^{\beta x_i}}{\left(1+e^{\beta x_i}\right)}-(\theta+1)\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}$$

$$\lim_{\beta\to\infty} g3=\lim_{\beta\to\infty}\left(\frac{n}{\beta}+n\bar{x}+(\theta-1)\sum\frac{x_i e^{\beta x_i}}{\left(1+e^{\beta x_i}\right)}-(\theta+1)\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}\right)=-\bar{x}n<0$$

Also

$$\lim_{\beta\to 0} g3=\lim_{\beta\to\infty}\left(\frac{n}{\beta}+n\bar{x}+(\theta-1)\sum\frac{x_i e^{\beta x_i}}{\left(1+e^{\beta x_i}\right)}-(\theta+1)\sum\frac{x_i e^{\beta x_i}}{\left(\alpha+e^{\beta x_i}\right)}\right)=\infty$$

$$\frac{\partial g3}{\partial\beta}<0, If \quad \theta\sum\frac{x_i^2 e^{-\beta x_i}}{\left(1+e-^{\beta x_i}\right)^2}<\frac{n}{\beta^2}+\sum\frac{x_i^2 e^{-\beta x_i}}{\left(1+e^{-\beta x_i}\right)^2}+(\theta+1)\sum\frac{x_i^2 e^{-\beta x_i}}{\left(1+\alpha e^{-\beta x_i}\right)^2}$$

There for their exist at least one root say $\beta\in(0,\infty)$ such that g3=0 provided condition is satisfied.

5. Order Statistics

Assume that $X_1, X_2, ..., X_n$ be a random sample from the population. Let $X_{i,n}$ denot the ith order statistics. The pdf of $X_{i,n}$ is

$$g_{i,n}(x)=\frac{n!}{(i-1)!(n-i)!}g(x)\left[G(x)\right]^{i-1}\left[\bar{G}(x)\right]^{n-i}$$

$$g_{i,n}(x)=\frac{n!}{(i-1)!(n-i)!}\frac{\alpha^{\theta}}{\left(\alpha^{\theta}-1\right)}\frac{\theta\beta(\alpha-1)e^{\beta x}\left(e^{\beta x}+1\right)^{\theta-1}}{\left[\alpha+e^{\beta x}\right]^{\theta+1}}$$

$$\left[1-\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\frac{\left(e^{\beta x}+1\right)^{\theta}}{\left(\alpha+e^{\beta x}\right)^{\theta}}-1\right]\right]^{i-1}\left[\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\frac{\left(e^{\beta x}+1\right)^{\theta}}{\left(\alpha+e^{\beta x}\right)^{\theta}}-1\right]\right]^{n-i}$$

$$g_{i,n}(x)=\frac{n}{\beta\left(n-1,i-1\right)}\frac{\alpha^{\theta(n-i+1)}\theta\beta(1-\alpha)}{\left(1-\alpha^{\theta}\right)^{n}}$$
$$\frac{e^{\beta x}\left(1-e^{\beta x}\right)^{\theta-1}\left[\left(\alpha+e^{\beta x}\right)^{\theta}-\alpha^{\theta}\left(1+e^{\beta x}\right)^{\theta}\right]^{i-1}\left[\left(1+e^{\beta x}\right)^{\theta}-\left(\alpha+e^{\beta x}\right)^{\theta}\right]^{n-i}}{\left(\alpha+e^{\beta x}\right)^{n\theta}}$$

Define minimum as $X_{(1)}=\min\left(X_1,X_2,...X_n\right)$ and maximum as $X_{(n)}=\max\left(X_1,X_2,...X_n\right)$ and the medium as $X_{(m+1)} \quad with \quad m=\frac{n}{2}$. Therefore the pdf of minimum, maximum and median are respectively

$$g_{1,n}(x)=\frac{n}{}\frac{\alpha^{\theta(n-2)}\theta\beta(1-\alpha)}{\left(1-\alpha^{\theta}\right)^{n}}\frac{e^{\beta x}\left(1-e^{\beta x}\right)^{\theta-1}\left[\left(1+e^{\beta x}\right)^{\theta}-\left(\alpha+e^{\beta x}\right)^{\theta}\right]^{n-1}}{\left(\alpha+e^{\beta x}\right)^{n\theta}}$$

$$g_{n,n}(x)=\frac{n}{}\frac{\alpha^{\theta}\theta\beta(1-\alpha)}{\left(1-\alpha^{\theta}\right)^{n}}\frac{e^{\beta x}\left(1-e^{\beta x}\right)^{\theta-1}\left[\left(\alpha+e^{\beta x}\right)^{\theta}-\alpha^{\theta}\left(1+e^{\beta x}\right)^{\theta}\right]^{n-1}}{\left(\alpha+e^{\beta x}\right)^{n\theta}}$$

$$g_{m+1,n}(x)=\frac{n}{\beta\left(n-1,m-1\right)}\frac{\alpha^{\theta(n-m)}\theta\beta(1-\alpha)}{\left(1-\alpha^{\theta}\right)^{n}}$$
$$\frac{e^{\beta x}\left(1-e^{\beta x}\right)^{\theta-1}\left[\left(\alpha+e^{\beta x}\right)^{\theta}-\alpha^{\theta}\left(1+e^{\beta x}\right)^{\theta}\right]^{m}\left[\left(1+e^{\beta x}\right)^{\theta}-\left(\alpha+e^{\beta x}\right)^{\theta}\right]^{n-m-1}}{\left(\alpha+e^{\beta x}\right)^{n\theta}}$$

6. Stochastic orders

For the last 40 years stochastic orders have been using for many applications. Its uses are seen in areas of probability and statistics such as reliability, survival analysis, queuing theory, biology economics, Insurance and actuarial science. (See Shaked and Shanthikumar (2007). Let X and Y be two random variable having distribution F and G respectively. Denote $\bar{F}=1-F$ and $\bar{G}=1-G$ as their survival functions. With pdf f and g. The random variable X is said to be smaller than Y (1) In stochastic order denoted as $X\leq_{st}Y$, If $\bar{F}(x)\leq\bar{G}(x)$ for all x. (2) Likelihood ratio order denoted as $X\leq_{Lr}Y$, IF $f(x)/g(x)$ is decreasing in $x\geq 0$. (3) Hazard rate order denoted by $X\leq_{hr}Y$ If $\bar{F}(x)/\bar{G}(x)$ is decreasing in $x\geq 0$. (4) Reversed hazard rate order denoted as $X\leq_{rhr}Y$, If $F(x)\leq G(x)$ is decreasing in $x\geq 0$. The four stochastic orders defined above are related to each other as the following implications.

$$X\leq_{rhr}Y\Leftarrow X\leq_{Lr}Y\Rightarrow X\leq_{hr}Y\Rightarrow X\leq_{st}Y$$
$$\frac{f(x)}{g(x)}=\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta}\frac{\alpha_2^{\theta}-1}{\alpha_1^{\theta}-1}\left(\frac{\alpha_2+e^{\beta x}}{\alpha_1+e^{\beta x}}\right)^{\theta+1}\frac{\alpha_1-1}{\alpha_2-1}$$

Since $\alpha_1<\alpha_2$

$$\left(\frac{f(x)}{g(x)}\right)^{/}=\beta\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta}\frac{\alpha_2^{\theta}-1}{\alpha_1^{\theta}-1}\frac{\alpha_1-1}{\alpha_2-1}\frac{\left(\alpha_2+e^{\beta x}\right)^{\theta}e^{\beta x}}{\left(\alpha_1+e^{\beta x}\right)^{\theta+2}}<0$$

Hence $f(x)/g(x)$ is decreasing in x. That is $X \leq_{Lr} Y$. The remaining statements follow from the implications above.

Application

Consider the real data Table 5, of the strength measured in GPA for single-carbon fibers data as an example

Table 5. The strength measured in GPA for single-carbon fibers data

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.532	2.575	2.614	2.616	2.618	2.624
2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937	2.977
2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243	3.264
3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.537	3.554
3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395
5.020	3.501	3.562							

Table 6. Parameter estimates for single-carbon fibers data

Distribution	Parameters	K-S	K–S *p*-value
LOGISTIC	$\theta = 0.1997$	0.7123	0.00
SLD	$\alpha = 0.54$, $\lambda = 2.68$, $\mu = 2.774$	0.0918	0.6632
NBLD	$\alpha = 3.34$, $\theta = 0.054$, $\beta = 0.18$	0.0693	0.8991

Conclusion

A special case of the logistic distribution, the NBL distribution is defined and studied. Discussed about Quantile method of estimation, Maximum likelihood estimation, and Order Statistics and Stochastic orders. Also test the goodness of fit for a real data set and found the logistic distribution fails and NBLD more suitable than skew logistic distribution (SLD). The NBL distribution provides a very flexible model for fitting of such kind of data. It is hoped that it will serve as an alternative to related but less versatile models that are currently in use for modeling data sets occurring in various areas of scientific investigation such as engineering, survival analysis, hydrology and economics.

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