Clique number and Girth of the Rough Co-zero Divisor Graph

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Abstract: This study aims to determine the Clique number, Girth and Maximal independence number of the Rough Co-zero divisor Graph $G(Z^*(J))$ corresponding to a Rough semiring (T, Δ, ∇). The methodology of these graph theoretical parameters are obtained using partition graph $P\big(Z^*(J)\big)$ of Rough co-zero divisor graph $G\big(Z^*(J)\big)$. Though the number of vertices in $G(Z^*(J))$ is $2^{n-m}.3^m - 2, 1 \leq m \leq n$, the partition graph plays a significant role in determining the results the Clique number of $G(Z^*(J))$ as $2^{n-m}.3^m - 2^m - 1$ and the Girth $G\big(Z^*(J)\big)$ is 3 and Maximal independence number of $G(Z^*(J))$ is $m + 1$. All these concepts are illustrated with suitable examples.

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1. Introduction

Rough set theory proposed by Zdzislaw Pawlak [8] in 1982. He defined Rough set as a formal approximation of a crisp set in terms of a pair of sets which give the lower and the upper approximations of the original set. Rough set theory is an extension of Fuzzy set theory. Rough sets have been proposed for a very wide variety of applications.

In particular, the rough set approach seems to be important for cognitive sciences and Artificial Intelligence, especially in knowledge discovery, machine learning, expert systems, data mining, pattern recognition, approximate reasoning etc., The concept of Rough Lattice was discussed by B. Praba and R. Mohan. [4-7] In this paper, the authors considered an information system and for any given information system a relation R on the set of all Rough sets T was defined. They have defined two operations $Praba\Delta$ and $Praba\nabla$.

Afkhami and Khashyarmanesh [1-3] introduced the co-zero divisor graph, denoted by $\Gamma'(R)$, on a commutative ring R. Let $W^*(R)$ be the set of all non-unit elements of R. The vertex set of $\Gamma'(R)$ is $W^*(R)$ and for two distinct vertices x and y in $W^*(R)$, x is connected to y if and only if $x \notin (yR)$ and $y \notin (xR)$, where (zR) is an ideal generated by the element z.

In this paper, our goal is to discover the clique number and girth of the Rough Co-zero divisor graph. This paper is systematized as follows:

In Sec. 2, we contribute preliminaries on Graph theory and Rough set theory.

In Sec. 3, we acquire the Clique number and Girth of the Rough Co-zero divisor graph and we illustrate with suitable examples.

In Sec. 4, we give the conclusion.

2. Preliminaries

2.1. Graph Theory

Definition 1.1.

A clique of a graph is a complete subgraph of it and the number of vertices in a greatest clique of G is called the clique number of G and is denoted by $\omega(G)$.

Definition 1.2.

The girth of G is the length of the shortest cycle in G, denoted by $gr(G)$.

Definition 1.3.

Let $G = (V, E)$. A subset I of V is called an independent set of G if no two vertices in I are adjacent. Independent vertex set I of G is said to be maximal if no other vertex of G can be added to I

2.2 Rough Set Theory

In this section we consider an approximation space $I = (U, R)$ where U is a non empty finite set of objects, called universal set and R be an equivalence relation defined on U.

Definition 1.3.

For any approximation space, the equivalence classes induced by R is defined by $[x] = \{y \in U \mid (x, y) \in R\}$. For any $X \subseteq U$, the lower approximation is defined as $R_{-}(X) = \{x \in U \mid [x] \subseteq X\}$ and the upper approximation is defined by $R^{-}(X) = \{x \in U \mid [x] \cap X = \phi\}$. The rough set corresponding to X is $RS(X) = (R_{-}(X), R^{-}(X))$.

Theorem 1.1

For any approximation space $I = (U, R), (T, \Delta, \nabla)$ is a semiring called the Rough semiring.

2.3 Rough Co-zero divisor Graph

In this section we consider an approximation space $I = (U, R)$ where U is the non empty finite set of objects and R is an equivalence relation on U. Let (T, Δ, ∇) be the rough semiring induced by I. Without loss of generality we also assume that there are m equivalence classes $\{X_1, X_2, \dots X_m\}$ with cardinality greater than 1 and the remaining $n - m$ equivalence classes $\{X_{m+1}, X_{m+2}, \dots X_n\}$ have cardinality equal to 1, where $1 < m \leq n$. Let B be the set of representative elements of X_i , $i = 1, 2, \dots, m$ and J be the rough ideal of T. We also assume that M is the union of none, one or more equivalence classes whose cardinality is equal to one and M' is the union of one or more equivalence classes whose cardinality is equal to one.

Definition 1.4.

Rough Co-zero divisor graph The Rough Co-zero divisor graph $G(Z^*(J)) = (V, E)$ where V is the set of vertices consisting of the elements of $T^* = T - \{RS(\emptyset), RS(U)\}$ and two elements $RS(X), RS(Y) \in T^*$ are adjacent iff $RS(X) \notin RS(Y)\nabla J$ and $RS(Y) \notin RS(X)\nabla J$.

2.4 Partition Graph

Partition graph $P(Z^*(J))$ is obtained by defining suitable partition in the vertices of $G(Z^*(J))$. Hence vertices having same degree will fall into same partition.

Definition 1.5.

Partition graph The partition graph $P(Z^*(J))$ is a graph whose vertices are the partitions on $V(Z^*(J))$ Hence the vertices of $P(Z^*(J))$ is the set $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ where

$P_1 = RS(x_i)$
$P_2 = RS(x_i \cup M') \cup RS(X_i \cup M)$
$P_3 = RS(Y) | Y \in M'$
$P_4 = RS\ (x_1, x_2, \dots x_r)$
$P_5 = RS\ (x_1, x_2, \dots x_m)$
$P_6 = RS\ x_1, x_2, \dots x_r \cup M' \cup\ RS\ (X_1, X_2, \dots X_r \cup M) \cup RS\ (Q_r \cup M)$
$P_7 = RS\ x_1, x_2, \dots x_m \cup\ M' \cup RS\ (X_1, X_2, \dots X_m \cup M)\ \cup RS\ (Q_m \cup M)$

Two vertices P_i and P_j in the partition graph are connected by an edge if the elements in P_i are adjacent to any of the elements in P_j by an edge in $G(Z^*(J))$.

The following Figure 1 represents the partition graph of $G(Z^*(J))$ for $n \neq m$

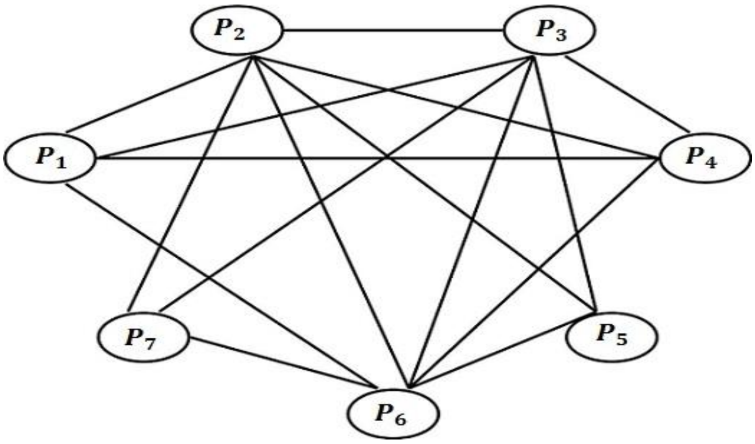

Figure 1. Partition Graph for $n \neq m$

When $n = m$, the corresponding partition graph of $G(Z^*(J))$ is given in Figure 2

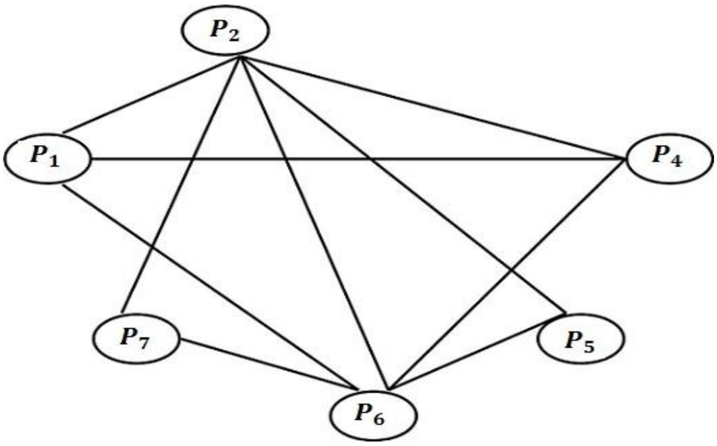

Figure 2. Partition Graph for $n = m$

3. Clique Number, Girth and Maximal Independence number of a Rough Co-zero divisor graph

In this section we acquire the Clique number, Girth and Maximal independence number of a Rough Co-zero divisor graph using partition graph.

3.1. Clique Number of a Rough Co-zero divisor graph

In this section the Clique of the Rough Co-zero divisor graph is obtained.

Theorem 3.1

The clique number of the Rough Co-zero divisor graph $G(Z^*(J))$ is $2^{n-m}.3^m - 2^m - 1$ for $1 \leq m \leq n$.

Proof:
Case 1: When $m < n$

We know that the Clique number of a complete graph is equal to the number of vertices in it. From the partition graph let us consider the set $A = \{P_2, P_3, P_6, P_7\}$

First, we prove that the elements of A forms a complete subgraph of $G(Z^*(J))$.The following observation suggest that the set A will form a complete subgraph of $G(Z^*(J))$.

Let, $P_2 = RS(x_i \cup M') \cup RS(X_i \cup M)$

For i, $RS(x_i \cup M')$ is connected to $RS(Y)$ as $RS(x_i \cup M') \notin RS(Y)\nabla J$ and $RS(Y) \notin RS(x_i \cup M')\nabla J$ where $Y \in RS(x_i \cup M') \cup RS(X_i \cup M)$and $RS(X_i \cup M')$ is connected to all $RS(Z)$ for $i \neq j, i, j = 1, 2, \ldots m$ as $RS(X_i \cup M') \notin RS(Z)\nabla J$ and $RS(x_j \cup M') \cup RS(Z)\nabla J$ where $Z \in RS(x_j \cup M') \cup RS(X_j \cup M)$.Which proves the elements of P_2 forms a complete graph.

Similarly in partition P_3, $\{RS(Q_j)|P(Q) - \emptyset, Q = X_{m+1}, X_{m+2}, \ldots X_n\}$, for each i RS$(X_i) \notin RS(X_j)\nabla J$ and $RS(X_j) \notin RS(X_i)\nabla J$, $RS(X_i), RS(X_j) \in P_3$. Also it is connected to all the elements in $V(Z^*(J))$. Hence elements of P_3 forms a complete graph.

Correspondingly in P_6, $P_6 = RS(x_1, x_2, \ldots x_r \cup M') \cup RS(X_1, X_2, \ldots X_r \cup M) \cup RS(Q_r \cup M)$.

For each i, $RS(Y) \notin RS(Z)\nabla J$ and $RS(Z) \notin RS(Y)$, $i = 1, 2..m, 1 < r < m$, where $Y \in RS(x_1, x_2, \ldots x_r) \cup M'$ and $Z \in RS(x_{r+1}, x_{r+2}, \ldots x_m) \cup M'$

For each i,
$RS(T) \notin RS(S)\nabla J$ and$RS(S) \notin RS(T)\nabla J$, $i = 1, 2..m$, $1 < r < m$, where $T \in RS(X_1, X_2, \ldots X_r) \cup M$ and $S \in RS(X_{r+1}, X_{r+2}, \ldots X_m) \cup M$

For each i,
$RS(V) \notin RS(W)\nabla J$ and $RS(W) \notin RS(V)\nabla J, i = 1, 2..m, 1 < r < m$, where $V \in RS(Q_1, Q_2, \ldots Q_r) \cup M$ and $W \in RS(Q_{r+1}, Q_{r+2}, \ldots Q_m) \cup M$

Henceforth the elements of P_6 forms a complete graph. Likewise the elements of P_7, $P_7 = RS(x_1, x_2, \ldots x_m \cup M') \cup RS(X_1, X_2, \ldots X_m \cup M) \cup RS(Q_m \cup M)$ forms a complete graph of $G(Z^*(J))$

For each j,
$RS(x_j \cup M') \cup RS(X_j \cup M) \notin RS(X_k)\nabla J$ and $RS(X_k) \notin RS(x_j \cup M') \cup RS(X_j \cup M)\nabla J$ and $RS(x_j \cup M') \cup RS(X_j \cup M) \notin RS(x_1, x_2, \ldots x_r \cup M') \cup RS(X_1, X_2, \ldots X_r \cup M) \cup RS(Q_r \cup M)\nabla J$ & $RS(x_1, x_2, \ldots x_r \cup M') \cup RS(X_1, X_2, \ldots X_r \cup M) \cup RS(Q_r \cup M)$ $RS(x_j \cup M') \cup RS(X_j \cup M)\nabla J$ and $RS(x_j \cup M') \cup RS(X_j \cup M) \notin RS\, x_1, x_2, \ldots x_m \cup M' \cup RS(X_1, X_2, \ldots X_m \cup M \cup RS(Q_m \cup M)\nabla J$ and $RS(x_1, x_2, \ldots x_m \cup M') \cup RS(X_1, X_2, \ldots X_m \cup M) \cup RS(Q_m \cup M) \notin RS(x_j \cup M') \cup RS(X_j \cup M)\nabla J$ which implies every elements of P_2 is connected to all the elements of P_3, P_6 and P_7 and similarly it can be prove that every elements of P_3 is connected to all the elements of P_6 and P_7.

Finally every elements of P_6 is connected to all the elements in P_7. Therefore elements of A will form a complete subgraph of $G(Z^*(J))$ for.

To prove A is maximal, we take another complete subgraph say $A' = \{P_2, P_3, P_5, P_6\}$. The total numbers of elements in each of these sets are

$$|A| = 2^{n-m}.3^m - 2^m - 1$$
$$A' = 2^{n-m}(2^m + m - 1) - m + 3(3^{m-1} - 2^m + 1)$$

Note that the number of elements in A is greater than$'$. Since $|P_5| < |P_7|$ This intimate that no other maximal complete graph can be formed in $G(Z^*(J))$. Hence the set A will form a maximal complete subgraph of $G(Z^*(J))$ Hence the clique $C = \{P_2, P_3, P_5, P_6\}$ and the clique number is $|P_2|+|P_3| + |P_5| + |P_6| = 2^{n-m}.3^m - 2^m - 1$

Case 2: When $n = m$

The partition $P_3 = \{RS(Y)|Y(X_{m+1}, X_{m+2}, \ldots X_n)\} = \emptyset$. Therefore $P_3 = 0$, also when $n = m$, the number of vertices in $P(Z^*(J))$ has only 6 vertices. Thus our clique is $C = \{P_2, P_6, P_7\}$ and the clique number is $|P_2|+|P_6| + |P_7| = 2^{n-m}.3^m - 2^m - 1$

Example 3.1

Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $\{X_1, X_2, X_3\}$ are the equivalence classes induced by an equivalence relation R on U such $X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_4, x_6\}$ and $X_3 = \{x_5\}$

$V\big(Z^*(J)\big) = \{RS(x_1), RS(x_2), RS(X_1), RS(X_2), RS(X_3), RS(x_1 \cup x_2), RS(X_1 \cup X_2), RS(X_1 \cup X_3), RS(X_2 \cup X_3), RS(x_1 \cup X_2), RS(X_1 \cup x_2), RS(x_1 \cup X_3), RS(x_2 \cup X_3), RS(x_1 \cup X_2 \cup X_3), RS(X_1 \cup x_2 \cup X_3), RS(x_1 \cup x_2 \cup X_3)\}$; $B = \{x_1, x_2\}, J = \{RS(x_1), RS(x_2), RS(x_1 \cup x_2)\}$

Figure 3 represents the Rough co-zero divisor graph for $n = 3$ and $m = 2$.

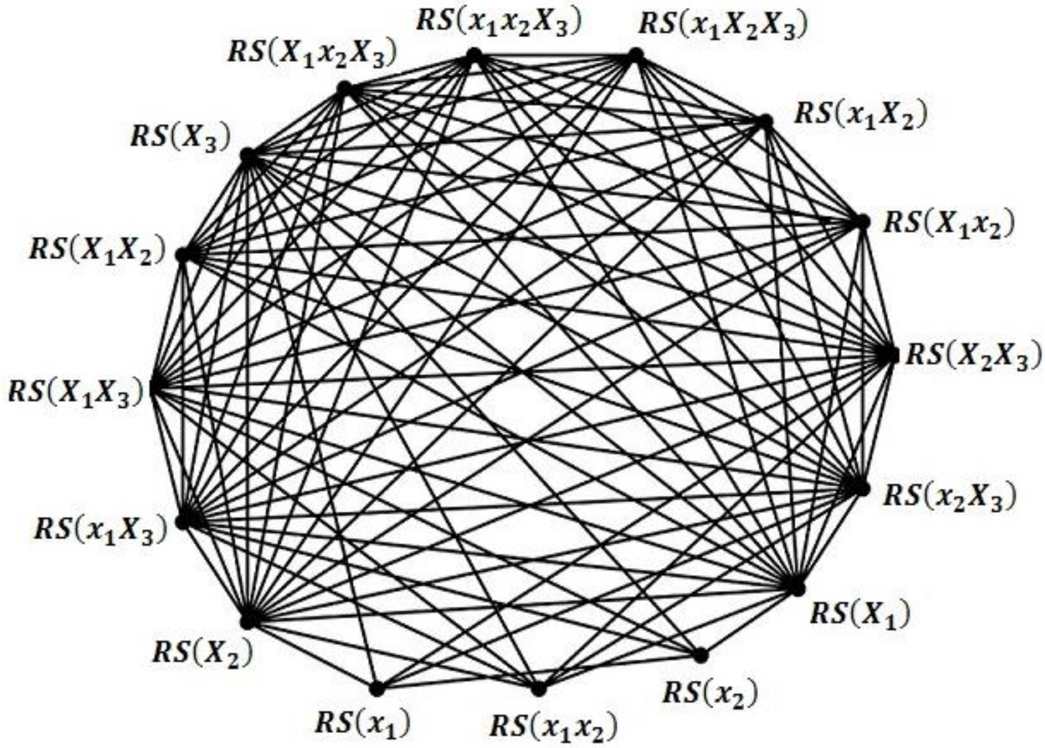

Figure 3. Rough co-zero divisor graph for $n = 3$ and $m = 2$

By theorem 3.1,

$A = \{, RS(X_1), RS(X_2), RS(X_3), RS(X_1 \cup X_2), RS(X_1 \cup X_3), RS(X_2 \cup X_3), RS(x_1 \cup X_2), RS(X_1 \cup x_2), RS(x_1 \cup X_3), RS(x_2 \cup X_3), RS(x_1 \cup X_2 \cup X_3), RS(X_1 \cup x_2 \cup X_3), RS(x_1 \cup x_2 \cup X_3)\}$

Clique number of the Rough Co-zero divisor Graph is $2^{n-m}.3^m - 2^m - 1 = 13$

Example 3.2

Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $\{X_1, X_2, X_3\}$ are the equivalence classes induced by an equivalence relation R on U such $X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_4\}$ and $X_3 = \{x_5, x_6\}$

$V\big(Z^*(J)\big) = \{RS(x_1), RS(x_2), RS(x_3), RS(X_1), RS(X_2), RS(X_3), RS(x_1 \cup x_2), RS(x_1 \cup x_3), RS(x_2 \cup x_3), RS(x_1 \cup x_2 \cup x_3), RS(X_1 \cup X_2), RS(X_1 \cup X_3), RS(X_2 \cup X_3), RS(x_1 \cup X_2), RS(X_1 \cup x_2), RS(x_1 \cup X_3), RS(X_1 \cup x_3), RS(x_2 \cup X_3), RS(X_2 \cup x_3), RS(x_1 \cup X_2 \cup X_3), RS(X_1 \cup x_2 \cup X_3), RS(X_1 \cup X_2 \cup x_3), RS(x_1 \cup x_2 \cup X_3), RS(x_1 \cup X_2 \cup x_3), RS(X_1 \cup x_2 \cup x_3)\}$
$B = \{x_1, x_2, x_3\}, J = \{RS(x_1), RS(x_2), RS(x_3), RS(x_1 \cup x_2), RS(x_1 \cup x_3), RS(x_2 \cup x_3) RS(x_1 \cup x_2 \cup x_3)\}$

Figure 4 represents the Rough co-zero divisor graph for $= m = 3$.

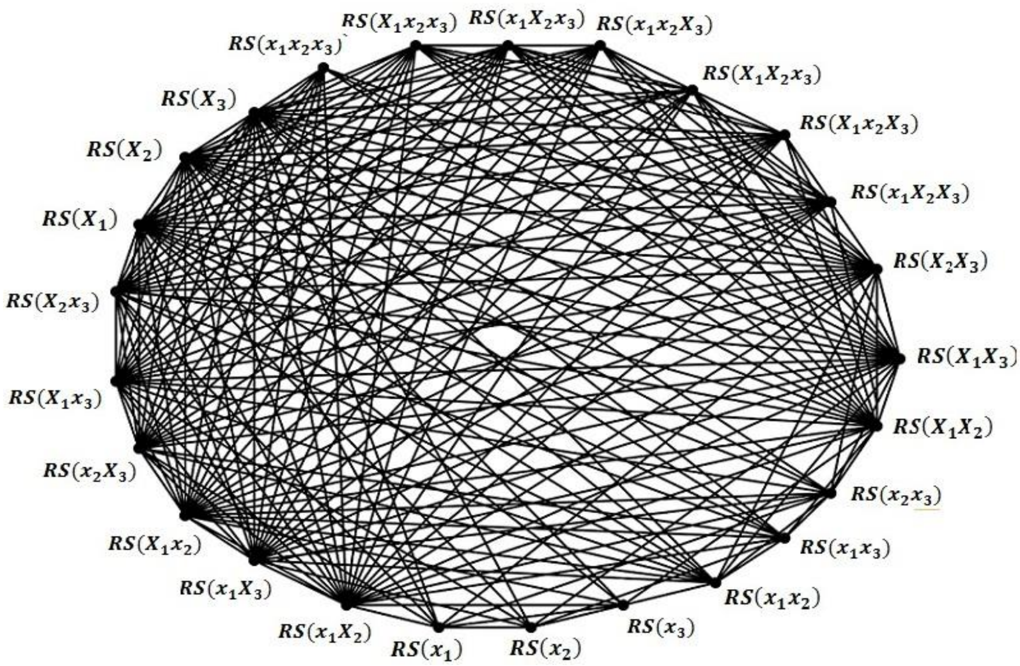

Figure 4. Rough co-zero divisor graph for $n = m = 3$

By theorem 3.1,

$A = \{ RS(X_1), RS(X_2), RS(X_3), RS(X_1 \cup X_2), RS(X_1 \cup X_3), RS(X_2 \cup X_3), RS(x_1 \cup X_2), RS(X_1 \cup x_2), RS(x_1 \cup X_3), RS(X_1 \cup x_3), RS(x_2 \cup X_3), RS(X_2 \cup x_3), RS(x_1 \cup X_2 \cup X_3), RS(X_1 \cup x_2 \cup X_3), RS(X_1 \cup X_2 \cup x_3), RS(x_1 \cup x_2 \cup X_3), RS(x_1 \cup X_2 \cup x_3), RS(X_1 \cup x_2 \cup x_3)\}$

Clique number of the Rough Co-zero divisor Graph is $2^{n-m}.3^m - 2^m - 1 = 18$

3.2. Girth of Rough Co-zero Divisor Graph

In this section the Girth of Rough Co-zero divisor graph is obtained.

Theorem 3.2

Girth of Rough Co-zero divisor graph $G(Z^*(J))$ is 3 for $1 \le m \le n$.

Proof:

Note that in any undirected graph the girth ≥ 3. If there exist a shortest cycle of length 3 in $G(Z^*(J))$ then girth of $G(Z^*(J))$ is 3.

Consider the set $Gr\left(G\left(Z^*(J)\right)\right) = \left\{RS(X_i), RS\left(X_j\right), RS(Q_k)\right\}$ $1 \le i, j \le m$, $Q_k = x_i \cup X_j$ or $X_i \cup x_j$

For each i the element $RS(X_i)$, $RS(X_i) \notin RS(X_j)\nabla J$ and $RS(X_j) \notin RS(X_i)\nabla J$ when $i \neq j$. Which indicates that $RS(X_i)$ is connected to $RS(X_j)$. Also element $RS(Q_k)$, is connected to $RS(X_i)$ and $RS(X_j)$. Since $RS(X_i) \notin RS(Q_k)\nabla J$ and $RS(Q_k) \notin RS(X_i)\nabla J$. Hence $Gr(G(Z^*(J))) = \{RS(X_i), RS\left(X_j\right), RS(Q_k)\}$ forms a shortest cycle of length 3 in $G\left(Z^*(J)\right)$.

Example 3.3 (From example 3.1). $n = 3$ and $m = 2$.
$Gr(G(Z^*(J))) = \{RS(X_1), RS(X_2), RS(x_1X_2)\}$
Girth of the Rough Co-zero divisor graph 3.

Example 3.4 (From example 3.2). $n = 3$ and $m = 3$.
$Gr(G(Z^*(J))) = \{RS(X_1), RS(X_2), RS(X_1x_2)\}$
Girth of the Rough Co-zero divisor graph 3.

Maximal independent set of a Rough Co-zero divisor graph

In this section the Maximal independent number of a Rough Co-zero divisor graph is obtained.

Theorem 3.3

Maximal independent number of the Rough Co-zero divisor graph is $m+1$ for $1 \leq m \leq n$

Proof:

From $G(Z^*(J))$, Consider the set

$I = \{RS(x_1), RS(x_1x_2), RS(x_1x_2x_3), \ldots RS(x_1x_2, \ldots x_m), RS(x_1x_2, \ldots x_mX_n)\}$

Note that $RS(x_1) \in P_1$
$RS(x_1x_2), RS(x_1x_2x_3), \ldots RS(x_1x_2, \ldots x_{m-1}) \in P_4$
$RS(x_1x_2, \ldots x_m) \in P_5$
$RS(x_1x_2, \ldots x_mX_n) \in P_7$

As $RS(x_1) \in RS(x_1x_2)\nabla J$ and $RS(x_1x_2) \notin RS(x_1)\nabla J$ implies $RS(x_1)$ is connected to $RS(x_1x_2)$. Similarly we can prove that $RS(x_1)$ is not connected to $RS(x_1x_2, \ldots x_r)$ for $2 \leq r \leq m$. Same way $RS(x_1x_2, \ldots x_r)$ is not connected to $RS(x_1x_2, \ldots x_{r+s})$, $s \geq 1$. By the similar discussion it is easy to verify that $RS(x_1x_2, \ldots x_r)$ is not connected to $RS(x_1x_2, \ldots x_mX_n)$, where $1 < r < m$.

(i.e)., Note that cardinality of I is $m+1$ and all the elements of not connected to each other. As elements of P_1, P_2, P_3, P_6 and P_7 forms a complete subgraph of $G(Z^*(J))$. None of these elements can be added to I. Note that $|P_5| = 1$ and this element is already added to I.

$P_4 = RS(x_1x_2, \ldots x_r)$, By the property of connectivity of the edges in $G(Z^*(J))$. None of the elements in P_4 can be added to I without asserting the property of independency.

Hence elements I will form a maximal independent set of $G(Z^*(J))$ and the number of elements in the maximal independent set of a Rough co-zero divisor graph is $m+1$.

Example 3.5 (From example 3.1). $n = 3$ and $m = 2$.
$I = \{RS(x_1), RS(x_1x_2), RS(x_1x_2X_3)\}$
Maximal independent number of the Rough Co-zero divisor graph $m+1 = 3$.

Example 3.6 (From example 3.2). $n = 3$ and $m = 3$.
$I = \{RS(x_1), RS(x_1x_2), RS(x_1x_2x_3), RS(x_1x_2x_3X_4)\}$
Maximal independent number of the Rough Co-zero divisor graph $m+1 = 4$.

4. Conclusion

In this article, the clique number, Girth and the Maximal independence number of the Rough Co-zero divisor Graph of the Rough Semiring (T, Δ, ∇) using partition graph are obtained. All the concepts are illustrated with appropriate examples. Future work is to derive the independence number of $G(Z^*(J))$. Also some real time applications to the graph theoretical approach using these parameters are obtain.

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