A Study on Stability Analysis of Non-linear System of a Real-time Dynamic Sub Structuring Model via Neutral Delay Differential Equation

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Abstract: In this article, necessary condition for existence of characteristic roots of Neutral Delay Differential equation to study the stability of a nonlinear system of a mass-spring-damper connected to a pendulum is derived from an real-time dynamic sub structuring model and illustrated with examples.

Keywords: Neutral Delay Differential Equation, Characteristic Equation, Stability theorem.

1. Introduction

Numerous seismic testing approach for reviewing the engineering model complexity, for the earthquake turmoil is studied. Out of which the real-time dynamic sub-structuring [1, 2] is a specific type where its structure is been divided into two [3], substructure and its numerical part by an electrically driven actuator that induces delay which leads to destabilization [1]. The system examined in this article, has an auto parametric pendulum with a mass-spring-damper (MSD) [2]. We analytically show how to stabilize using the theorem to retrieve its stability. The concept of a Delay Differential Equations (DDEs) of the model for was presented in [4] to examine a single mass-spring oscillator. The delay acts as an important part in examining the stability of the simple linear system and to validate the outcomes is shown by the author.

NDDEs are a special category of DDEs, where the delay appears in the highest derivative of the DDEs. Research has been done on the stability of one-dimensional wave equation using Lambert W function [5]. This article enhance on the stability of the solution by applying sufficient condition for the for a second order linear neutral delay differential equation.

2. Mathematical Formulation of the Model

The model consists of a mass M fastened with a linear spring, connected to a pendulum with a mass m_{pend} of weightless rod of length l is given by

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m_p\ddot{y}(t-\tau) + m_{pend}l\left[\ddot{\theta}(t-\tau)sin\theta(t-\tau) + \dot{\theta}^2(t-\tau)cos\theta(t-\tau)\right] = F_{ext}, \quad (1)$$

$$m_{pend}l^2\ddot{\theta}(t-\tau) + k_{pend}\dot{\theta}(t-\tau) + m_{pend}glsin\theta(t-\tau) + m_{pend}l\ddot{y}(t-\tau)sin\theta(t-\tau) = 0 \quad (2)$$

Where τ denotes the time lag`. The force in the model is been named by the state delay for the MSD of the numerical system, where F_{ext} is the external force used in the y direction, K and C are the coefficients of stiffness and damping respectively. The position, velocity and acceleration of MSD at time t are denoted by y(t), $\dot{y}(t)$ and $\ddot{y}(t)$ respectively. The MSD attached to a pendulum is simulated, as a result $F_{ext} = 0$, and this alters model (1) into an autonomous model of second order Neutral Delay Differential equation.

3. Stability Analysis

When the above model decouples, as θ (θ ≪1) tends to zero and the equation (2) associated to decaying oscillations of the pendulum. While we focus on the equation, that denotes the pendulum's vertical motion in an MSD system, the above mathematical model is reduced as second order neutral delay differential equation given by

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m_{pend}\ddot{y}(t-\tau) = 0.$$

The non-dimensionalized form for the above is described as

$$y''(t) + py''(t-\tau) = -2\varsigma y'(t) - y(t), t \geq 0, \quad (3)$$
$$y(t) = \phi(t), -\tau \leq t \leq 0 \quad (4)$$

and the constraints are rescaled as $\hat{t} = w_n t, \hat{\tau} = w_n \tau, w_n = \sqrt{\frac{K}{M}}, p = \frac{m_{pend}}{M}, \varsigma = \frac{C}{2\sqrt{MK}}$.

The solution of (3) of the form $y(t) = e^{\lambda t}$for $t \in IR$,where λ is a root of the characteristic equation(3)

$$\lambda^2(1+p) = -2\varsigma\lambda - 1 \quad (5)$$

Assume y be the solution of (3), which is define as $x(t) = e^{-\lambda_0 \tau} y(t)$, for $t \in [-\tau, \infty)$,

where λ_0 is a real root of the characteristic equation (3). Therefore, for all $t \geq 0$, we get by [6] as

$$x''(t) + 2\lambda_0 x'(t) + {\lambda_0}^2 x(t) + pe^{-\lambda_0 \tau}\left(x''(t-\tau) + 2\lambda_0 x'(t-\tau) + {\lambda_0}^2 x(t-\tau)\right)$$
$$= -2\varsigma x'(t) - 2\varsigma x(t) - x(t)$$

Or

$$x'(t) + 2\lambda_0 x(t) + 2\varsigma x(t) + pe^{-\lambda_0 \tau}\left(x'(t-\tau) + 2\lambda_0 x(t-\tau)\right)$$
$$= {\lambda_0}^2 x(t) - 2\varsigma x(t) - x(t) + pe^{-\lambda_0 \tau}{\lambda_0}^2 x(t-\tau)$$
$$\left(x'(t) + (2\lambda_0 + 2\varsigma)x(t) + pe^{-\lambda_0 \tau}x'(t-\tau) + pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau)\right)' = (-{\lambda_0}^2 - 2\varsigma - 1)x(t) - pe^{-\lambda_0 \tau}{\lambda_0}^2 x(t-\tau) \quad (6)$$

Furthermore, the initial state (4) can be identically described as

$$x(t) = e^{-\lambda_0 \tau}\phi(t), \text{ for } t \in [-\tau, \infty), \quad (7)$$

By applying λ_0 is a characteristic roots of(5) and using (7), which confirms that (6) is equal to

$$x'(t) + (2\lambda_0 + 2\varsigma)x(t) + pe^{-\lambda_0 \tau}x'(t-\tau) + pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) = (-{\lambda_0}^2 - 2\varsigma - 1)\int_0^t x(s)ds - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_0^t x(s-\tau)ds + x'(0) + pe^{-\lambda_0 \tau}x'(-\tau) + (2\lambda_0 + 2\varsigma)x(0) + pe^{-\lambda_0 \tau}2\lambda_0 x(-\tau)$$

$$x'(t) + pe^{-\lambda_0 \tau}x'(t-\tau) = -(2\lambda_0 + 2\varsigma)x(t) - pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) + (-{\lambda_0}^2 - 2\varsigma - 1)\int_0^t x(s)ds - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_0^t x(s-\tau)ds + x'(0) + pe^{-\lambda_0 \tau}x'(-\tau) + (2\lambda_0 + 2\varsigma)x(0) + pe^{-\lambda_0 \tau}2\lambda_0 x(-\tau)$$

$$x'(t) + pe^{-\lambda_0 \tau}x'(t-\tau) = -(2\lambda_0 + 2\varsigma)x(t) - pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) + (-{\lambda_0}^2 - 2\varsigma - 1)\int_0^t x(s)ds - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_{-\tau}^{t-\tau} x(s)ds + \phi'(0) - \lambda_0\phi(0) + p(\phi'(-\tau) - \lambda_0\phi(-\tau)) + (2\lambda_0 + 2\varsigma)\phi(0) + p2\lambda_0\phi(-\tau)$$

$$x'(t) + pe^{-\lambda_0 \tau}x'(t-\tau)$$
$$= -(2\lambda_0 + 2\varsigma)x(t) - pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) + (-{\lambda_0}^2 - 2\varsigma - 1)\int_0^t x(s)ds$$
$$- pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_0^{t-\tau} x(s)ds + L(\lambda_0; \phi)$$

$$x'(t) + pe^{-\lambda_0 \tau}x'(t-\tau)$$
$$= -(2\lambda_0 + 2\varsigma)x(t) - pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_0^t x(s)ds - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_0^{t-\tau} x(s)ds$$
$$+ L(\lambda_0; \phi)$$
$$x'(t) + pe^{-\lambda_0 \tau}x'(t-\tau) = -(2\lambda_0 + 2\varsigma)x(t) - pe^{-\lambda_0 \tau}2\lambda_0 x(t-\tau) - pe^{-\lambda_0 \tau}{\lambda_0}^2 \int_{t-\tau}^t e^{-\lambda_0 \tau}x(s)ds + L(\lambda_0; \phi) \quad (8)$$

where

$$L(\lambda_0;\phi) = \phi'(0) - \lambda_0\phi(0) + p(\phi'(-\tau) - \lambda_0\phi(-\tau)) + (2\lambda_0 + 2\varsigma)\phi(0) + 2p\lambda_0\phi(-\tau) - pe^{-\lambda_0\tau}{\lambda_0}^2\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds \quad (9)$$

$$\beta_{\lambda_0} = -pe^{-\lambda_0\tau}{\lambda_0}^2\tau + (2\lambda_0 + 2\varsigma) + pe^{-\lambda_0\tau}2\lambda_0 \neq 0$$

Define $z(t) = x(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}, for t \geq -\tau$

Then equation (6) diminishes to the subsequent form as

$$z'(t) + pe^{-\lambda_0\tau}z'(t-\tau) = -(2\lambda_0 + 2\varsigma)z(t) - pe^{-\lambda_0\tau}2\lambda_0 z(t-\tau) - pe^{-\lambda_0\tau}{\lambda_0}^2\int_{t-\tau}^{t} z(s)ds \quad (10)$$

If equation (8) has its solution of the form $z(t) = e^{\delta t}$ for $t \in IR$, thenδ is a root of the next characteristic equation

$$\delta\left(1 + pe^{-(\lambda_0+\delta)\tau}\right) = -(2\lambda_0 + 2\varsigma) - 2p\lambda_0 e^{-(\lambda_0+\delta)\tau} + pe^{-\lambda_0\tau}{\lambda_0}^2\delta^{-1}(1 - e^{-\delta t}) \quad (11)$$

But, the initial condition (5) can be written as

$$z(t) = \emptyset(t)e^{-\lambda_0 t} - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}, t \in [-\tau, 0] \quad (12)$$

Let $F(\delta)$ is defined by the characteristic function of (9), i.e.,

$$F(\delta) = \delta\left(1 + pe^{-(\lambda_0+\delta)\tau}\right) + (2\lambda_0 + 2\varsigma) + 2p\lambda_0 e^{-(\lambda_0+\delta)\tau} + pe^{-\lambda_0\tau}{\lambda_0}^2\delta^{-1}(e^{-\delta t} - 1)$$

Since removable singularity $\delta = 0$ in $F(\delta)$, we consider $F(\delta)$ as an entire function with

$$F(0) = 2\lambda_0 + 2\varsigma + 2p\lambda_0 e^{-\lambda_0\tau} + pe^{-\lambda_0\tau}{\lambda_0}^2\tau \equiv \beta_{\lambda_0}$$

Since by the definition $\beta_{\lambda_0} \neq 0$, a root of the characteristic equation (11) will have $\delta_0 \neq 0$

Consider z be the solution of (10)-(12) and δ_0 be a real root of the characteristic equation (11).

Express $\delta_0 \neq 0$, then $v(t) = e^{-\delta_0 t}z(t)$,for all $t \in [-\tau, \infty)$

Hence for every $t \geq 0$, we have

$$v'(t) + \delta_0 v(t) + pe^{-(\lambda_0+\delta_0)\tau}v'(t-\tau) = -(2\lambda_0 + 2\varsigma)v(t) - pe^{-(\lambda_0+\delta_0)\tau}2\lambda_0 v(t-\tau) - pe^{-\lambda_0\tau}{\lambda_0}^2\int_{t-\tau}^{t} e^{-\delta_0 s}v(t-s)ds$$

$$v'(t) + pe^{-(\lambda_0+\delta_0)\tau}v'(t-\tau) = -(2\lambda_0 + 2\varsigma - \delta_0)v(t) - pe^{-\lambda_0\tau}(2\lambda_0 + \delta_0)v(t-\tau) + pe^{-\lambda_0\tau}{\lambda_0}^2\int_{t-\tau}^{t} e^{-\delta_0 s}v(t-s)ds \quad (13)$$

Furthermore, the initial condition (10) can be written equivalently as

$$v(t) = \emptyset(t)e^{-(\lambda_0+\delta_0)t} - e^{-\delta_0 t}\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}, t \in [-\tau, 0] \quad (14)$$

Moreover, by applying$\delta_0 \neq 0$ is a real root of (9) and considering (14), we can prove that (13) is identical to

$$v(t) + pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau) = -(2\lambda_0 + 2\varsigma - \delta_0)\int_0^t v(s)ds - pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0 + \delta_0)\int_0^t v(s-\tau)ds + pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^t v(u-s)du\right\}ds + v(0) + pe^{-(\lambda_0+\delta_0)\tau}v(-\tau)$$

$$v(t)+pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau)$$
$$=(-2\lambda_0-2\varsigma-\delta_0)\int_0^t v(s)ds-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_{-\tau}^{t-\tau}v(s)ds$$
$$+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}\left\{\int_{-s}^{t-s}v(u)du\right\}ds+v(0)+pe^{-(\lambda_0+\delta_0)\tau}v(-\tau)$$
$$v(t)+pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau)$$
$$=+(-2\lambda_0-2\varsigma-\delta_0)\int_0^t v(s)ds-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\left\{\int_{-\tau}^{0}v(s)ds+\int_0^{t-\tau}v(s)ds\right\}$$
$$+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}\left\{\int_{-s}^{0}v(u)du+\int_0^{t-s}v(u)du\right\}ds+v(0)+pe^{-(\lambda_0+\delta_0)\tau}v(-\tau)$$
$$v(t)+pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau)$$
$$=(-2\lambda_0-2\varsigma-\delta_0)\int_0^t v(s)ds-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\left\{\int_0^{t-\tau}v(s)ds\right\}$$
$$-pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}\left\{\int_0^{t-s}v(u)du\right\}ds+R(\lambda_0,\delta_0;\emptyset),$$
$$v(t)+pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau)$$
$$=pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_0^t v(s)ds$$
$$-pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^t e^{-\delta_0 s}ds\int_0^t v(s)ds-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\left\{\int_0^{t-\tau}v(s)ds\right\}$$
$$+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}\left\{\int_0^{t-s}v(u)du\right\}ds+R(\lambda_0,\delta_0;\emptyset),$$

$$v(t)+pe^{-(\lambda_0+\delta_0)\tau}v(t-\tau)=pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_{t-\tau}^t v(s)ds-pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^t e^{-\delta_0 s}ds\int_{t-s}^t v(s)ds+R(\lambda_0,\delta_0;\emptyset), \tag{15}$$

Where $R(\lambda_0,\delta_0;\emptyset)=\emptyset(0)+p\emptyset(-\tau)-e^{-\delta_0 t}\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\left(1+pe^{-\lambda_0\tau}\right)-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_{-\tau}^0 e^{-\delta_0 s}\left(\emptyset(s)e^{-\lambda_0 s}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)ds+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}ds\left\{\int_{-s}^0 e^{-\delta_0 u}\left(\emptyset(u)e^{-\lambda_0 u}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)du\right\}ds$ (16)

Then we define

$$w(t)=v(t)-\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}} \text{ for } t\geq-\tau \tag{17}$$

Where $\eta_{\lambda_0,\delta_0}\equiv 1+pe^{-(\lambda_0+\delta_0)\tau}-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\tau+\delta^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)pe^{-\lambda_0\tau}{\lambda_0}^2$ (18)

Then (13) reduces to the correspondent equation as

$$w(t)+pe^{-(\lambda_0+\delta_0)\tau}w(t-\tau)=-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_{t-\tau}^t w(s)ds-pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^\tau e^{-\delta_0 s}\left\{\int_{t-s}^t w(u)du\right\}ds, t\geq 0 \tag{19}$$

Moreover, the initial condition (12) can be denoted as

$$w(t)=\emptyset(t)e^{-(\lambda_0+\delta_0)\tau}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}, \text{for } t\in[-\tau,0]. \tag{20}$$

Theorem 1: let λ_0 and $\delta_0(\delta_0\neq 0)$be real characteristic roots of the equations (5) and (11). Assume that the roots λ_0 and δ_0 have the resulting property

$$\mu_{\lambda_0,\delta_0}\equiv(-|p|+|p(2\lambda_0+\delta_0)\tau|)e^{-(\lambda_0+\delta_0)\tau}+\delta^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)|-p{\lambda_0}^2|e^{-\lambda_0\tau}<1 \tag{21}$$
and $\beta_{\lambda_0}=-pe^{-\lambda_0\tau}{\lambda_0}^2\tau+(2\lambda_0+2\varsigma)+pe^{-\lambda_0\tau}2\lambda_0\neq 0$

Then, for any $\emptyset\in C^1([-\tau,0],R)$, the solution y of (1)-(2) satisfies

$$\left|y(t)e^{-(\lambda_0+\delta_0)\tau}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}\right|\leq M(\lambda_0,\delta_0;\emptyset),\text{ for all } t\geq 0, \quad (22)$$

where $L(\lambda_0;\phi), R(\lambda_0,\delta_0;\emptyset)$ and $\eta_{\lambda_0,\delta_0}$ were given in (9),(16) and (18), respectively and

$$M(\lambda_0,\delta_0;\emptyset)=\max_{-\tau\leq t\leq 0}\left|\emptyset(t)e^{-(\lambda_0+\delta_0)\tau}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}\right| \quad (23)$$

Proof: The property (21) assures that $\eta_{\lambda_0,\delta_0}>0$.

By using the descriptions of x, z, v and w, we get that (22) is equal to

$$|w(t)|\leq M(\lambda_0,\delta_0;\emptyset)\mu_{\lambda_0,\delta_0},\forall t\geq 0. \quad (24)$$

Hence we conclude (24)

From (20) and (23) it trails that

$$|w(t)|\leq M(\lambda_0,\delta_0;\emptyset), for\ t\in[-\tau,0] \quad (25)$$

To prove$M(\lambda_0,\delta_0;\emptyset)$ is a bound of w on the entire interval [-τ,∞].

Specifically$|w(t)|\leq M(\lambda_0,\delta_0;\emptyset), for\ t\in[-\tau,\infty]$ (26)

Assume an arbitrary constant $\varepsilon>0$. We define that

$$|w(t)|<M(\lambda_0,\delta_0;\emptyset)+\varepsilon,\text{ for every } t\in[-\tau,\infty) \quad (27)$$

otherwise, by (23) there exists a $t^*>0$,where $|w(t)|<M(\lambda_0,\delta_0;\emptyset)+\varepsilon$,

when $t<t^*$ and $|w(t^*)|<M(\lambda_0,\delta_0;\emptyset)+\varepsilon$.

Then by applying (19), we get

$$\begin{aligned}M(\lambda_0,\delta_0;\emptyset)+\varepsilon&=|w(t^*)|\\&\leq|p|e^{-(\lambda_0+\delta_0)\tau}|w(t^*-\tau)|\\&\quad+|p(2\lambda_0+\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t}|w(s)|ds+|-p\,{\lambda_0}^2|e^{-\lambda_0\tau}\int_0^\tau e^{-\delta_0 s}\left\{\int_{t-s}^{t}|w(u)|du\right\}ds\\&\leq\left\{|p|+|p(2\lambda_0+\delta_0)\tau|e^{-(\lambda_0+\delta_0)\tau}+{\delta_0}^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)|-p\,{\lambda_0}^2|e^{-\lambda_0\tau}\right\}[M(\lambda_0,\delta_0;\emptyset)+\varepsilon]\\&<[M(\lambda_0,\delta_0;\emptyset)+\varepsilon]\end{aligned}$$

But this contradicts, which we assume in equation (21). So, our assumption is correct.

Therefore (27) is true for all $\varepsilon>0$,it trails that (26) is confirm invariably.

By applying (26) and (19), we derive

$$\begin{aligned}|w(t)|&\leq|p|e^{-(\lambda_0+\delta_0)\tau}|w(t-\tau)|\\&\quad+|p(2\lambda_0+\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t}|w(s)|ds+|-p\,{\lambda_0}^2|e^{-\lambda_0\tau}\int_0^\tau e^{-\delta_0 s}\left\{\int_{t-s}^{t}|w(u)|du\right\}ds\end{aligned}$$

$|w(t)|\leq|p|e^{-(\lambda_0+\delta_0)\tau}|w(t-\tau)|+|p(2\lambda_0+\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t}|w(s)|ds+$ $|\,p\,{\lambda_0}^2|e^{-\lambda_0\tau}\int_0^\tau e^{-\delta_0 s}\left\{\int_{t-s}^{t}|w(u)|du\right\}ds\ \leq\left\{|p|+|p(2\lambda_0+\delta_0)\tau|e^{-(\lambda_0+\delta_0)\tau}+{\delta_0}^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)|-p\,{\lambda_0}^2|e^{-\lambda_0\tau}\right\}M(\lambda_0,\delta_0;\emptyset)=M(\lambda_0,\delta_0;\emptyset)\mu_{\lambda_0,\delta_0}$, for all $t\geq 0$.that means (24) holds.

Theorem 2: Let λ_0 and $\delta_0(\delta_0\neq 0)$be real roots of the characterisitic equations (5) and (11). Consider β_{λ_0} as in theorem 1. Then, for any $\emptyset\in C^1([-\tau,0],R)$,the solution y of (3)-(4) satisfies

$\lim_{n\to\infty}\left\{y(t)e^{-(\lambda_0+\delta_0)\tau}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\}=\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}$,

Where $L(\lambda_0;\phi), R(\lambda_0,\delta_0;\emptyset)\ and \eta_{\lambda_0,\delta_0}$ were given in (9),(16) and (18) respectively.

Proof: By the definitions of x,z,v and w, we have to prove that

$$\lim_{n\to\infty} w(t)=0. \tag{28}$$

In the end of the proof we will establish (28). By using (19) and taking into account (24) and (26), one can show, by an easy induction, that w satisfies

$$|w(t)|\le(\mu_{\lambda_0,\delta_0})^n\, M(\lambda_0,\delta_0;\emptyset), \text{ For all } t\ge n\tau-\tau, (n=0,1,..) \tag{29}$$

But, (20) guarantees that $0<\mu_{\lambda_0,\delta_0}<1$. thus from (29) it follows immediately that w tends to zero as $t\to\infty$,i.e (28) holds.

The proof of the theorem 2 is completed.

Theorem 3: Let λ_0 and $\delta_0(\delta_0\neq 0)$be real roots of the characterisitic equations (5) and (11) and also the conditions in theorem 1 β_{λ_0} and μ_{λ_0,δ_0}be provided. Then, for any $\emptyset\in C^1([-\tau,0],IR)$, the solution y of (3)-(4) satisfies for all $t\ge 0$

$$|y(t)|\le\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}N(\lambda_0,\delta_0;\emptyset)e^{\lambda_0 t}+\left[\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}+\left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right)\mu_{\lambda_0,\delta_0}\right]N(\lambda_0,\delta_0;\emptyset)e^{(\lambda_0+\delta_0)t}, \tag{30}$$

Where $\eta_{\lambda_0,\delta_0}$ was given in (18),

$$k_{\lambda_0}=1+|\lambda_0|+|p|(1+|\lambda_0|)+|(2\lambda_0+2\varsigma)|+|2p\lambda_0|e^{-\lambda_0\tau}+|-p\,\lambda_0{}^2|e^{-\lambda_0\tau}\tau \tag{31}$$

$$h_{\lambda_0,\delta_0}=1+|p|+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\left(1+|p|e^{-\lambda_0\tau}\right)+\delta_0{}^{-1}\left(e^{-\delta_0\tau}-1\right)|p(-2\lambda_0-\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)+$$
$$\delta_0{}^{-2}\left(\delta_0\tau+e^{-\delta_0\tau}-1\right)|p\,\lambda_0{}^2|e^{-\lambda_0\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right) \tag{32}$$

$$e_{\delta_0}=\max_{-\tau\le t\le 0}\left\{e^{-\delta_0 t}\right\} \tag{33}$$

And

$$N(\lambda_0,\delta_0;\emptyset)=\max\{\max_{-\tau\le t\le 0}\left|e^{-\lambda_0 t}\emptyset(t)\right|,\max_{-\tau\le t\le 0}\left|e^{-(\lambda_0+\delta_0)t}\emptyset(t)\right|,\max_{-\tau\le t\le 0}|\emptyset'(t)|,\max_{-\tau\le t\le 0}|\emptyset(t)|\} \tag{34}$$

Furthermore, when $\lambda_0\le 0, \lambda_0+\delta_0\le 0$, the trivial outcome of (1)is stable ,when $\lambda_0<0, \lambda_0+\delta_0<0$,it is asymptotically stable and when $\lambda_0>0, \lambda_0+\delta_0>0$,we say that it is unstable.

Proof: By theorem 1, equation (22) is proved, where$M(\lambda_0,\delta_0;\emptyset)$ and $L(\lambda_0;\emptyset)$ are described by (23) and (9) correspondingly.

Equation (22) leads that $e^{-(\lambda_0+\delta_0)t}|y(t)|\le\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}e^{-\delta_0 t}+\frac{|R(\lambda_0,\delta_0;\emptyset)|}{\eta_{\lambda_0,\delta_0}}+M(\lambda_0,\delta_0;\emptyset)\mu_{\lambda_0,\delta_0}$ (35)

Furthermore, by using (31), (32), (33) and (34), from (9), (16) and (23), we obtain

$|L(\lambda_0;\phi)|\le|\phi'(0)|+|\lambda_0||\phi(0)|+|p|(|\phi'(-\tau)|+|\lambda_0||\phi(-\tau)|)+|2\lambda_0+2\varsigma||\phi(0)|+|p2\lambda_0||\phi(-\tau)|+|-p\,\lambda_0{}^2|e^{-\lambda_0\tau}\int_{-\tau}^{0}e^{-\lambda_0 s}|\phi(s)|ds$
$\le\left(1+|\lambda_0|+|p|(1+|\lambda_0|)+|(2\lambda_0+2\varsigma)|+|2p\lambda_0|e^{-\lambda_0\tau}+|-p\,\lambda_0{}^2|e^{-\lambda_0\tau}\tau\right)N(\lambda_0,\delta_0;\emptyset)=k_{\lambda_0}N(\lambda_0,\delta_0;\emptyset)$

$$|R(\lambda_0,\delta_0;\emptyset)| \le |\emptyset(0)| + |p||\emptyset(-\tau)| - \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\left(1+|p|e^{-\lambda_0\tau}\right)$$
$$+|p(-2\lambda_0-\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{-\delta_0 s}\left(|\emptyset(s)|e^{-\lambda_0 s}+\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\right)ds$$
$$+|p\,{\lambda_0}^2|e^{-\lambda_0\tau}\int_0^t e^{-\delta_0 s}ds\left\{\int_{-s}^{t} e^{-\delta_0 u}\left(|\emptyset(u)|e^{-\lambda_0 u}+\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\right)du\right\}ds$$
$$\le \left[1+|p|+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\left(1+|p|e^{-\lambda_0\tau}\right)+{\delta_0}^{-1}\left(e^{-\delta_0\tau}-1\right)|p(-2\lambda_0-\delta_0)|e^{-(\lambda_0+\delta_0)\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)+\right.$$
$$\left.{\delta_0}^{-2}\left(\delta_0\tau+e^{-\delta_0\tau}-1\right)\left|p\,{\lambda_0}^2\right|e^{-\lambda_0\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)\right]N(\lambda_0,\delta_0;\emptyset)=h_{\lambda_0,\delta_0}N(\lambda_0,\delta_0;\emptyset),$$

$$M(\lambda_0,\delta_0;\emptyset) \le \max_{-\tau\le t\le 0}\left\{e^{-(\lambda_0+\delta_0)t}|\emptyset(t)|\right\} + \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\max_{-\tau\le t\le 0}\left\{e^{-\delta_0 t}\right\} + \frac{|R(\lambda_0,\delta_0;\emptyset)|}{\eta_{\lambda_0,\delta_0}}$$
$$\le \left\{1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right\}N(\lambda_0,\delta_0;\emptyset) \qquad (36)$$

For every $t \ge 0$.since $\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} > 1$, by taking into account the fact that $\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}\right)\mu_{\lambda_0,\delta_0} + \left(1+\mu_{\lambda_0,\delta_0}\right)\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} > 1$,we have

$$|y(t)| \le \left\{\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}+\left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}\right)\mu_{\lambda_0,\delta_0}+\left(1+\mu_{\lambda_0,\delta_0}\right)\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right\}N(\lambda_0,\delta_0;\emptyset), \text{for all } t\in[-\tau,\infty),$$

Which implies that equation (3) has stable trivial solution (at 0).

Next, if $\lambda_0 < 0$ and $\lambda_0+\delta_0 < 0,$ implies that (30) proves that $\lim_{t\to\infty} y(t) = 0$ and equation (3) has asymptotically stable trivial solution (at 0).

Finally, if $\delta_0 > 0, \lambda_0+\delta_0 > 0$. then the trivial solution of (3) is unstable (at 0). Otherwise, there exists a number $l \equiv l(1) > 0$ such that, for any $\emptyset \in C^1([-\tau,0],R)$with $\|\emptyset\| < l$,the solution y of problem (3)-(4) satisfies

$$|y(t)| < 1 \text{ for all } t \ge -\tau \qquad (37)$$

Define $\emptyset_0(t) = e^{(\lambda_0+\delta_0)t} - e^{\lambda_0 t}$ for $t\in[-\tau,0]$

Furthermore, by the definition of $L(\lambda_0;\emptyset)$ and $R(\lambda_0,\delta_0;\emptyset)$, by using (11), we have

$$L(\lambda_0;\phi) = \delta_0 + p\delta_0 e^{-(\lambda_0+\delta)\tau} + 2p\lambda_0\left(e^{-(\lambda_0+\delta)\tau}-e^{\lambda_0\tau}\right) - pe^{-\lambda_0\tau}{\lambda_0}^2\left(\int_{-\tau}^{0} e^{\delta s}\mathrm{ds}-\tau\right)$$
$$= -(2\lambda_0+2\varsigma) - 2p\lambda_0 e^{-\lambda_0\tau} + pe^{-\lambda_0\tau}{\lambda_0}^2\tau = -\beta_{\lambda_0}$$
$$R(\lambda_0,\delta_0;\emptyset) = \emptyset(0) + p\emptyset(-\tau) - e^{-\delta_0 t}\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\left(1+pe^{-\lambda_0\tau}\right)$$
$$-pe^{-(\lambda_0+\delta_0)\tau}(2\lambda_0+\delta_0)\int_{-\tau}^{0} e^{-\delta_0 s}\left(\emptyset(s)e^{-\lambda_0 s}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)ds$$
$$+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^t e^{-\delta_0 s}ds\left\{\int_{-s}^{t} e^{-\delta_0 u}\left(\emptyset(u)e^{-\lambda_0 u}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)du\right\}ds$$
$$R(\lambda_0,\delta_0;\emptyset) = 1 + pe^{-(\lambda_0+\delta_0)\tau}$$
$$+\left(pe^{-(\lambda_0+\delta_0)\tau}(-2\lambda_0-\delta_0)\right)\int_{-\tau}^{0} e^{-\delta_0 s}\left(e^{-\lambda_0 s}\left(e^{-(\lambda_0+\delta)\tau}-e^{\lambda_0\tau}\right)+1\right)ds$$
$$+pe^{-\lambda_0\tau}{\lambda_0}^2\int_0^{\tau} e^{-\delta_0 s}ds\left\{\int_{-s}^{0} e^{-\delta_0 u}\left(e^{-\lambda_0 u}\left(e^{-(\lambda_0+\delta)\tau}-e^{\lambda_0\tau}\right)+1\right)du\right\}ds$$
$$=1+pe^{-(\lambda_0+\delta_0)\tau}+\left(pe^{-(\lambda_0+\delta_0)\tau}(-2\lambda_0-\delta_0)\tau+{\delta_0}^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)p\,{\lambda_0}^2e^{-\lambda_0\tau} \equiv \eta_{\lambda_0,\delta_0} > 0.\right.$$

Let $\emptyset \in C^1([-\tau,0],R)$ be defined by $\emptyset = \frac{l_1}{\|\emptyset_0\|}\emptyset_0$, Where l_1 is a number with $0 < l_1 < l$. Moreover, let y be the solution of (3)-(4). From theorem 2 it follows that y satisfies

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\}=\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t)-\frac{l_1}{\|\emptyset_0\|}e^{-\delta_0 t}\right\}=\frac{R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}$$
$$=\frac{\left(\frac{l_1}{\|\emptyset_0\|}\right)R(\lambda_0,\delta_0;\emptyset)}{\eta_{\lambda_0,\delta_0}}=\frac{l_1}{\|\emptyset_0\|}>0.$$

But, we have $\|\emptyset_0\| = l_1 < l$ and hence from (37) and conditions $\delta_0 > 0, \lambda_0 + \delta_0 > 0$ it follows that

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\}=0$$

This is a contradiction. The proof of theorem 3 is completed.

Example 1:

Consider $y''(t)-(\frac{1}{6})y''\left(t-\frac{1}{2}\right)=-2y'(t)-y(t),\ t\ >\ 0,$ (38)

$y(t)=\emptyset(t), -1/2\ <\ t\ <0,$

where $\emptyset(t)$ is an arbitrary continuously differentiable initial function on the interval $[-\frac{1}{2},0]$. In this example we apply the characteristic equations (5) and (11).

That is, the characteristic equation (5) is

$$\lambda^2\left(1-\frac{1}{6}e^{\frac{-\lambda_0}{2}}\right)=-2\lambda-1 \quad (39)$$

and using Newton Raphson method λ = -0.7101 is a root of (39). Then, for $\lambda_0\ =\ -0.7101$ the characteristic equation (11) is

$$\delta\left(1-\frac{1}{6}e^{-(-0.7101+\delta)\frac{1}{2}}\right)=-(2(-0.7101)+2)-\frac{1}{3}0.7101e^{-(-0.7101+\delta)\frac{1}{2}}+(-\frac{1}{6})e^{+\frac{0.7101}{2}}(-0.7101)^2\delta^{-1}(1-e^{-\delta(-0.7101)}))$$

Therefore, $\delta\ =\ \delta_0=-0.3524$ is a root, and the conditions of Theorems 3 are satisfied.

That is, $\mu_{\lambda_0,\delta_0}=\mu_{-0.7101,-0.3524}=0.6016<1$

$\beta_{\lambda_0}=\beta_{-0.7101}=0.9773\neq 0$

Since λ_0= - 0.7101 <0 and $\lambda_0+\delta_0$=-1.0625 <0, the zero solution of (38) is asymptotically stable.

Example 2: Consider

$$y''(t)+(\frac{1}{2e})y''\left(t-\frac{1}{2}\right)=4y'(t)-y(t)\ t\ >\ 0, \quad (40)$$
$y(t)=\emptyset(t), -1/2\ <\ t\ <0,$

where $\emptyset(t)$ is an arbitrary continuously differentiable initialfunction on $[-\frac{1}{2}$,0]. The characteristic equation (5) is

$$\lambda^2\left(1+\frac{1}{2e}e^{\frac{-\lambda_0}{2}}\right)=+4\lambda-1 \quad (41)$$

and we see easily that λ = 0.2719 is a root of (41). Taking $\lambda_0=0.2719$, the characteristic equation (11) is

$$\delta\left(1-\frac{1}{2e}e^{-(0.2719+\delta)\frac{1}{2}}\right) = -(2(0.2719)-4)+\frac{1}{e}0.2719e^{-(0.2719+\delta)\frac{1}{2}}+(-\frac{1}{2e})e^{+\frac{0.2719}{2}}(0.2719)^2\delta^{-1}(1-e^{-\delta(0.2719)})$$

Therefore, we find that $\delta = \delta_0 = 0.3279$ is a root.

Corresponding to the roots $\lambda_0 = 0.2719$ and $\delta_0 = 0.3279$, the conditions of Theorem 3 are satisfied. Since $\lambda_0 > 0$ and $\lambda_0 + \delta_0 > 0$, the zero solution of (38) is unstable.

4. Conclusion

The stability is examined for the real-time sub-structuring testing method to a mass-spring-damper system attached with a pendulum. Numerically, the system is modeled and necessary conditions are derived through Neutral Delay Differential Equations (NDDEs).

References

1. Blakeborough, M.S. Williams, A.P. Darby and D.M. Williams, The development of real-time sub-structure testing, *Proc. R. Soc. A* 359, 1869-1891 (2001).
2. M.S. Williams, A. Blakeborough, Laboratory testing of structures under dynamic loads: an introductory review, *Proc. R. Soc. A* 359, 1651-1669 (2001).
3. Y.N. Kyrychko, K.B. Blyuss, A. Gonzalez-Buelga, S.J. Hogan and D.J. Wagg, Stability Switches in a Neutral Delay Differential Equation with Application to Real-Time Dynamic Substructuring, Applied Mechanics and Materials, *5-6 ,79-84(2006).*
4. M.I. Wallace, J. Sieber, S.A. Neild, D.J. Wagg and B. Krauskopf, *Stability analysis of real-time.*
5. dynamic sub-structuring using delay differential equations models, *Earthq. Eng. Struct. Dyn.* 34,1817-1832 (2005).
6. D. Piriadarshani, K. Sasikala, Beena James Lamberts w function approach on the stability analysis of one-dimensional wave equation via second order neutral delay differential equation, *PalArch's Journal of Archaeology of Egypt/Egyptology, PJAEE,* 17 (7) (2020)
7. Ali Fuat Yeniçerioğlu, Barış and Demir, on the stability of second order neutral delay differential equation, *Beykent University Journal of Science and engineering* volume, 109-129(2012).