On The Homogeneous Third Degree Diophantine Equation With Four Unknowns $x^3 + y^3 = 42zw^2$

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ABSTRACT : The homogeneous third degree equation with four unknowns represented by the Diophantine equation

$$x^3 + y^3 = 42zw^2$$

is considered for its patterns of non – zero integral solutions. A few fascinating properties among the solutions and special integer are presented.

KEYWORDS : Third degree equation with four unknowns, Integral solutions.

I. INTRODUCTION

The Diophantine equation offer an unlimited fieldfor research due to their change [1-3]. In particular, one may denote [4-15] for third degree equation with four unknowns. This communication concern withso far another interesting equation $x^3 + y^3 = 42zw^2$demonstrating the homogeneous third degree equation with four unknowns for defining its infinitely many non – zero integral points. Varies interesting properties among the values x, y, z and w are presented.

II. NOTATION USED

- $t_{m,n}$ = Polygonal integer of order n with size m
- P_n^m= pyramidal integer of order n with size m
- P_r^n= pronic integer of order n
- So_n = Stella octangular integer of order n
- j_n= Jacobsthallucas integer of order n
- J_n = Jacobsthal integer of order n
- Gno_n = Gnomic integer of order n
- M_n = Mersenne integer of order n
- HG_n = Hexagonal integer of order n
- PP_n = Pentagonal pyramidal integer of order n
- SP_n = Square pyramidal integer of order n
- OH_n = Octohedral integer of order n
- FN_n^4 = Four dimensional figurate integer whose generating polygonal is a square

1.1 METHOD OF ANALYSIS

The Third degree Diophantine equation with four unknowns to be solved for obtaining non-zero integral solution is

$$x^3 + y^3 = 42zw^2 \qquad (1)$$

On substituting the linear transformations

$x = u + v, y = u - v, z = u$ (2)

In (1) leads to

$$u^2 + 3v^2 = 21w^2 \quad (3)$$

We obtain unlike pattern of integral solutions to (1) through solving (3) which are explained as follows:

1.1.1 PATTERN - I

Assume $w = a^2 + 3b^2$ (4)

Write $21 = \frac{(3n+2ni\sqrt{3})(3n-2ni\sqrt{3})}{n^2}, \forall n = 1,2,3,...$ (5)

Using (4), (5) in (3) and employing factorization it is expressed as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = \frac{(3n+2ni\sqrt{3})(3n-2ni\sqrt{3})}{n^2}(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2$$

which is corresponding to the system of equations

$$(u + i\sqrt{3}v) = \frac{(3n+2ni\sqrt{3})}{n}(a + i\sqrt{3}b)^2 \quad (6)$$

$$(u - i\sqrt{3}v) = \frac{(3n-2ni\sqrt{3})}{n}(a - i\sqrt{3}b)^2 \quad (7)$$

Comparing the positive and negative parts either in (6) or (7), we have

$$u = 3a^2 - 9b^2 - 12ab$$
$$v = 2a^2 - 6b^2 + 6ab \quad (8)$$

In sight of (2), the non-zero different integral solutions of (1) are

$$x = 5a^2 - 15b^2 - 6ab$$
$$y = a^2 - 3b^2 - 18ab$$
$$z = 3a^2 - 9b^2 - 12ab$$
$$w = a^2 + 3b^2$$

PROPERTIES :

1. $x(a+1, a-1) - Star_a + 22T_{4,a} \equiv 5 \pmod{46}$
2. $y(a^2, a+1) - T_{4,a^2} + 36P_a^5 + 3Pro_a \equiv 3 \pmod{3}$
3. $z(a, 2a^2 - 1) + 432FN_a^4 + 12So_a - 3T_{4,a} + 9 = 0$
4. $3x(a, a+1) - y(a, a+1) + 42(Obl)_a - 14T_{4,a} \equiv 42 \pmod{42}$
5. $w(a^2, a^2) - 4T_{4,a^2} = 0$
6. $x(a^2, a^2 - 1) + 10T_{4,a^2} - 30T_{4,a} + 72FN_a^4 + 15 = 0$
7. $x(a^2, 2a - 1) + 12CP_a^6 - 60FN_a^4 + 49T_{4,a} \equiv 15 \pmod{60}$
8. $z(2, a+1) + 9Pro_a \equiv 21 \pmod{33}$
9. $y(a, 2a-1) - T_{4,a} + 18HG_a + 12(Obl)_a \equiv 3 \pmod{24}$
10. $w(a, a+2) - 4Pro_a \equiv 12 \pmod{8}$

1.1.2 PATTERN - II

Equation (3) can also be written as

$$u^2 + 3v^2 = 21w^2 * 1 \quad (9)$$

Put 1 as

$$1 = \frac{(n+in\sqrt{3})(n-in\sqrt{3})}{(2n)^2}, \forall n = 1,2,3,... \quad (10)$$

Using (4), (5) and (10) in (9) and using the method of factorization as in pattern - I, the equivalentintegral solutions are given by

$$x = \frac{1}{2}[2a^2 - 6b^2 - 36ab]$$

$$y = \frac{1}{2}[-8a^2 + 24b^2 - 24ab]$$

$$z = \frac{1}{2}[-3a^2 + 9b^2 - 306ab]$$

$$w = a^2 + 3b^2$$

As our plan is to find integral solutions, take a and b suitably so that the solutions are in integers. In particular, the choice $a = 2A,\ \ b = 2B$ leads to the integer solution to equation (1) are given by,

$$x = 4A^2 - 12B^2 - 72AB$$

$$y = -16A^2 + 48B^2 - 48AB$$

$$z = -6A^2 + 18B^2 - 60AB$$

$$w = 4A^2 + 12B^2$$

PROPERTIES :

1) $y(A+1, A-1) + 16Pro_A \equiv 80(mod\ 112)$
2) $z(A+1, A+2) + 48Pro_A + M_6 \equiv 9(mod\ 72)$
3) $x(A^2, A+1) - 4T_{4,A^2} - Star_A + 18Pro_A + 144P_A^5 + 13 = 0$
4) $w(A, A) - 16T_{4,A} = 0$
5) $y(2B-1, B) - 16T_{4,A} + 48HG_B \equiv 16\ (mod\ 64)$
6) $z(7A^2 - 4, A) + 180CP_A^{14} - 354T_{4,A} + 294T_{4,A^2} + M_6 + 33 = 0$
7) $x(A, 2A^2 + 1) + 92T_{4,A} + 216(OH)_B + 576FN_A^4 + 12 = 0$
8) $x(A, A^2) + 48FN_A^4 - 8T_{4,A} - 72CP_A^6 = 0$
9) $z(A, A(A+1)) - (So_A * Gno_A) - 48FN_A^4 - 10T_{4,A^2} - 38CP_A^6$
$+120P_A^5 \equiv 18\ (mod\ 1)$
10) $y(A,\ 8A-7) + 2688(obl)_A + 400T_{4,A} + J_{13} - M_8 + 48T_{18,A} = 124$

1.1.3 PATTERN– III

Let 21 can be written as $(7 * 3)$

In equation (3) can be written as,

$$u^2 + 3v^2 = (7 * 3)w^2 \qquad \text{s} \quad (11)$$

Write (7) and (3) as

$$7 = \frac{(2n + ni\sqrt{3})(2n - ni\sqrt{3})}{n^2}, \forall n = 1,2,3, \ldots (12)$$

$$3 = \frac{(3n + ni\sqrt{3})(3n - ni\sqrt{3})}{(2n)^2}, \forall n = 1,2,3 \ldots (13)$$

Using (4), (12) and (13) in (11) and employing factorization, it is expressed as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = \frac{(2n + ni\sqrt{3})(2n - ni\sqrt{3})}{n^2} \frac{(3n + ni\sqrt{3})(3n - ni\sqrt{3})}{(2n)^2}$$

$$(a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2$$

which is corresponding to the system of equations

$$(u + i\sqrt{3}v) = \frac{(2n + ni\sqrt{3})}{n} \frac{(3n + ni\sqrt{3})}{2n} (a + i\sqrt{3}b)^2 \quad (14)$$

$$(u - i\sqrt{3}v) = \frac{(2n - ni\sqrt{3})}{n} \frac{(3n - ni\sqrt{3})}{2n} (a - i\sqrt{3}b)^2 \quad (15)$$

Comparing the positive and negative parts either in (14) or (15), we have

$$\left.\begin{aligned} u &= \frac{1}{2} 3a^2 - 9b^2 - 30ab \\ v &= \frac{1}{2} 5a^2 - 15b^2 + 6ab \end{aligned}\right\} \quad (16)$$

In sight of (2), the non-zero different integral solutions of (1) are

$$x = \frac{1}{2}(8a^2 - 24b^2 - 24ab)$$

$$y = \frac{1}{2}(-2a^2 + 6b^2 - 36ab)$$

$$z = \frac{1}{2}(3a^2 - 9b^2 - 30ab)$$

$$w = a^2 + 3b^2$$

As our plan is to find integral solutions, take a and b suitably so that the solutions are in integers. In particular, the choice $a = 2A,\ b = 2B$ leads to the integer solution to equation (1) are given by,

$$x = 16A^2 - 48B^2 - 48AB$$

$$y = -4A^2 + 12B^2 - 72AB$$

$$z = 6A^2 - 18B^2 - 60AB$$

$$w = 4A^2 + 12B^2$$

PROPERTIES :

1) $z(A+1, A-1) + 72T_{4,A} \equiv 48 (mod\ 48)$
2) $x(2B^2 - 1, B) + 48T_{4,B} - 768FN_B^4 + 48So_A = 16$
3) $y(B, 2B^2 + 1) - 48Biq_B + 52T_{4,B} + 72(OH)_B = 12$
4) $x(A, A+1) + 4y(A, A+1) + 336Pro_A = 0$
5) $w(A+1, A+1) - 16(Obl)_A \equiv 16 (mod\ 16)$
6) $z(B, B^2 + 1) + 24CP_B^6 - 60T_{4,B} + 18T_{4,B^2} + 72P_B^5 \equiv 18\ (mod\ 60)$
7) $x(A, 2A - 1) + 192T_{4,A} + 48(HG)_A - 16Pro_A \equiv 48\ (mod\ 192)$
8) $x(A^2, A+1) - 16Biq_A + 48T_{4,A} + 96PP_A \equiv 48\ (mod\ 96)$
9) $y(\ A(A+1), A+2) + 16P_A^5 + 4T_{4,A^2} + 432P_A^3 - 16Pro_A \equiv 48\ (mod\ 32)$
10) $z(A, A+2) + 72Pro_A \equiv 72\ (mod\ 120)$

1.1.4 PATTERN – IV

Let 21 can be written as $(7*3)$

$$u^2+3v^2=21w^2*1$$

$$u^2+3v^2=(7*3)w^2*1 \quad (17)$$

Using (4), (10),(12) and (13) in (17) and employing factorization, it is expressed as

$$(u+i\sqrt{3}v)(u-i\sqrt{3}v)$$

$$=\frac{(2n+ni\sqrt{3})(2n-ni\sqrt{3})}{n^2}\frac{(3n+ni\sqrt{3})(3n-ni\sqrt{3})}{(2n)^2}$$

$$\frac{(n+in\sqrt{3})(n-in\sqrt{3})}{(2n)^2}(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

which is corresponding to the system of equations

$$(u+i\sqrt{3}v)=\frac{(2n+ni\sqrt{3})}{n}\frac{(3n+ni\sqrt{3})}{2n}\frac{(n+ni\sqrt{3})}{2n}(a+i\sqrt{3}b)^2 \quad (18)$$

$$(u-i\sqrt{3}v)=\frac{(2n-ni\sqrt{3})}{n}\frac{(3n-ni\sqrt{3})}{2n}\frac{(n-ni\sqrt{3})}{2n}(a-i\sqrt{3}b)^2 \quad (19)$$

Comparing the positive and negative parts either in (18) or (19), we have

$$u=\frac{1}{4}(-12a^2+36b^2-48ab$$

$$v=\frac{1}{4}(8a^2-24b^2-24ab) \quad (20)$$

In sight of (2), the non-zero different integral solutions of (1) are

$$x=\frac{1}{4}(-4a^2+12b^2-72ab)$$

$$y=\frac{1}{4}(-20a^2+60b^2-24ab)$$

$$z=\frac{1}{4}(-12a^2+36b^2-48ab)$$

$$w=a^2+3b^2$$

As our plan is to find integral solutions, take a and b suitably so that the solutions are in integers. In particular, the choice $a=4A,\ b=4B$ leads to the integer solution to equation (1) are given by,

$$x=-16A^2+48B^2-288AB$$

$$y=-80A^2+240B^2-96AB$$

$$z=-48A^2+144B^2-192AB$$

$$w=16A^2+48B^2$$

PROPERTIES :

1) $x(A+1,A^2)-32Biq_A-144FN_A^4+576P_A^5\equiv 16\ (mod\ 32)$
2) $w(A,A^2)-48Biq_A-16T_{4,A}=0$
3) $z(A,2A-1)-528T_{4,A}+192(HG)_A-M_7\equiv 17\ (mod\ 576)$
4) $y(A+1,A)+96(Obl)_A-160T_{4,A}\equiv 80(mod\ 160)$
5) $x(\ A(A+1),2A+1)+1728P_A^4+32T_{4,A}+32CP_A^6+144FN_A^4-192Pro_A=48$
6) $y(2A^2+1,A)+240Biq_A+288(OH)_A+80=0$

7) $x(A+2,2) + w(A+2,2) + M_9 + j_8 \equiv 8 \ (mod\ 572)$

8) $w(A^2, A^2) - 64Biq_A = 0$

9) $z(A, A^2+1) + 192CP_A^6 - 144T_{4,A^2} - 240Pro_A \equiv 17 \ (mod\ 432)$

10) $x(A+2, A+1) + 256T_{4,A} + M_9 \equiv 81 \ (mod\ 592)$

III. CONCLUSION

In conclusion, one may study other methods of third degree equation with four unknowns and examine for their integer solutions.

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