Types of Filter and Ultra-filter with applications

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Abstract
The aim of this paper is to study the notion classes of filter and ultra-filter with application. In section one, types of filter have been introduced
Principle, non-principle, maximal and prime filter with some basic properties are studied and we establish a proof of some important properties. If $\mathcal{F}$ be a filter on the set $\mathcal{M}$, and let $p \subseteq \mathcal{M}$, either:There is some $q \in \mathcal{F}$, s.t $p \cap q = \phi$ or $\{c \subseteq \mathcal{M}$: there is some $q \in \mathcal{F}$, $p \cap q \subseteq c\}$ is a filter on M. Frechet filter is also introduced in this paper. In section, two of this paper is the major contribution; we introduced two important application with new proof of ultra-filter in additive measure theory and Boolean algebra. There are one to one corresponding of ultra-filters on $\mathcal{M}$ and finitely additive measure and Boolean algebra defined on $P(\mathcal{M})$.

Keywords: Filter, Ultra-filter, Frechet filter, Maximal filter, Prime filter, additive measure, Boolean algebra.

1- Preliminaries.

Definition 2-1:[6] Let $\mathcal{M}$ be any set, a filter on a set $\mathcal{M}$ is a non-empty set $\mathcal{F}$ with the following properties:

1- $\phi \notin \mathcal{F}$.

2- If p and $q \in \mathcal{F}$ then $p \cap q \in \mathcal{F}$.

3- If $p \in \mathcal{F}$, and $p \subseteq q \subseteq \mathcal{M}$, then $q \in \mathcal{F}$, ($\mathcal{F}$ is closed superset).

Next, we will come up with two filter examples on topology and set theory.

Example 2-1: The set $\mathcal{F}$ of a neighborhood of a point b in a topological space X is a filter. Clearly $\phi \notin \mathcal{F}$ and if $p = (b - \frac{\in}{2},\ b + \frac{\in}{2})$, $q = (b - \frac{\in}{4}\ , b + \frac{\in}{4})$ in $\mathcal{F}$ then $p \cap q \in \mathcal{F}$, Also a neighborhood N for any point in X, such that $p \subseteq$ N $\subseteq$ X implies N $\in \mathcal{F}$.

Example 2-2: Let $\mathcal{M}$ be infinite set, and consider the set $\mathrm{T} = \{ \mathrm{A} \subseteq \mathcal{M}$ s.t $\mathcal{M}$ / A is finite $\}$ the set of all cofinte subset of $\mathcal{M}$ is a cofinite filter this filter is called Frechet filter on $\mathcal{M}$ which is denoted by FR.

Note: One can see the Frechet filter is not an ultra-filter on infinite set.

Definition 2-2:[2] Let $\mathcal{M}$ be a non-empty set and $D \subseteq \rho(\mathcal{M})$ be a collection of subsets of a set $\mathcal{M}$. We say that D has a finite intersection property (FIP) if the finite intersection for any specific subset of D is not empty.

Remarks 2-1: 1- Filter is closed under the finite intersection property.
2- Every filter and thus any subset of a filter has finite intersection property. Induce that we can get filter including $\mathcal{M}$ iff $\mathcal{M}$ satisfy the finite intersection property.
Remark: If $\emptyset \neq K \subseteq \mathcal{M}$, the set $\{ D \subset \mathcal{M}: K \subset D\}$ is filter generated by a set K denoted by $< \{K\} >$. If K is singleton subset, i.e. $K = \{c\}$ where $c \in \mathcal{M}$, then it's called a principle filter generated by K consisting all subsets containing c.

Lemma 2-4: Let $\mathcal{M}$ be a finite set then any ultra-filter over $p(\mathcal{M})$ is principle.

Example 2-3: Let $\mathcal{M}$ be a non-empty set and let x $\subseteq \mathcal{M}$. Then $\mathcal{F}$= {D $\subseteq \mathcal{M}$: x $\subseteq$ D} is a filter generated by x. In fact it's a proper filter if x = {1,2,3} then $F = \{D \subseteq N: \{1,2,3\} \subseteq D\}$ is principle proper filter generated by {1,2,3}.

Definition 2-3:[1] $\mathcal{F}$ is prime filter if for any, $q \subseteq \mathcal{M}$, satisfies $p \cup q \in \mathcal{F}$, either $p \in \mathcal{F}$ or q $\in \mathcal{F}$.

Lemma 2-1: Let $\mathcal{F}$ be an ultra-filter, if $p \cup q \in \mathcal{F}$, then either $p \in \mathcal{F}$ or $q \in \mathcal{F}$.

Proof: Suppose $p \notin \mathcal{F}$, and $q \notin \mathcal{F}$, then $p^c, q^c \in \mathcal{F}$, it follows that $p^c \cap q^c = (p \cup q)^c \in \mathcal{F}$, then therefore $p \cup q \notin \mathcal{F}$, contradiction.

Definition 2-4:[5] A filter Ƒ on $\mathcal{M}$ is called an ultra-filter if it is not properly contained in any other filter.

An ultra-filter on M is non-principal if it is not principle.

Example 2-4: The trivial filter $\{\mathcal{M}\}$ on $\mathcal{M}$ is not ultra-filter unless $\mathcal{M}$ is singleton. Also the Frechet filter is not ultra-filter if $\mathcal{M}$ is infinite, since there are infinite cofiinite subsets in $\mathcal{M}$. For example if $\mathcal{M} = \mathbb{Z}$, then neither the set of positive integer numbers neither its complement is contained in $\mathcal{M}$ which is not ultrafilter according to the next following lemma.

Another characteristic for ultra-filter show in the next lemma.

Lemma 2-2: Let $\mathcal{M}$ be a non-empty set and $\mathcal{F}$ be a filter on $\mathcal{M}$. Then $\mathcal{F}$ is an ultra-filter if for every p $\subseteq \mathcal{M}$ either p or $\mathcal{M}\backslash p$ is an element on $\mathcal{F}$.

Proof: For the first direction, it is direct proof by (2) of definition.
Conversely, let $\mathcal{F}$ be an ultra-filter in $\mathcal{M}$. Assume that for p $\subseteq \mathcal{M}$ neither p nor its complement $\mathcal{M}\backslash p$ belong to $\mathcal{F}$. Case (1): p and $\mathcal{M}\backslash p \in \mathcal{F}$ implies by definition of filter p $\cap\, \mathcal{M}\backslash p = \emptyset \in \mathcal{F}$, which is a contradiction. Case (2): we have p and $\mathcal{M}\backslash p \notin \mathcal{F}$. Note that $\mathcal{F} \cup$ p and $\mathcal{F} \cup \mathcal{M}\backslash p$ both are filter and $\mathcal{F} \subseteq \mathcal{F} \cup p$ and $\mathcal{F} \subseteq \mathcal{F} \cup \mathcal{M}\backslash p$ which is a contradiction. Therefore p or $\mathcal{M}\backslash p \in \mathcal{F}$.

Definition 2-5:[6] A filter $\mathcal{F}$ on $\mathcal{M}$ is maximal filter if for any $p \subseteq \mathcal{M}$ and $p \notin \mathcal{F}, then\ \mathcal{F} \cup \{p\}$ is not a filter.

Proposition 2-1: A filter $\mathcal{F}$ on a non-empty set $\mathcal{M}$ is an ultra-filter if only if it is maximal filter.

In the following theory, we can prove that we had a non-empty set that contains the filter and ultra-filter, and so the filter is part of the ultra-filter within this set.

Theorem 2-1: Every filter $\mathcal{F}''$ on anon-empty set $\mathcal{M}$ there exists an ultra-filter $\mathcal{F}\ on\ \mathcal{M}$ such that $\mathcal{F}'' \subseteq \mathcal{F}$

Proof: Let $S = \{ F : F\ is\ a\ filter\ and\ \mathcal{F}'' \subseteq F \}$ and take the partially ordered set (S, ⊆). Now consider a chain $L \subseteq S$, the set of union $\cup L$ of this collection of filter indicted with ⊆ is clearly a filter on $\mathcal{M}$ and containing $\mathcal{F}''$ which is an upper bound for L. Hence, (S, ⊆) satisfies the hypotheses of Zorn's lemma implies has maximal element $\mathcal{F}$ which is maximal element of (S, ⊆) . We claim that $\mathcal{F}\ is$ an ultra-filter. If not then there exists A $\subseteq \mathcal{M}$ such that A $\notin \mathcal{F}$ and $\mathcal{M} \backslash A \notin \mathcal{F}$. Consider the collection C = $\mathcal{F} \cup$ {A }. We claim that the set C has finite intersection property. Let $y_1, y_2, \dots, y_n \in C$ ..

Case1: Suppose that $y_i \in \mathcal{F}$ for every $1 \leq i \leq n.$ since $\mathcal{F}$ has finite intersection property then $y_1 \cap y_2 \cap \dots \cap y_n \in F \subseteq C.$

Case 2: Suppose that $y_i \notin \mathcal{F}$ for some $1 \leq i \leq n$. By changing these sets without changing them intersection, we can suppose without loss of generality that y_1 = A and $y_2, \dots, y_n \in \mathcal{F}$. As $\mathcal{F}$ has finite intersection property, we have that $y_2 \cap \dots \cap y_n \in \mathcal{F}$. It follows that any superset of $y_2 \cap \dots \cap y_n$ is in $\mathcal{F}$. From the other side, $\mathcal{M}\backslash A \notin \mathcal{F}$ and hence $y_2 \cap \dots \cap y_n \nsubseteq \mathcal{M}\backslash A$, that is A $\cap\, y_2 \cap \dots \cap y_n \neq \emptyset$. Therefore, C has finite intersection property and can be extension to a filter.Hence $F \subseteq C \subseteq \mathcal{F}$ which is a contradiction by Zorn's lemma $\mathcal{F}$ is maximal.

Theorem 2-2: Let $\mathcal{M}$ be a set, $\mathcal{M} \neq \emptyset$ then $p \subseteq \mathcal{M}$, and let $\mathcal{F}$ be a filter on $\mathcal{M}$. Then there is some $q \in \mathcal{F}$, such that $p \cap q = \phi$ or if there exist
and some $q \in \mathcal{F}, p \cap q \subseteq C\ is\ a\ filter\ on\ \mathcal{M}. C \subseteq \mathcal{M}$

Proof: Suppose a for all $q \in \mathcal{F}, p \cap q \neq \phi$, we need to show the set $k = \{C \subseteq \mathcal{M}, \exists$ some $q \in \mathcal{F}$ with $p \cap q \subseteq C\}$ is actually a filter. Let $p_1, p_2 \in k$, then there exist $q_1, q_2 \in \mathcal{F}\ s.t\ p \cap q_1 \subseteq p_1\ , p \cap q_2 \subseteq p_2$.

Then $p \cap (q_1 \cap q_2) \subseteq p_1 \cap p_2 \in k$. If $p' \in k$ take $q \in \mathcal{M}$ such that $p' \subseteq q \subseteq \mathcal{M}$ to show that that $q \in k$. Now $p' \in k$ implies that $\exists q' \in \mathcal{F}$ such that $\cap q' \subseteq p'$, hence $p \cap q' \subseteq q$ and then $q \in k$. By negative condition of (a) then $\phi \notin k$.
The next lemma it is easy to show, as a one of an important fact for the ultra-filter.
Lemma 2-3: Let $\mathcal{M}$ be a set and $\mathcal{F}$ and L be ultra-filters on $\mathcal{M}$ then $\mathcal{F} = \mathrm{L}$ if and only if $\mathcal{F} \subseteq \mathrm{L}$.

Proof: The first direction is clear. Conversely, let $\mathcal{F} \subseteq \mathrm{L}$, $\mathcal{F}$ is ultra-filter. Then by definition of ultra-filter $\mathcal{F} = L$.

The next theory it's called ultra-filter theorem, the content of this theory that the filter can be extended to ultra-filter.

Theorem 2-3: Any filter $\mathcal{F}$ on a non-empty set $\mathcal{M}$ be expansion to an ultra- filter.

Proof: For some filter $\mathcal{F}$, suppose that $A = \{\mathrm{F}' \subseteq \mathcal{M}: \mathrm{F}' \text{ is filter s.t } \mathcal{F} \subseteq F'\}$. Note that A is nonempty since $\mathcal{F} \in \mathrm{A}$. Claim that for each chain $\{ \mathcal{F}_\alpha : \alpha \in I \}$ in A , their union $\bigcup_{\propto \in I} \mathcal{F}_\alpha$ is still a filter in A. It is clear that $\emptyset \notin \bigcup_{\propto \in I} \mathcal{F}_\alpha$. For an element B $\in \bigcup_{\propto \in I} \mathcal{F}_\alpha$, $B \in \mathcal{F}_\alpha$ for some $\alpha \in I$. Then for all C such that $B \subseteq C$, $C \in \mathcal{F}_\alpha \subset \bigcup_{\propto \in I} \mathcal{F}_\alpha$. Similarly, for P and $Q \in \bigcup_{\propto \in I} \mathcal{F}_\alpha$ $P \in \mathcal{F}_\alpha$ and $\mathrm{C} \in \mathcal{F}_\beta$. Without loss of generality, assume $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$; thus, $\mathrm{C} \in \mathcal{F}_\alpha$ *and* $\mathrm{B} \cap \mathrm{C} \in \mathcal{F}_\alpha \subseteq \bigcup_{\propto \in I} \mathcal{F}_\alpha$. Hence, by Zorn's lemma, there exists a maximal element in A and by proposition {2-1}; it is an ultra-filter.

Remark 2-2: The Frechet filter in infinite set $\mathcal{M}$ is non–principle. In the next proposition show whatever an ultra-filter is principle or non-principle by checking if it have Frechet filter.

Lemma 2- 4:[6] Let $\mathcal{M}$ be a finite set then any ultra-filter over $p(\mathcal{M})$ is principle.

Proposition 2-2: Let $\mathcal{M}$ be an infinite set and $\mathcal{F}$ be an ultrafilter on $\mathcal{M}$. Then $\mathcal{F}$ is non-principle if and only if it include the Frechet filter.

Proof: Assume that $\mathcal{F}$ is principal, let it be $\mathcal{F} = \{ B \subseteq \mathcal{M}: b \in B\}$. Then, since $\{b\} \in \mathcal{F}$, we have $\mathcal{M} - \{b\} \notin \mathcal{F}$. On the other hand, $\mathcal{M} - \{b\}$ is cofinite. Hence, $\mathcal{M}$ dose not have the frechet filter. Suppose that $\mathcal{F}$ dose not include the frechet filter. Then there are a cofinite set $B \subseteq \mathcal{M}$ such that $B \notin \mathcal{F}$ and hence $\mathcal{M} - B \in \mathcal{F}$. Redefine the set $\mathcal{M} - B$, say, $\mathcal{M} - \mathrm{B} = \{d_1, d_2, \dots\dots, d_n\}$. If $\mathcal{M} - \{di\} \in \mathcal{F}$ for every $1 \leq i \leq n$, then we would have

$$\bigcap_{i=1}^{n} \mathcal{M} - \{d_i\} = \{d_1, d_2, \dots\dots, d_n\} \in F$$

Which is a contradiction, also the intersection of this set and $\mathcal{M} - B$ is empty. Therefore, there exists $1 \leq i \leq n$ such that $\mathcal{M} - \{d_i\} \} \notin \mathcal{F}$and hence $\{di\} \in \mathcal{F}$ .Therefore , as in the proof of lemma {2-4 } we should have $\{ B \subseteq \mathcal{M}: b \in B\} = \mathcal{M}$.

2- Ultra-filter application.

In the present section of this paper, we will cove two main application of ultra-filter.

Definition 2-6:[3] A finitely-additive measure on X is a function $\mu : 2^X \to \{0, 1\}$ that satisfies

1. $\mu(X) = 1$, $\mu(\emptyset) = 0$

2. If $A_1, \dots, A_n$ are pairwise disjoint, then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.

The next theorem give us an application of ultra-filter by showing that the ultra-filter can significant as a finitely additive measure.

Theorem 2- 4: Let $\mathcal{M}$ be any set, show that the ultra-filters on $\mathcal{M}$ are one to one corresponding with finitly additive measurs defined on $\mathrm{P}(\mathcal{M})$ which takes values in {0,1} and are not identically zero.

Proof: Define a map μ: $\mathrm{P}(\mathcal{M}) \to \{0,1\}$ by

$$\begin{cases} 1 & if \quad A \in \mathcal{F} \\ 0 & if \quad A \notin \mathcal{F} \end{cases} \qquad \mu(A) =$$

Since F is a filter then $\emptyset \notin \mathcal{F}$ then $\mathcal{M} \in \mathcal{F}$, by define implies $\mu\,(\mathcal{M})$ = 1. It is enough to prove that for disjoint set A and B, $\mu(A \cup B\,) = \mu\,(A) + \mu\,(B)$. Case (1): if $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ or vice versa, then by definition it's clear that they are equal. Case (2): Since A and B are disjoint then $A \cap B = \emptyset$. Therefore Aor $B \in \mathcal{F}$ but not both, led to

, implies that $A^c \cap B^c = (\,A \cup B)^c \in \mathcal{F}$. We get $A \cup B \notin \mathcal{F}$ and therefore $\mu(A \cup A^c and\; B^c \in \mathcal{F}$
$B\,) = \mu\,(A) + \mu\,(B) = 0$.

For the other direction, suppose we has non-zero finitely additive measure. Clearly by definition $\mu(\emptyset)$ = 0. Let $A, B \in \mathcal{F}$, by definition μ (A) = μ (B) =1 therefore (A ∪ A^c) = μ (A) + μ (A^c)= μ (M)

$$\Rightarrow 1 + \mu\,(A^c) = 1$$

$$\Rightarrow \mu\,(A^c) = 0$$

Similarly one can get μ (B^c) =0. Now if A ∪ B ∈ $\mathcal{F}$ then $\mu\,(A \cup B) = \mu\,(A) + \mu\,(B) - \mu\,(\,A \cap B\,)$

$$\Rightarrow 1 = 1 + 1 - \mu\,(A \cap B\,)$$

$$\Rightarrow \mu\,(A \cap B\,) = 1$$

Finally, let A∈ $\mathcal{F}$ such that $A \subseteq B \subseteq \mathcal{M}\,, B \in \mathcal{M}$. Since A ⊆ $B\; then\;\; \mu$ (A) $\leq \mu(\,B)$. But μ (A) = 1 and hence μ (B) = 1.

Theorem 2-5: Let $\mathcal{M}$ be a non-empty set. The ultra-filters on $\mathcal{M}$ are in one-to-one corresponding with the Boolean algebra homomorphism mapping (p ($\mathcal{M}$), ∪, ∩) on to Bolean algebra ({0, 1}, ∨, ∧).

Solution: Let $f: (p(\mathcal{M}),\cup,\cap) \rightarrow (\{0,1\},\vee,\wedge)$ defined by:

$$f(A) = \begin{cases} 1 & if\; A \in \mathcal{F} \\ 0 & if\; A \notin \mathcal{F} \end{cases}$$

Where $A \in p(\mathcal{M})$ and $\mathcal{F}$ is some ultra-filter on $\mathcal{M}$. We claim that f is onto and Boolean homomorphism. If $A \in \mathcal{F}$ then $f(A) = 1$ and if $A \notin \mathcal{F}$ then $f(A^c) = 0$ this implies f is onto. For f is Boolean homomorphism, take $A, B \in p(\mathcal{M})$ then:

Case1: If $A, B \in U$. Note $A \in \mathcal{F} \subseteq A \cup B \subseteq p(\mathcal{M})$ then A∪ B ∈ $\mathcal{F}$ and therefore $f(A \cup B) = 1$

$= 1 \vee 1$

$= f(A) \cup f(B)$.

Also since A, B ∈ $\mathcal{F}$ then $A \cap B \in \mathcal{F}$. Hence $f(A \cap B) = 1$

$=1 \wedge 1$

$= f(A) \wedge f(B)$.

Case2: If $A, B \notin \mathcal{F}$ then $A \cap B \notin \mathcal{F}$ otherwise it's a contradiction.

Therefore $f(A \cap B) = 0$

$= 0 \wedge 0$

$= f(A) \wedge f(B)$.

Also A∪ B ∈ $\mathcal{F}$. Hence $f(A \cup B) = 1$

$= 1 \vee 1$

$= f(A) \cup f(B)$.

Case 3: If $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ then it will be similar to case 2.

Conversely, to show that $\mathcal{F}$ is a filter. Let $A \notin \mathcal{F}$ then $f(A \cup \emptyset) = f(A) \vee f(\emptyset) = f(A)$. Since $f(A) = 0$ then $f(A) \vee f(\emptyset) = 0$ i.e. it must be $f(\emptyset) = 0$. Let $B \in \mathcal{F}$, since f is homo then $f(A \cap B) = f(A) \cap f(B) = 1 \wedge 1 = 1$. Hence $A \cap B \in \mathcal{F}$. To obtain a filter we need also show if $A \in \mathcal{F}$ s.t $A \subseteq B \subseteq p(\mathcal{M})$ then B$\in \mathcal{F}$. Note that because f is homo then $f(A \cap B) = f(A) \cap f(B) = f(A)$
Since $f(A) = 1$ then $f(A) \cap f(B) = 1 \wedge 1 = 1$ one must obtain $f(B) = 1$, therefore $B \in \mathcal{F}$. Finally for ultra-filter, note that $f(A \cap A^c) = f(A) \wedge f(A^c) = f(\emptyset) = 0$. So $f(A^c) = 0$. To show either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Suppose $A \in U$ then $(A \cup Ac) = f(A) \vee f(A^c) = f(\mathcal{M})$. But $f(\mathcal{M}) = 1$ and $f(A) = 0$. Therefore $f(A^c) = 0$.

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