Some Algebraic Structures On Poset And Loset

[1]Reyaz Ahmed, [2]Rashmi Rani
[1]College of Mathematics and Statistics, Emirates Aviation, Academic city, Dubai, UAE
[2]College of Engineering and Computing, Al Ghurair university, Academic city, Dubai, UAE

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ABSTRACT: K. Is´eki and S. Tanaka [4] has introduced a new concept BCK algebras in 1966. Many researchers have developed various concepts using BCK/BCI algebras.Numerous results have been established using BCK/ BCH algebras and their properties. In this paper the existence of some systems has been established which are neither a BCK/BCI algebra nor a BCH–algebra Such system has been named as weakBCK/BCI – algebras. Some properties of poset and losethas also discussed Weak BCK or BCH-algebra.

Keywords: BCK/ BCI Algebra, BCH algebra, poset, loset.

1. INTRODUCTION

After the introduction of systems likeBCK/BCI- algebras by Imai and Iseki in 1966 some more systems have been developed and studied by a number of authors. The BCK-operation * is an analogue of the set theoretical difference.Here we see that there exist systems which are different from the existing systems. Such systems give way to study different systems which are named as weak BCK/BCI–algebras.

2. LITERATUREREVIEW/EXPERIMENTAL DETAILS

Definition (2.1):-

(a) A system (X, *, 0)consisting of a non – empty set X, a binary operation * and a fixed element 0 is called a weak BCI – algebra if the elements of X satisfy the following conditions:

(i) $x * 0 = x$

(ii) $x * x = 0$

(iii) $(x * (x * y)) * y = 0$

(iv) $x * y = 0 = y * x \Rightarrow x = y$

for all x, y, z $\in$ X.

(b) If, in addition to the above conditions, the condition

(v) $0 * x = 0$ for all $x \in X$

is also satisfied then the system (X, *, 0) is called a weak BCK – algebra.

Definition (2.2):- A weak BCI (resp. weak BCK) algebra (X, *, 0) is a BCI (resp.BCK) algebra if the condition

(vi) $((x * y) * (x * z)) * (z * y) = 0$ $(x, y, z \in X)$ is also satisfied.

Definition (2.3):- A system (X, *, 0) is called

(a) a BCH – algebra if conditions (i), (iv) and

(vii) $(x * y) * z = (x * z) * y$ (for all $x, y, z \in X$) are satisfied,

(b) a BCC – algebra if conditions (i), (iv),(v) and

(viii) $((x * y) * (z * y)) * (x * z) = 0$ (for all $x, y, z \in X$) are satisfied,

(c) a weak BCC–algebra if conditions (i), (ii), (iv) and (viii) are satisfied.

First, we examine the existence of systems in definition (2.1) (a) and (b).

Example (2.4):- Let E = {0, a, b, c, d, e} and let ‘o’ be a binary operation defined on E and given by the table

o	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	a	c	a	0
b	b	0	0	b	0b	
c	cc	b	0	c	c	
d	d	0	d	0	0	0
e	e	a	0	0	e	0

Table 1

Here Table 1 represents a weak BCK–algebra.Now,

$((a * b) * (a * e)) * (e * b) = (a * 0) * 0 = a \neq 0,$

$((a * b) * (e * b)) * (d * e) = (d * 0) * 0 = d \neq 0.$

Also, $(a * b) * c = a * c = c$ and $(a * c) * b = c * b = b$

Which means that $(a * b) * c \neq (a * c) * b$.

So, table 1 does not represent a BCK – algebra, a BCC/weak BCC – algebra and a BCH – algebra.

Example (2.5):- Let $(E, *, 0)$ be a system where $E = \{0, a, b, c\}$ and binary operation '*'given by

*	0	a	b	c
0	0	0	0	c
a	a	0	b	c
b	bb	0	c	
c	cc	0	0	

Table 2

Table 2 represents a weak BCI–algebra which is not a BCI–algebra because

$((a * b) * (a * 0)) * (0 * b) = (b * a) * 0 = b * 0 = b \neq 0$

Also $(a * b) * c = b * c = c$ and $(a * c) * b = c * b = 0$ imply Table 2 does not represents a BCH–algebra.

Remark (2.6):- The concepts of weak BCK/ BCI algebras aregeneralizations of the concepts BCK/BCI–algebras.

3. RESULT AND DISCUSSION

In this section, some properties of poset and losetusing BCK or BCH algebra has discussed.

Theorem (3.1):- Every finite partially ordered set (poset) can be made into a weak BCK–algebra by adjoining an element and defining a binary operation suitably.

Proof: Let $E = \{a_1, a_2, ..., a_n\}$ be a poset. We choose an element $a_o \notin E$ and take it as zero element. Let $E_1 = E \cup \{a_o\}$. We index a binary operation '*' on E_1as $a_o, a_1, a_2, ..., a_n$.

We define a binary operation '*' on E_1 as follows:

(i) $a_i * a_o= a_i$ (1)

(ii) $a_o* a_i = a_i$ (2)

(iii) $a_i * a_i = a_o$ (3)

for $i = 0, 1, 2,....,n$.

If a_i and a_j are not comparable, we define

Either $a_i * a_j= a_i$ and $a_j* a_i= a_o$

or $a_i * a_j= a_o$ and $a_j* a_i = a_i$(4)

Where $i \neq j$, and $i, j = 1, 2,...,n$.

If a_i and a_j are comparable, we take

$a_i * a_j = \min \{a_i, a_j\}$ (5)

Where $i \neq j$, and $i, j = 1, 2,........,n$.

From the above definitions, it follows that conditions (i), (ii),(iv) and (v) of a weak BCK–algebrasaresatisfied. So, we need to check condition (iii) only.

In case $a_i<a_j$, $z\neq j$,

$(a_i* (a_i* a_j))* a_j= (a_i* a_i)* a_j = a_o* a_j= a_o$

and $(a_j* (a_j* a_j))* a_j= (a_j* a_j)* a_j = a_i* a_j= a_o$.

in case a_i and $a_j(i \neq j)$ are not comparable, we chose $a_i * a_j = a_i$ and $a_j * a_i = 0$.

Then $(a_i * (a_i * a_j)) * a_j = (a_i * a_{i)} * a_j = a_o * a_j = a_o$

and $(a_j * (a_j* a_i)) * a_i = (a_j * a_o) * a_j = a_j * a_j = a_o$.

Further, $(a_o * (a_o* a_i)) * a_i = (a_o * a_o) * a_j = a_o * a_j = a_o$

and $(a_i * (a_i * a_o)) * a_o = (a_i * a_o) * a_o = a_o * a_o = a_o$.

$i = 1,2,........,n$.

Thus condition (iii) of a weak BCK – algebra is satisfied in all cases. Hence $(E_1, *, a_o)$ is a weak BCK – algebra.

Remark (3.2): -

(a) We choose a_i, a_j,a_k different from a_o0which are not pair wise comparable and such that

$a_i * a_j= a_j, a_j * a_i = a_o;$

$a_i * a_k = a_o, a_k * a_i = a_k;$

$a_k * a_j = a_o, a_j * a_k = a_j.$ (6)

Then $((a_i * a_j) * (a_i * a_k)) * (a_k * a_j) = (a_i * a_o) * a_o = a_i\neq a_o$.

(b) Let $a_i<a_j$ and set a_i and a_j are not comparable with a_k satisfying condition (6).

Then $((a_i * a_j) * (a_i * a_k)) * (a_k * a_j) = (a_i * a_o) * a_o = a_i\neq a_o$.

This means that $(E_1, *, a_o)$ is not a weak BCK – algebra.

Remark (3.3): -We choose $a_i<a_j$ and a_i, a_j not comparable with a_k satisfying condition (6).

Then

$(a_j * a_k) * a_i = a_j * a_i = a_i$ and $(a_j * a_i) * a_k = a_i * a_k = a_o$.

This means that $(E_1, *, a_o)$ is not a BCK–algebra

Remark (3.4): - If we replace relation (2) by $a_o * a_i = a_k$, such that $a_o * a_k = a_k$ for at least one i and k, then the system $(E_1, *, 0)$ is a weak BCI – algebra.

As an illustration, we have the following example.

Example (3.5):- Let X = (a, b, c} and let E = Q(X) which is a posetwith respect to set inclusion. Let

A = ϕ,

B = {a},

C = {b},

D= {c},

E = {a, b},

F ={a, c},

G = {b, c} and

H = X.

We choose $0 \notin E$ let $E_1 = E \cup \{0\}$and define the binary operation according to the relations (1) to (5) given in theorem then the binary operation table is given by

*	0	A	B	C	D	E	F	G	H
0	0	0	0	0	0	0	0	0	0
A	A	0	A	A	A	A	A	A	A
B	B	A	0	B	0	B	B	A	B
C	C	A	0	0	C	C	C	C	C
D	D	A	D	0	0	D	D	D	D
E	E	A	B	C	0	0	E	E	E
F	F	A	B	0	D	0	0	F	F
G	G	A	G	C	D	0	0	0	G
H	H	A	B	C	D	E	F	G	0

Table 3

Then $(E_1, *, 0)$ is a weak BCK–algebra.

Now ((B * C) * (B * D)) * (D * C) = (B * 0) * 0 = B ≠ 0implies $(E_1, *, 0)$ is not a BCK–algebra.Further, (E * C) * B = C * B = 0

and (E * B)* C = B * C = B.imply $(E_1, *, 0)$ is not a BCH – algebra.

Theorem (3.6):- Every finite linearly ordered set (loset) can be made into a BCH–algebra by adjoining one element and defining a binary operation suitably which is not aBCK–algebra.

Proof: - Let $E = \{a_1, a_2, \ldots\ldots, a_n\}$ be a linearly ordered set such that $a_i < a_j$ for i < j. Let $a_o \notin E$ and let $E_1 = \{a_o\} \cup E$. We define abinary operation'*' in E_1 as

$$a_o * a_i = a_o \text{ for } i = 1, 2, \ldots, n \quad (7)$$

$$a_i * a_o = a_i \text{for } i = 1, 2, \ldots, n \quad (8)$$

$$a_i * a_i = a_o \text{for } i = 0, 1, 2, \ldots, n \quad (9)$$

$$a_i * a_j = \min \{a_i, a_j\} \text{ for } i < j, \quad i, j = 1, 2, \ldots, n \quad (10)$$

conditions (i) and (iv) of a BCH – algebra are satisfied from the relations defined above. It remains to veritycondition (vii).

For i < j < k, we have

$(a_i * a_j) * a_k = a_i * a_k = a_i$ and $(a_i * a_k) * a_i = a_i * a_j = a_i$;

For i < j.

$(a_o * a_i) * a_j = a_o * a_j = a_o$, $(a_o * a_j) * a_i = a_o * a_i = a_o$;

$(a_i * a_o) * a_j = a_i * a_j = a_i$, $(a_i * a_j) * a_o = a_i * a_o = a_i$;

$(a_j * a_i) * a_o = a_i * a_o = a_i$, $(a_j * a_o) * a_i = a_i * a_i = a_i$;

So, $(E_1, *, a_o)$ is a BCH – algebra. But $(E_1, *, a_o)$ is not aBCK – algebra.

$((a_2 * a_1) * (a_2 * a_o)) * (a_o * a_1) = (a_1 * a_2) * a_o = a_1 * a_o = a_1 \neq a_o$.

4. CONCLUSION

Form the above results we have proved some theorems on poset and loset. The theorems have also verified with some specific examples. we have discussedhow a poset can be made into a weak BCK–algebra by adjoining an element and defining a binary operation suitably. Similarly, result has proved for loset also. This idea can also be used as extension of properties of BCK and BCH algebra in future.

5. REFERENCES

1. R. A. Borzooei, J. Shohani "*BCK – algebra of fraction, Sci. Math. Japonica"*72, No 3 (2010) 265 – 276.
2. Y. Imai, K. Iseki. *"On axiom systems of propositional Calculi"* XIV, proc. Japan. Academy, 42(1966),19- 22.
3. K. Iseki "*On bounded BCK – algebras"*, Math Seminar Notes, Kobe University, 5 III, (1975).
4. K. Iseki, S. Tanaka "*An introduction to the theory of BCK – algebras*" Math Japonica,23, No 1 (1978).
5. Yonlin Liu, Jie.Meng "*Quotient BCK – algebra by a fuzzy BCK – filter*" South east Asian Bulletin of Maths.(2002), 26, 825 – 834.
6. Meenakshi Sinha *"A study to the theory of generalized BCK – algebras"* Ph. D. Thesis, Magadh University, Bodh- Gaya (2013).
7. Rashmi Rani, Pooja "*BCK/BCH Algebras with Mutually Disjoint elements*" ICCIASH 2020.