Congruence Of Convex Polygons

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 10 May 2021

Abstract: The study aimed to determine the conditions for the congruence of convex polygons by using direct proof, specifically the two-column proof under Euclidean geometry. Patterns of the existing postulates and theorems on triangles and convex quadrilaterals were discovered. Moreover, another proof of the theorems on the congruence of convex quadrilaterals were formulated by establishing Nth Angle Theorem. Furthermore, since the patterns on the congruence of triangles and convex quadrilaterals are perfectly correlated, conjectures were derived from these arrays. That is, two convex n-gons are congruent if and only if their corresponding: n-2 consecutive sides and n-1 angles are respectively congruent; n-1 sides and n-2 included angles are respectively congruent; and n sides and n-3 angles are respectively congruent. In addition, to verify the conjectures theorems on the congruence of convex pentagons and convex hexagons were proven. Since there is strong evidence that holds for the conjectures of the congruence of convex polygons, it opens portal for other researchers to exactly predict and prove the congruence of other convex n-gons, where n >6.

Keywords: Congruence, Convex Polygons, Convex Quadrilaterals, Convex Pentagons, Convex Hexagons, direct proof, two-column proof

1. Introduction

If there is a one-to-one correspondence among the vertices of two convex polygons such that all pairs of correspondent angles and all pairs of correspondent sides are congruent, and consecutive vertices correspond to consecutive vertices, then the two polygons are said to be congruent (Anatriello, Laudano, & Vincenz, 2018). Nevertheless, it does not need to measure all the corresponding parts to conclude that the two polygons are congruent. At least how many corresponding parts are needed to justify that the two polygons are congruent?

The congruencies of some polygons have been analyzed by some mathematicians beginning thousand years ago. Moise and Downs (1975), and Rich and Thomas (2009), for instance, compiled postulates and theorem on the congruence of triangles, such as ASA, SSS, SAS and SAA. Vance (1982), Lee (2013), Laudano and Vincenzi (2017) accumulated theorems for the quadrilateral congruence, which include: ASASA, AASAS, AAASS, SASAS, and SSSSA.

However, a simple and shorter proof of the theorems on the congruence of convex quadrilaterals is necessary for convenience. In this regard, new theorem could be utilized to minimize the number of steps, as lesser as possible.

In addition, there were undiscovered patterns based on the corresponding parts needed for the congruence of triangles and convex quadrilaterals. Hence, conjectures for the congruence of convex polygons could be drawn from the patterns. To verify the conjectures, theorem on the congruence of convex pentagons and hexagons would be formulated.

2.Methodology

The method used in proving the theorems on the congruence of convex polygons is direct proof, specifically the two-column proof. A new theorem was constructed prior to the congruence of convex polygons for convenience. Such theorem is called Nth Angle Theorem, which is an extension from Third Angle Theorem. In proving the Nth Angle Theorem, a paragraph proof was utilized.

3.Results And Discussions

The postulates and theorems on the congruence of convex polygons such as triangles, and quadrilaterals were categorized as shown below.

Table 1. Patterns of the Given Parts for the Congruence of Convex Polygons such as Triangles and Quadrilaterals

Triangle Congruence	Convex Quadrilateral Congruence
S - 2A (i.e. SAA and ASA)	2S - 3A (i.e. ASASA, AASAS, and AAASS)
2S – A (i.e. SAS) Note: the angle is included by the two sides.	3S - 2A (i.e. SASAS) Note: the two angles are included by the three sides.
3S (i.e. SSS)	4S – A (i.e. SSSSA)

3.1.Another Proof of the Theorems on the Congruence of Convex Quadrilaterals

In providing another proof of the theorems on the congruence of quadrilaterals, a new theorem is established. Such theorem is called Nth Angle Theorem.

3.2.Nth Angle Theorem

If the $n-1$ angles of one n-gon are congruent respectively to the corresponding $n-1$ angles of another n-gon, then the corresponding nth angle of two n-gons, with the same number of sides, are congruent.

Note: This theorem is an extension from Third Angle Theorem. This is applicable to all polygons, not only triangles, with the same number of sides.

Proof:

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_{n-1}, \alpha_n,$ and $\beta_1, \beta_2, \beta_3, \beta_4, \ldots, \beta_{n-1}, \beta_n,$ be the angles of the two polygons respectively. To prove $\alpha_n \cong \beta_n$.

Since the $n-1$ number of corresponding angles of the two polygons are congruent respectively, then

$\alpha_1 \cong \beta_1$, $\alpha_2 \cong \beta_2$, $\alpha_3 \cong \beta_3$, $\alpha_4 \cong \beta_4$, … , $\alpha_{n-1} \cong \beta_{n-1.}$

Since the sum of the measures of the interior angles of a polygon is $(n-2)\,180°$, then

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots + \alpha_{n-1} + \alpha_n \cong (n-2)\,180°$,

and

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \ldots + \beta_{n-1} + \beta_n \cong (n-2)180°.$$

By transitivity,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots + \alpha_{n-1} + \alpha_n \cong \beta_1 + \beta_2 + \beta_3 + \beta_4 + \ldots + \beta_{n-1} + \beta_{n\,.}$$

By subtraction property of equality,

$$\alpha_n \cong \beta_n.$$

QED

3.3.2S-3A Convex Quadrilateral Congruence Theorem

Two convex quadrilaterals are congruent if and only if their corresponding two consecutive sides and three angles are respectively congruent.

Note: This is based on ASASA, AASAS, and AAASS Theorems of Vance (1982) and Lee (2013).

Proof:

(⇒) If two convex quadrilaterals are congruent, then their corresponding two consecutive sides and three angles are respectively congruent, by CPCQC.

(⇐) If the corresponding two consecutive sides and three angles of two convex quadrilaterals are respectively congruent, then the two convex quadrilaterals are congruent as shown in the following.

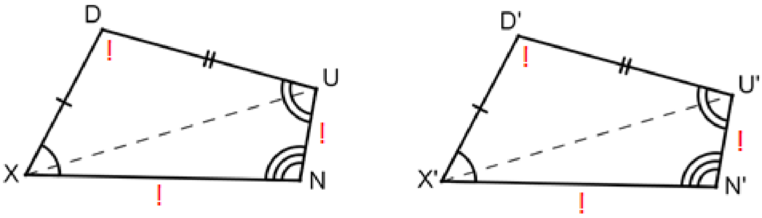

Figure 1. $\square$ DUNX $\cong \square$ D′U′N′X′ by 2S-3A Convex Quadrilateral Congruence Theorem

Given: $\overline{XD} \cong \overline{X'D'}$, $\overline{DU} \cong \overline{D'U'}$, $\angle DUN \cong \angle D'U'N'$, $\angle UNX \cong \angle U'N'X'$, and $\angle NXD \cong \angle N'X'D'$

Prove: $\square$ DUNX $\cong \square$ D′U′N′X′

Plan: Prove $\overline{UN} \cong \overline{U'N'}$, $\overline{XN} \cong \overline{X'N'}$, and $\angle XDU \cong \angle X'D'U'$

Table 2. Proof of 2S-3A Convex Quadrilateral Congruence Theorem

Statements	Reasons
1. $\angle DUN \cong \angle D'U'N'$, $\angle UNX \cong \angle U'N'X'$, and $\angle NXD \cong \angle N'X'D'$	1. Given
2. $\angle XDU \cong \angle X'D'U'$	2. Nth Angle Theorem
3. $\overline{XD} \cong \overline{X'D'}$, and $\overline{DU} \cong \overline{D'U'}$	3. Given
4. XU and X′U′ are line segments.	4. Two distinct points determine a line.
5. $\Delta DUX \cong \Delta D'U'X'$	5. Steps 2, 3, and SAS Congruence Postulate
6. $\overline{XU} \cong \overline{X'U'}$	6. CPCTC
7. $\angle DUX + \angle XUN \cong \angle DUN$, and $\angle D'U'X' + \angle X'U'N' \cong \angle D'U'N'$	7. Angle Addition Postulate
8. $\angle DUX + \angle XUN \cong \angle D'U'X' + \angle X'U'N'$	8. Steps 1, 7, and Substitution
9. $\angle DUX \cong \angle D'U'X'$	9. Step 5, and CPCTC
10. $\angle XUN \cong \angle X'U'N'$	10. Steps 8, 9, and Subtraction Property
11. $\Delta XUN \cong \Delta X'U'N'$	11. Steps 1, 6, 10, and SAA Congruence Theorem
12. $\overline{UN} \cong \overline{U'N'}$, and $\overline{XN} \cong \overline{X'N'}$	12. CPCTC
13. $\square$ DUNX $\cong \square$ D′U′N′X′	13. Steps 1, 2, 3, 12, and Definition of Congruent Polygons

QED

3.4.3S-2A Convex Quadrilateral Congruence Theorem

Two convex quadrilaterals are congruent if and only if their corresponding three sides and two included angles are respectively congruent.

Note: This is based on the SASAS Theorem of Vance (1982) and Lee (2013).

Proof:

(⇒) If two convex quadrilaterals are congruent, then their corresponding three sides and two included angles are respectively congruent, by CPCQC.

(⇐) If the corresponding three sides and two included angles of two convex quadrilaterals are respectively congruent, then the two convex quadrilaterals are congruent as shown in the following.

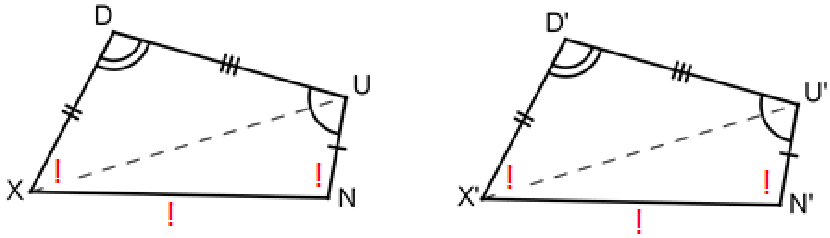

Figure 2. □ DUNX ≅ □ D′U′N′X′ by 3S-2A Convex Quadrilateral Congruence Theorem

Given: $\overline{XD} \cong \overline{X'D'}$, $\overline{DU} \cong \overline{D'U'}$, $\overline{UN} \cong \overline{U'N'}$, $\angle XDU \cong \angle X'D'U'$, and $\angle DUN \cong \angle D'U'N'$

Prove: □ DUNX ≅ □ D′U′N′X′

Plan: Prove $\overline{XN} \cong \overline{X'N'}$, $\angle UNX \cong \angle U'N'X'$, and $\angle NXD \cong \angle N'X'D'$

Table 3. Proof of 3S-2A Convex Quadrilateral Congruence Theorem

Statements	Reasons
1. XU and X′U′ are line segments.	1. Two distinct points determine a line.
2. $\overline{XD} \cong \overline{X'D'}$, and $\overline{DU} \cong \overline{D'U'}$	2. Given
3. $\angle XDU \cong \angle X'D'U'$	3. Given
4. $\Delta XDU \cong \Delta X'D'U'$	4. SAS Congruence Postulate
5. $\overline{XU} \cong \overline{X'U'}$	5. CPCTC
6. $\angle DUN \cong \angle D'U'N'$	6. Given
7. $\angle DUX + \angle XUN \cong \angle DUN$, and $\angle D'U'X' + \angle X'U'N' \cong \angle D'U'N'$	7. Angle Addition Postulate
8. $\angle DUX + \angle XUN \cong \angle D'U'X' + \angle X'U'N$	8. Steps 6, 7, and Substitution
9. $\angle DUX \cong \angle D'U'X'$	9. Step 4, and CPCTC
10. $\angle XUN \cong \angle X'U'N'$	10. Steps 8, 9, and Subtraction Property
11. $\overline{UN} \cong \overline{U'N'}$	11. Given
12. $\Delta XUN \cong \Delta X'U'N'$	12. Steps 5, 10, 11, and SAS Congruence Postulate
13. $\angle UNX \cong \angle U'N'X'$	13. CPCTC
14. $\angle NXD \cong \angle N'X'D'$	14. Steps 3, 6, 13, and Nth Angle Theorem
15. $\overline{XN} \cong \overline{X'N'}$	15. Step 12, and CPCTC
16. □ DUNX ≅ □ D′U′N′X′	16. Steps 2, 3, 6, 11, 13, 14, 15, and Definition of Congruent Polygons

QED

3.5.4S-A Convex Quadrilateral Congruence Theorem

Two convex quadrilaterals are congruent if and only if their corresponding four sides and one angle are respectively congruent.

Note: This is based on SSSSA Theorem of Vance (1982) and Lee (2013).

Proof:

(⇒) If two convex quadrilaterals are congruent, then their corresponding four sides and one angle are respectively congruent, by CPCQC.

(⇐) If the corresponding four sides and one angle of two convex quadrilaterals are respectively congruent, then the two convex quadrilaterals are congruent as shown in the following.

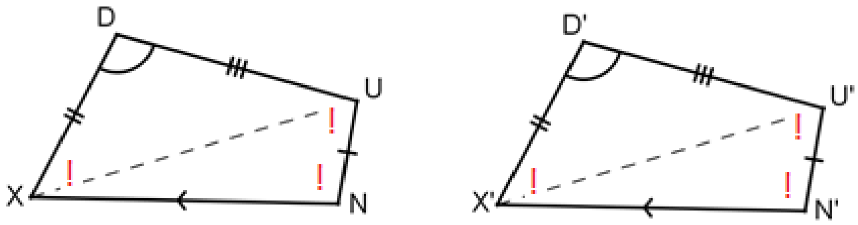

Figure 3. $\square$ DUNX $\cong \square$ D′U′N′X′ by 4S-A Convex Quadrilateral Congruence Theorem

Given: $\overline{DU} \cong \overline{D'U'}$, $\overline{UN} \cong \overline{U'N'}$, $\overline{NX} \cong \overline{N'X'}$, $\overline{XD} \cong \overline{X'D'}$, and $\angle XDU \cong \angle X'D'U'$

Prove: $\square$ DUNX $\cong \square$ D′U′N′X′

Plan: Prove $\angle DUN \cong \angle D'U'N'$, $\angle UNX \cong \angle U'N'X'$, and $\angle NXD \cong \angle N'X'D'$

Table 4. Proof of 4S-A Convex Quadrilateral Congruence Theorem

Statements	Reasons
1. XU and X′U′ are line segments.	1. Two distinct points determine a line.
2. $\overline{DU} \cong \overline{D'U'}$, and $\overline{XD} \cong \overline{X'D'}$	2. Given
3. $\angle XDU \cong \angle X'D'U'$	3. Given
4. $\Delta XDU \cong \Delta X'D'U'$	4. SAS Congruence Postulate
5. $\overline{XU} \cong \overline{X'U'}$	5. CPCTC
6. $\overline{UN} \cong \overline{U'N'}$, and $\overline{NX} \cong \overline{N'X'}$	6. Given
7. $\Delta UNX \cong \Delta U'N'X'$	7. Steps 5, 6, and SSS Congruence Postulate
8. $\angle NXU \cong \angle N'X'U'$	8. CPCTC
9. $\angle UXD \cong \angle U'X'D'$	9. Step 4, and CPCTC
10. $\angle NXU + \angle UXD \cong \angle N'X'U' + \angle U'X'D'$	10. Steps 8, 9, and Addition Property
11. $\angle NXU + \angle UXD \cong \angle NXD$, and $\angle N'X'U' + \angle U'X'D' \cong \angle N'X'D'$	11. Angle Addition Postulate
12. $\angle NXD \cong \angle N'X'D'$	12. Steps 10, 11, and Substitution
13. $\angle UNX \cong \angle U'N'X'$	13. Step 7, and CPCTC
14. $\angle DUN \cong \angle D'U'N'$	14. Steps 3, 12, 13, and Nth Angle Theorem
15. $\square$ DUNX $\cong \square$ D′U′N′X′	15. Steps 2, 3, 6, 12, 13, 14, and Definition of Congruent Polygons

QED

Table 7. Congruency of Convex n-gons

Triangle Congruence	Convex Quadrilateral Congruence	..	**Convex n-gon Congruence Conjecture**
S - 2A	2S - 3A	..	**[(n-2)S] - [(n-1)A]**
2S – A	3S - 2A	..	**[(n-1)S] - [(n-2)A]** Note: the n-2 angles should be included by the n-1 sides.
3S	4S – A	..	**[(n)S] - [(n-3)A]**

Note: the conjectures for the congruence of the convex n-gons above do not mean that these are only true for convex polygons. These may or may not be true to non-convex polygons.

(a). [(n-2)S] - [(n-1)A] Convex Polygon Congruence Conjecture

Two convex n-gons are congruent if and only if their corresponding *(n-2)* consecutive sides and *(n -1)* angles are respectively congruent.

(b). [(n-1)S] - [(n-2)A] Convex Polygon Congruence Conjecture

Two convex n-gons are congruent if and only if their corresponding *(n-1)* sides and *(n-2)* included angles are respectively congruent.

(c). [(n)S] - [(n-3)A] Convex Polygon Congruence Conjecture

Two convex n-gons are congruent if and only if their corresponding *n* sides and *(n-3)* angles are respectively congruent.

3.6.Theorems on the Congruence of Convex Pentagons and Convex Hexagons

3.6.1.Congruence of Convex Pentagons

In symbols, pen ABCDE ≅ pen VWXYZ, if pentagon ABCDE is congruent to pentagon VWXYZ.

Corresponding Parts of Congruent Pentagons are Congruent (CPCPC)

Specifically, if two pentagons are congruent, then their corresponding parts, angles and sides are congruent.

3.7.3S-4A Convex Pentagon Congruence Theorem

Two convex pentagons are congruent if and only if their corresponding three consecutive sides and four angles are respectively congruent.

Proof:

(⇒) If two convex pentagons are congruent, then their corresponding three consecutive sides and four angles are respectively congruent, by CPCPC.

(⇐) If the corresponding three consecutive sides and four angles of two convex pentagons are respectively congruent, then the two convex pentagons are congruent as shown in the following.

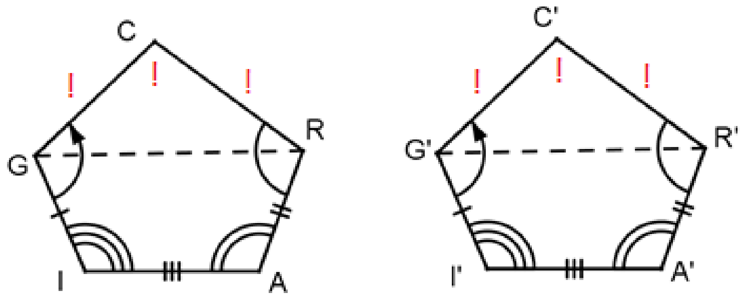

Figure 4. pen CRAIG ≅ pen C′R′A′I′G′ by 3S-4A Convex Pentagon Congruence Theorem

Given: $\overline{RA} \cong \overline{R'A'}$, $\overline{AI} \cong \overline{A'I'}$, $\overline{IG} \cong \overline{I'G'}$, ∠CRA ≅ ∠C′R′A′, ∠RAI ≅ R′A′I′, ∠AIG ≅ ∠A′I′G′, and ∠IGC ≅ ∠I′G′C′

Prove: pen CRAIG ≅ pen C′R′A′I′G′

Plan: Prove $\overline{GC} \cong \overline{G'C'}$, $\overline{CR} \cong \overline{C'R'}$, and ∠GCR ≅ ∠G′C′R′

Table 5. Proof of 3S-4A Convex Pentagon Congruence Theorem

Statements	Reasons
1. ∠CRA ≅ ∠C′R′A′, ∠RAI ≅ ∠R′A′I′, ∠AIG ≅ ∠A′I′G′, and ∠IGC ≅ ∠I′G′C′	1. Given
2. ∠GCR ≅ ∠G′C′R′	2. Nth Angle Theorem
3. GR and G′R′ are line segments.	3. Two distinct points determine a line.
4. $\overline{RA} \cong \overline{R'A'}$, $\overline{AI} \cong \overline{A'I'}$, and $\overline{IG} \cong \overline{I'G'}$	4. Given
5. ◻ RAIG ≅ ◻ R′A′I′G′	5. Steps 1, 4, and 3S-2A Convex Quadrilateral

	Congruence Theorem
6. $\overline{GR} \cong \overline{G'R'}$	6. CPCQC
7. $\angle CRG + \angle GRA \cong \angle CRA$, and $\angle C'R'G' + \angle G'R'A' \cong \angle C'R'A'$	7. Angle Addition Postulate
8. $\angle CRG + \angle GRA \cong \angle C'R'G' + \angle G'R'A'$	8. Steps 1, 7, and Substitution
9. $\angle GRA \cong \angle G'R'A'$	9. Step 5, and CPCQC
10. $\angle CRG \cong \angle C'R'G'$	10. Steps 8, 9, and Subtraction Property
11. $\Delta CRG \cong \Delta C'R'G'$	11. Steps 2, 6, 10, and SAA Congruence Theorem
12. $\overline{GC} \cong \overline{G'C'}$, and $\overline{CR} \cong \overline{C'R'}$	12. CPCTC
13. pen CRAIG $\cong$ pen $C'R'A'I'G'$	13. Steps 1, 2, 4, 12, and Definition of Congruent Polygons

QED

3.8.5S-2A Convex Pentagon Congruence Theorem

Two convex pentagons are congruent if and only if their corresponding five sides and two angles are respectively congruent.

Proof:

(⇒) If two convex pentagons are congruent, then their corresponding five sides and two angles are respectively congruent, by CPCPC.

(⇐) If the corresponding five sides and two angles of two convex pentagons are respectively congruent, then the two convex pentagons are congruent as shown in the following.

Statements	Reasons
1. GR and $G'R'$ are line segments.	1. Two distinct points determine a line.
2. $\overline{CR} \cong \overline{C'R'}$, and $\overline{GC} \cong \overline{G'C'}$	2. Given
3. $\angle GCR \cong \angle G'C'R'$	3. Given
4. $\Delta GCR \cong \Delta G'C'R'$	4. SAS Congruence Postulate
5. $\overline{GR} \cong \overline{G'R'}$	5. CPCTC
6. $\overline{RA} \cong \overline{R'A'}$, $\overline{AI} \cong \overline{A'I'}$, and $\overline{IG} \cong \overline{I'G'}$	6. Given
7. $\angle RAI \cong \angle R'A'I'$	7. Given
8. □ GRAI $\cong$ □ $G'R'A'I'$	8. Steps 5, 6, 7, and 4S-A Convex Quadrilateral Congruence Theorem
9. $\angle AIG \cong \angle A'I'G'$	9. CPCQC
10. $\angle GRA \cong \angle G'R'A'$	10. CPCQC
11. $\angle CRG \cong \angle C'R'G'$	11. Step 4, and CPCTC
12. $\angle CRG + \angle GRA \cong \angle C'R'G' + \angle G'R'A'$	12. Steps 10, 11, and Addition Property
13. $\angle CRG + \angle GRA \cong \angle CRA$, and $\angle C'R'G' + \angle G'R'A' \cong \angle C'R'A'$	13. Angle Addition Postulate
14. $\angle CRA \cong \angle C'R'A'$	14. Steps 12, 13, and Substitution
15. $\angle IGC \cong \angle I'G'C'$	15. Steps 3, 7, 9 14, and Nth Angle Theorem
16. pen CRAIG $\cong$ pen $C'R'A'I'G'$	16. Steps 2, 3, 6, 7, 9, 14, 15, and Definition of

	Congruent Polygons

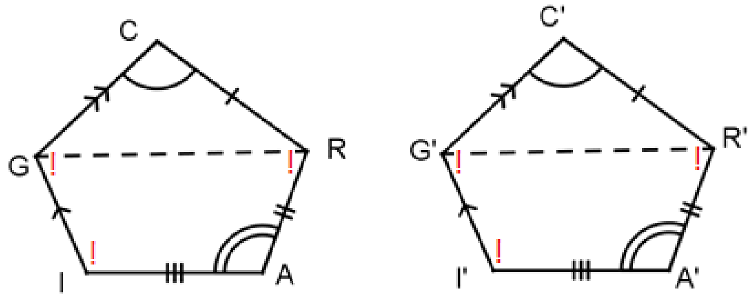

Figure 5. pen CRAIG $\cong$ pen C′R′A′I′G′ by 5S-2A Convex Pentagon Congruence Theorem

Given: $\overline{CR} \cong \overline{C'R'}, \overline{RA} \cong \overline{R'A'}, \overline{AI} \cong \overline{A'I'}, \overline{IG} \cong \overline{I'G'}, \overline{GC} \cong \overline{G'C'}, \angle GCR \cong \angle G'C'R'$, and $\angle RAI \cong \angle R'A'I'$

Prove: pen CRAIG $\cong$ pen C′R′A′I′G′

Plan: Prove $\angle CRA \cong \angle C'R'A', \angle AIG \cong \angle A'I'G'$, and $\angle IGC \cong \angle I'G'C'$

Table 6. Proof of 5S-2A Convex Pentagon Congruence Theorem

QED

From the above conjectures, another theorem could be established for the congruence of convex pentagons, and theorems for congruence of convex hexagons. Such theorems are: 4S-3A Convex Pentagon Congruence Theorem; and 4S-5A, 5S-4A, and 6S-3A Convex Hexagon Congruence Theorems.

3.9.4S-3A Convex Pentagon Congruence Theorem

Two convex pentagons are congruent if and only if their corresponding four sides and three included angles are respectively congruent.

Proof:

(⇒) If two convex pentagons are congruent, then their corresponding four sides and three included angles are respectively congruent, by CPCPC.

(⇐) If the corresponding four sides and three included angles of two convex pentagons are respectively congruent, then the two convex pentagons are congruent as shown in the following.

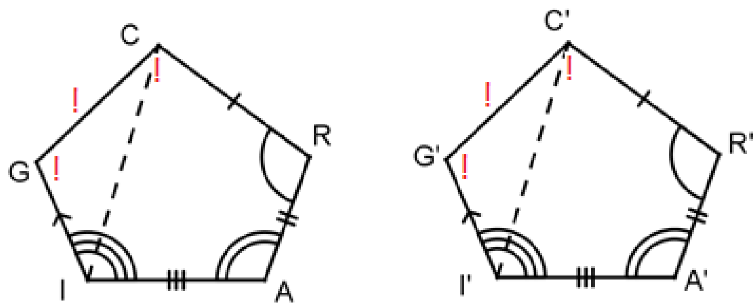

Figure 6. pen CRAIG $\cong$ pen C′R′A′I′G′ by 4S-3A Convex Pentagon Congruence Theorem

Given: $\overline{CR} \cong \overline{C'R'}, \overline{RA} \cong \overline{R'A'}, \overline{AI} \cong \overline{A'I'}, \overline{IG} \cong \overline{I'G'}, \angle CRA \cong \angle C'R'A', \angle RAI \cong R'A'I'$, and $\angle AIG \cong \angle A'I'G'$

Prove: pen CRAIG $\cong$ pen C′R′A′I′G′

Plan: Prove $\overline{GC} \cong \overline{G'C'}, \angle GCR \cong \angle G'C'R'$, and $\angle IGC \cong \angle I'G'C'$

Table 8. Proof of 4S-3A Convex Pentagon Congruence Theorem

Statements	Reasons
1. IC and I′C′ are line segments.	1. Two distinct points determine a line.

2. $\overline{CR} \cong \overline{C'R'}, \overline{RA} \cong \overline{R'A'}$, and $\overline{AI} \cong \overline{A'I'}$	2. Given
3. $\angle CRA \cong \angle C'R'A'$, and $\angle RAI \cong R'A'I'$	3. Given
4. $\square\ CRAI \cong \square\ C'R'A'I'$	4. 3S-2A Convex Quadrilateral Congruence Theorem
5. $\overline{IC} \cong \overline{I'C'}$	5. CPCQC
6. $\angle AIG \cong A'I'G'$	6. Given
7. $\angle AIC + \angle CIG \cong \angle AIG$, and $\angle A'I'C' + \angle C'I'G' \cong \angle A'I'G'$	7. Angle Addition Postulate
8. $\angle AIC + \angle CIG \cong \angle A'I'C' + \angle C'I'G'$	8. Steps 6, 7, and Substitution
9. $\angle AIC \cong \angle A'I'C'$	9. Step 4, and CPCQC
10. $\angle CIG \cong \angle C'I'G'$	10. Steps 8, 9, and Subtraction Property
11. $\overline{IG} \cong \overline{I'G'}$	11. Given
12. $\Delta CIG \cong \Delta C'I'G'$	12. Steps 5, 10, 11, and SAS Congruence Postulate
13. $\angle IGC \cong \angle I'G'C'$	13. CPCTC
14. $\angle GCR \cong \angle G'C'R'$	14. Steps 3, 6, 13, and Nth Angle Theorem
15. $\overline{GC} \cong \overline{G'C'}$	15. Step 12, CPCTC
16. pen CRAIG $\cong$ pen $C'R'A'I'G'$	16. Steps 2, 3, 6, 11, 13, 14, 15, and Definition of Congruent Polygons

QED

3.10.Congruence of Convex Hexagons

In symbols, hex ABCDEF ≅ hex UVWXYZ, if hexagon ABCDEF is congruent to hexagon UVWXYZ.

Corresponding Parts of Congruent Hexagons are Congruent (CPCHC)

Specifically, if two hexagons are congruent, then their corresponding parts, angles and sides, are congruent.

3.11.4S-5A Convex Hexagon Congruence Theorem

Two convex hexagons are congruent if and only if their corresponding four consecutive sides and five angles are respectively congruent.

Proof:

(⇒) If two convex hexagons are congruent, then their corresponding four consecutive sides and five angles are respectively congruent, by CPCHC.

(⇐) If the corresponding four consecutive sides and five angles of two convex hexagons are respectively congruent, then the two convex hexagons are congruent as shown in the following.

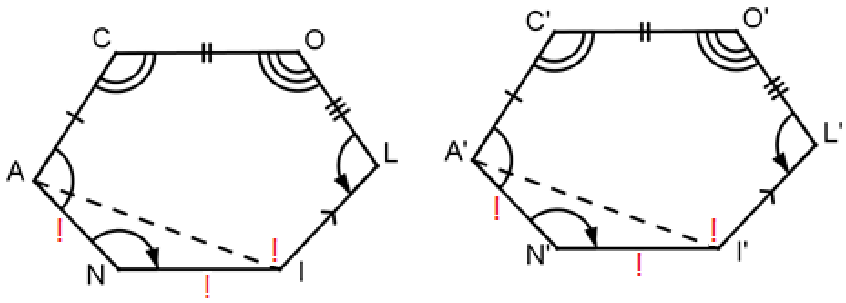

Figure 7. hex $COLINA \cong$ hex $C'O'L'I'N'A'$ by 4S-5A Convex Hexagon Congruence Theorem

Given: $\overline{CO} \cong \overline{C'O'}, \overline{OL} \cong \overline{O'L'}, \overline{LI} \cong \overline{L'I'}, \overline{AC} \cong \overline{A'C'}, \angle ACO \cong \angle A'C'O', \angle COL \cong \angle C'O'L', \angle OLI \cong \angle O'L'I', \angle INA \cong \angle I'N'A'$, and $\angle NAC \cong \angle N'A'C'$

Prove: hex $COLINA \cong$ hex $C'O'L'I'N'A'$

Plan: Prove $\overline{IN} \cong \overline{I'N'}, \overline{NA} \cong \overline{N'A'}$, and $\angle LIN \cong \angle L'I'N'$

Table 9. Proof of 4S-5A Convex Hexagon Congruence Theorem

Statements	Reasons
1. $\angle ACO \cong \angle A'C'O', \angle COL \cong \angle C'O'L', \angle OLI \cong \angle O'L'I', \angle INA \cong \angle I'N'A'$, and $\angle NAC \cong \angle N'A'C'$	1. Given
2. $\angle LIN \cong \angle L'I'N'$	2. Nth Angle Theorem
3. AI and $A'I'$ are line segments.	3. Two distinct points determine a line.
4. $\overline{C'O'} \cong \overline{C'O'}, \overline{OL} \cong \overline{O'L'}, \overline{LI} \cong \overline{L'I'}$, and $\overline{AC} \cong \overline{A'C'}$	4. Given
5. pen $ACOLI \cong$ pen $A'C'O'L'I'$	5. Steps 1, 4, and 4S-3A Convex Pentagon Congruence Theorem
6. $\overline{AI} \cong \overline{A'I'}$	6. CPCPC
7. $\angle NAI + \angle IAC \cong \angle NAC$, and $\angle N'A'I' + \angle I'A'C' \cong \angle N'A'C'$	7. Angle Addition Postulate
8. $\angle NAI + \angle IAC \cong \angle N'A'I' + \angle I'A'C'$	8. Steps 1, 7, and Substitution
9. $\angle IAC \cong \angle I'A'C'$	9. Step 5, and CPCPC
10. $\angle NAI \cong \angle N'A'I'$	10. Steps 8, 9, and Subtraction Property
11. $\Delta INA \cong \Delta I'N'A'$	11. Steps 1, 6, 10, and SAA Congruence Theorem
12. $\overline{IN} \cong \overline{I'N'}$, and $\overline{NA} \cong \overline{N'A'}$	12. CPCTC
13. hex $COLINA \cong$ hex $C'O'L'I'N'A'$	13. Steps 1, 2, 4, 12, and Definition of Congruent Polygons

QED

3.12.5S-4A Convex Hexagon Congruence Theorem

Two convex hexagons are congruent if and only if their corresponding five sides and four included angles are respectively congruent.

Proof:

(⇒) If two convex hexagons are congruent, then their corresponding five sides and four included angles are respectively congruent, by CPCHC.

(⇐) If the corresponding five sides and four included angles of two convex hexagons are respectively congruent, then the two convex hexagons are congruent as shown in the following.

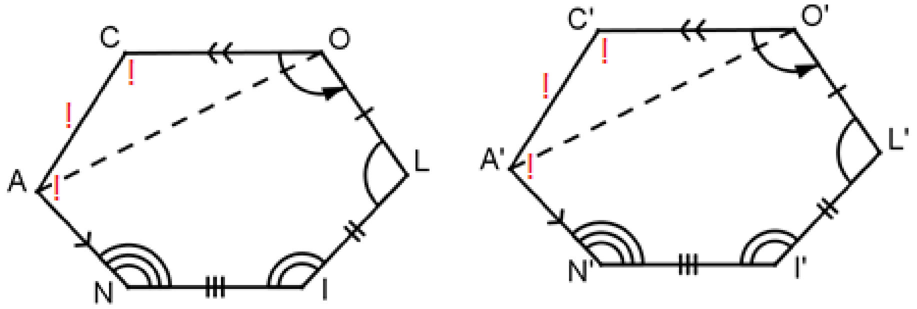

Figure 8. hex $COLINA \cong$ hex $C'O'L'I'N'A'$ by 5S-4A Convex Hexagon Congruence Theorem

Given: $\overline{CO} \cong \overline{C'O'}, \overline{OL} \cong \overline{O'L'}, \overline{LI} \cong \overline{L'I'}, \overline{IN} \cong \overline{I'N'}, \overline{NA} \cong \overline{N'A'}, \angle COL \cong \angle C'O'L', \angle OLI \cong \angle O'L'I', \angle LIN \cong \angle L'I'N'$, and $\angle INA \cong \angle I'N'A'$

Prove: hex $COLINA \cong$ hex $C'O'L'I'N'A'$

Plan: Prove $\overline{AC} \cong \overline{A'C'}, \angle ACO \cong \angle A'C'O'$, and $\angle NAC \cong \angle N'A'C'$

Table 10. Proof of 5S-4A Convex Hexagon Congruence Theorem

Statements	Reasons
1. AO and $A'O'$ are line segments.	1. Two distinct points determine a line.
2. $\overline{OL} \cong \overline{O'L'}, \overline{LI} \cong \overline{L'I'}, \overline{IN} \cong \overline{I'N'}$, and $\overline{NA} \cong \overline{N'A'}$	2. Given
3. $\angle OLI \cong \angle O'L'I'$, $\angle LIN \cong \angle L'I'N'$, and $\angle INA \cong \angle I'N'A'$	3. Given
4. pen $OLINA \cong$ pen $O'L'I'N'A'$	4. 4S-3A Convex Pentagon Congruence Theorem
5. $\overline{AO} \cong \overline{A'O'}$	5. CPCPC
6. $\overline{CO} \cong \overline{C'O'}$	6. Given
7. $\angle COA + \angle AOL \cong \angle COL$, and $\angle C'O'A' + \angle A'O'L' \cong \angle C'O'L'$	7. Angle Addition Postulate
8. $\angle COL \cong \angle C'O'L'$	8. Given
9. $\angle COA + \angle AOL \cong \angle C'O'A' + \angle A'O'L'$	9. Steps 7, 8, and Substitution
10. $\angle AOL \cong \angle A'O'L'$	10. Step 4, and CPCPC
11. $\angle COA \cong \angle C'O'A'$	11. Steps 8, 9, and Subtraction Property
12. $\Delta COA \cong \Delta C'O'A'$	12. Steps 5, 6, 11, and SAS Congruence Postulate
13. $\angle ACO \cong \angle A'C'O'$	13. CPCTC
14. $\angle NAC \cong \angle N'A'C'$	14. Steps 3, 8, 13, and Nth Angle Theorem
15. $\overline{AC} \cong \overline{A'C'}$	15. Step 12, and CPCTC
16. hex $COLINA \cong$ hex $C'O'L'I'N'A'$	16. Steps 2, 3, 6, 8, 13, 14, 15, and Definition of Congruent Polygons

QED

3.13.6S-3A Convex Hexagon Congruence Theorem

Two convex hexagons are congruent if and only if their corresponding six sides and three angles are respectively congruent.

Proof:

(⇒) If two convex hexagons are congruent, then their corresponding six sides and three angles are respectively congruent, by CPCHC.

(⇐) If the corresponding six sides and three angles of two convex hexagons are respectively congruent, then the two convex hexagons are congruent as shown in the following.

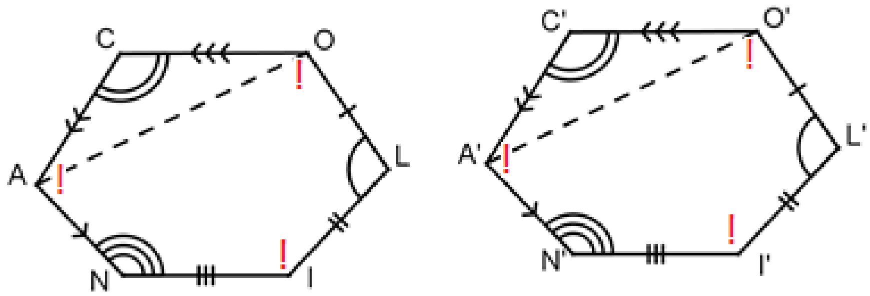

Figure 9. hex $\mathrm{COLINA} \cong$ hex $\mathrm{C'O'L'I'N'A'}$ by 6S-3A Convex Hexagon Congruence Theorem

Given : $\overline{\mathrm{CO}} \cong \overline{\mathrm{C'O'}}, \overline{\mathrm{OL}} \cong \overline{\mathrm{O'L'}}, \overline{\mathrm{LI}} \cong \overline{\mathrm{L'I'}}, \overline{\mathrm{IN}} \cong \overline{\mathrm{I'N'}}, \overline{\mathrm{NA}} \cong \overline{\mathrm{N'A'}}, \overline{\mathrm{AC}} \cong \overline{\mathrm{A'C'}}, \angle\mathrm{ACO} \cong \angle\mathrm{A'C'O'}, \angle\mathrm{OLI} \cong \angle\mathrm{O'L'I'}$, and $\angle\mathrm{INA} \cong \angle\mathrm{I'N'A'}$

Prove: hex $\mathrm{COLINA} \cong$ hex $\mathrm{C'O'L'I'N'A'}$

Plan: Prove $\angle\mathrm{COL} \cong \angle\mathrm{C'O'L'}, \angle\mathrm{LIN} \cong \angle\mathrm{L'I'N'}$, and $\angle\mathrm{NAC} \cong \angle\mathrm{N'A'C'}$

Table 11. Proof of 6S-3A Convex Hexagon Congruence Theorem

Statements	Reasons
1. AO and A′O′ are line segments.	1. Two distinct points determine a line.
2. $\overline{\mathrm{CO}} \cong \overline{\mathrm{C'O'}}$, and $\overline{\mathrm{AC}} \cong \overline{\mathrm{A'C'}}$	2. Given
3. $\angle\mathrm{ACO} \cong \angle\mathrm{A'C'O'}$	3. Given
4. $\Delta\mathrm{ACO} \cong \Delta\mathrm{A'C'O'}$	4. SAS Congruence Postulate
5. $\overline{\mathrm{AO}} \cong \overline{\mathrm{A'O'}}$	5. CPCTC
6. $\overline{\mathrm{OL}} \cong \overline{\mathrm{O'L'}}, \overline{\mathrm{LI}} \cong \overline{\mathrm{L'I'}}, \overline{\mathrm{IN}} \cong \overline{\mathrm{I'N'}}$, and $\overline{\mathrm{NA}} \cong \overline{\mathrm{N'A'}}$	6. Given
7. $\angle\mathrm{OLI} \cong \angle\mathrm{O'L'I'}$, and $\angle\mathrm{INA} \cong \angle\mathrm{I'N'A'}$	7. Given
8. pen $\mathrm{OLINA} \cong$ pen $\mathrm{O'L'I'N'A'}$	8. Steps 5, 6, 7, and 5S-2A Convex Pentagon Congruence Theorem
9. $\angle\mathrm{LIN} \cong \angle\mathrm{L'I'N'}$	9. CPCPC
10. $\angle\mathrm{NAO} \cong \angle\mathrm{N'A'O'}$	10. CPCPC
11. $\angle\mathrm{OAC} \cong \angle\mathrm{O'A'C'}$	11. Step 4, and CPCTC
12. $\angle\mathrm{NAO} + \angle\mathrm{OAC} \cong \angle\mathrm{N'A'O'} + \angle\mathrm{O'A'C'}$	12. Steps 10, 11, and Addition Property
13. $\angle\mathrm{NAO} + \angle\mathrm{OAC} \cong \angle\mathrm{NAC}$, and $\angle\mathrm{N'A'O'} + \angle\mathrm{O'A'C'} \cong \angle\mathrm{N'A'C'}$	13. Angle Addition Postulate
14. $\angle\mathrm{NAC} \cong \angle\mathrm{N'A'C'}$	14. Steps 12, 13, and Substitution
15. $\angle\mathrm{COL} \cong \angle\mathrm{C'O'L'}$	15. Steps 3, 7, 9, 14, and Nth Angle Theorem
16. hex $\mathrm{COLINA} \cong$ hex $\mathrm{C'O'L'I'N'A'}$	16. Steps 2, 3, 6, 7, 9, 14, 15, and Definition of Congruent Polygons

QED

4.Conclusions

The additional theorem for the congruence of convex pentagons and the congruence of convex hexagons, aligned with the conjectures obtained from the congruence of triangles and convex quadrilaterals. Therefore, there is a strong evidence that the two convex n-gons are congruent if and only if their corresponding:

a. *(n-2)* consecutive sides and *(n -1)* angles are respectively congruent;

b. *(n-1)* sides and *(n-2)* included angles are respectively congruent; and

c. *n* sides and *(n-3)* angles are respectively congruent.

5.Recommendations

Since there is strong evidence that holds for the conjectures of the congruence of convex polygons, this gives a gateway for future researchers to prove the congruence of other convex n-gons, where $\mathrm{n} > 6$, using the established conjectures.

References

1. Anatriello, G., Laudano, F., & Vincenzi, G. (2018). Pairs of congruent-like quadrilaterals that are not congruent. In Forum Geometricorum (Vol. 18, pp. 381-400). Retrieved from http://forumgeom.fau.edu/FG2018volume18/FG201844.pdf

2. Laudano, F., & Vincenzi, G. (2017). Congruence theorems for quadrilaterals. J. Geom. Graphics, 21(1), 45-59. Retrieved from https://scholar.google.com/scholar?hl=en&as_sdt=0%2C5&q=Congruence+theorems+for+quadrilaterals&btnG=
3. Lee, J. M. (2013). Axiomatic geometry (Vol. 21). American Mathematical Soc. Retrieved from https://scholar.google.com/scholar?hl=en&as_sdt=0%2C5&q=Lee%2C+J.+M.+%282013%29.+Axiomatic+geometry+%28Vol.+21%29.+American+Mathematical+Soc&btnG=
4. Moise, E. E., & Downs, F. (1975). GEOMETRY. Addison-Wesley Publishing Co., Inc.
5. Rich, B., & Thomas, C. (2009). Schaum's outline of geometry. Retrieved from https://www.academia.edu/34775708/schaums_outline_of_geometry_4th_edition.pdf?auto=download
6. Vance, I. E. (1982). Minimum Conditions for Congruence of Quadrilaterals. School Science and Mathematics, 82(5), 403-415. https://doi.org/10.1111/j.1949-8594.1982.tb11556.x