Research Article

Study Of Hermite-Fejer Type Interpolation Polynomial

Mousa Makey Khrajan[1]

[1]Department of Mathematics and Computer Applications College of Science, University of AL_Muthanna, Iraq

[1]mmkrady@gmail.com mmkrady@mu.edu.iq

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Abstract: Given $\boldsymbol{f} \in \boldsymbol{C}\,[-\mathbf{1},\mathbf{1}]$ and **n** points (node) in $[-\mathbf{1},\mathbf{1}]$**,** the Hermite-Fejer type (HFT) interpolation polynomial is the polynomial of degree at most (2n-1) that agree with $\boldsymbol{f}$ and has zero derivative at each of the nodes. The aim of this paper is to investigate HFT interpolation polynomial of **n** such that **n** is an even number of Chebyshev of the first kind. Mathematics Subject classification: 2010 primary 41A05, Secondary 41A10

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1. Introduction

Suppose that an function $\boldsymbol{f(x)}$ are continuous in $[-\mathbf{1},\mathbf{1}]$ denoted by **C;**
$\boldsymbol{f} \in \boldsymbol{C}\,[-\mathbf{1},\mathbf{1}]$**,** and let

$$X = \{k_{k,n}\}_{k=0}^{n=1}\,, k = 0,1,2,\dots\ , n = 1,2,3,\dots \quad \dots(1)$$

be an infinite triangular matrix of nodes such that, for all **n**

$$-1 \le x_{n-1,n} < \cdots < x_{1,n} < x_{0,n} \le 1 \dots\dots\dots..(2)$$

The well known Lagrange interpolation polynomial of $\boldsymbol{f}$ is the polynomial $L_n(X,f)(x) = L_n(X,f,x)$ of degree at most **(n-1)** which satisfies

$$L_n(X,f,x_{k,n}) = f(x_{k,n});\ \ k = 0,1,\dots,n-1$$

we further denote by $H_n(f,X.x)$, the polynomial of degree **2n-1** that is uniquely determined by the following conditions

$$H_n(f,X,x_{kn}) = f(x_{kn});\ \ H'_n(f,X,x_{kn}) = 0\,,$$

$$k = 0,1,2,\dots,(n-1) \text{ and } x_{k,n} \equiv x_k$$

The process $\{H_n(f,X,x_k)\}_{n=0}^{\infty}$ is called a Hermite-Fejer Type interpolation polynomial **(HFT).**

Faber showed that [1] for any X there exists $f \in C\,[-1,1]$ so that $L_n(X,f,x)$ does not converge uniformly to f on $[-1,1]$ as $\to \infty$.

Let the points $\{x_{kn}\}$ are the roots of the **n-th** Chebyshev nodes of the first kind

$$\mathrm{T} = \{\ x_{kn} = \cos\left(\frac{2k+1}{2n}\right)\pi\ \ ;\ k=0,1,\dots,(n-1)\ ;\ n=1,2,3,\dots\}\dots\dots\dots\dots (3)$$

Where Chebyshev polynomial defined as $T_n(x) = \cos(n\, arc\ \cos x)$, $|x| \le 1$

This result states that if the modulus of continuity $\omega(\delta,f)$ of f is defined by

$\omega(\delta) = \omega(\delta,f) = Sup_{|x-y|\le\delta}\{|f(x)-f(y)|\}$ **,** this value $\omega(\delta)$ is said to be

Modulus of continuity of the function ***f(x)*** , then $L_n(T,f)$ converges uniformly to f with $\omega\left(\frac{1}{n},f\right)\log n \to 0$ as $n \to \infty$.

Ageneralization of Lagrange interpolation is provided by Hermite –Fejer interpolation process. Given a non- negative integer $\boldsymbol{m}$ and nodes **X** defined by **[1,2],** the **HFT** interpolation polynomial $\mathrm{H}_{m,n}(X,f)(x) = \mathrm{H}_{m,n}(X,f,x)$ of f is the unique polynomial of degree at most **(m+1)(n-1)** which satisfies the **(m+1)(n)** *conditions*: $\mathrm{H}_{m,n}(X,f,x_{kn}) = f(x_{kn})\ ;\ 0 \le k \le n-1$

$$\mathrm{H}_{m,n}^{(r)}(X,f,x_{k,n}) = 0\ ;\ \ 1 \le r \le m,\ \ 0 \le k \le n-1$$

J. BYRNE and J.SMITH [8] focus on an aspect of **HFT** that has become known as Berman's phenomenon occurs if the Chebyshev nodes are augmented by the end point of [-1,1], that is for the case of nodes

$$\left.\begin{matrix} x_{k,n+2} \equiv x_k = \cos\left(\frac{2k+1}{2n}\right)\pi\ ,k=1,2,\dots,n \\ x_{0,n+2} \equiv x_0 = 1\ ;\ x_{n+1,n+2} \equiv x_{n+1} = -1 \end{matrix}\right\}\dots\dots\dots.(4)$$

Obtained by adding the nodes ∓ 1 to the node **(3)** .D.L.Berman[1] it is show that process constructed for ***f(x)***=$|x|$ **diverges at** ***x=0*** , while in **[2]** he showed that for $f(x)=x^2$, the process $\mathrm{H}_{1,n}(T_{\mp1},\mathrm{f},0)$diverges every where in $(-1,1)$. An explanation for Berman's phenomenon was provided by Bojanic as follows

Theorem: (Bojanic) [5].If $\mathbf{f} \in \boldsymbol{c}[-\mathbf{1},\mathbf{1}]$has left and right derivatives $\boldsymbol{f'_L}(\mathbf{1})$ and $\boldsymbol{f'_R}(-\mathbf{1})$ at 1 and -1 , respectively, then $\mathbf{H}_{\mathbf{1},\boldsymbol{n}}(\boldsymbol{T}_{\mp\mathbf{1}},\mathbf{f})$ converges uniformly to **f** on **[-1,1] if and only if** $\boldsymbol{f'_L}(\mathbf{1})$= $\boldsymbol{f'_R}(-\mathbf{1})$=**0**.Cook and Mils[6] in 1975, who showed that if $f(x)=(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{\mathbf{3}}$ then $\mathbf{H}_{\mathbf{3},\boldsymbol{n}}(\boldsymbol{T}_{\mp\mathbf{1}},\mathbf{f},\mathbf{0})$ diverges. The result in **[6]** later extended by my paper [7] that showed $\mathbf{H}_{\mathbf{3},\boldsymbol{n}}(\boldsymbol{T}_{\mp\mathbf{1}},\mathbf{f},\mathbf{x})$ diverges at each point in **(-1,1).** Byrne and Smith [8] investigate Berman's phenomenon in the set of **(0,1,2)**

HFI, where the interpolation polynomial agree with $\boldsymbol{f}$ and vanishing first and second derivatives at each node.G.Mastroianni and I.Notarangelo [9] study the uniform and L^P convergence of Hermite and Hermite-Fejer interpolation.

It is obvious that when $\boldsymbol{n}$ is odd, the nodes $x_{k,n} \equiv x_k^n = \cos\left(\frac{2k+1}{2n}\right)\pi$ include the point ***x=0*** . Therefore it will be assume that $\boldsymbol{n}$ is an even number **(*say n=2m*)** and this the aim of paper.

Consider the matrix of nodes

$$x_k^{2m} = \cos\left(\frac{2k-1}{4m}\right)\pi \ ,k=1,2,\dots,2m \ ; x=0 , m=1,2,\dots \qquad \dots\dots\dots\dots.(5)$$

and study Hermite – Fejer Type(**HFT**) interpolation polynomial constructed at these nodes of degree **4m+1** is a uniquely determined by the following conditions:

$$H_{2m}(f,X,0)=f(0);\ H'_{2m}(f,X,0)=0$$
$$H_{2m}(f,,0)=f(0);\ H'_{2m}(f,,0)=0 \ , k=1,2,\dots 2m$$

Therefore $H_{2m}(f,X,x)$ can be written as:

$$H_{2m}(f,X,x)=\sum_{k=1}^{2m} f(x_k)\,\frac{x^2}{x_k^2}\,\frac{T_{2m}^2(x)(1-x_k^2)}{4m^2\,(x-x_k)^2}\left[1-\frac{2-x_k^2}{x_k\,(1-x_k^2)}\,(x-x_k)\right]+f(0)\,T_{2m}^2(x).$$

--- **(6)**

Theorem :The HFT interpolation polynomial $\{H_{2m}(f,X,x)\}$ constructed with the matrix (5) for :

(i) $f(x)=x^2$ is convergent at all points of **(-1,1).**

(ii) $f(x)=x$ is divergent for all points $\mathbf{x}\neq \mathbf{0}$ **in (-1,1).**

2. Technical Preliminaries

We shall quite frequently make use the following results before proof theorem[7]

Lemma: **(i)** $\sum_{k=1}^{n}\frac{1}{(1-x_k^2)} = n^2$ **(ii)** $\sum_{k=1}^{n}\frac{1}{(1+x_k)}=\sum_{k=1}^{n}\frac{1}{(1-x_k)}=n^2$

(iii) $\sum_{k=1}^{n}\frac{1}{x_k^2} = n^2$ **(iv)** $\sum_{k=1}^{n}\frac{1}{(1+x_k)^2}=\sum_{k=1}^{n}\frac{1}{(1-x_k)^2}=\frac{2n^4+n^2}{3}$

(v) $\sum_{k=1}^{n}\frac{1}{(1-x_k^2)^2}=\frac{n^4+2n^2}{3}$ **(vi)** $\sum_{k=1}^{n}\frac{x_k^2}{(1-x_k^2)^2}=\frac{n^4-n^2}{3}$

(vii) $\sum_{k=1}^{n}\frac{1}{x_k^4}=\frac{n^4+2n^2}{3}$.

3. Proof of theorem

For $f(x)=x^2$, the formula (6)becomes

$$H_{2m}(z^2,X,x)\equiv H_{2m}(z^2,x)$$
$$= x^2\sum_{k=1}^{2m} l_k^2\,(x)-x^2\sum_{k=1}^{2m}\frac{(2-x_k^2)}{x_k\,(1-x_k^2)}\,l_k^2(x)(x-x_k) \text{ ------------------ (7)}$$

Where $l_k(x)=\frac{T_n(x)}{T'_n(x_k)(x-x_k)}$ **and** $T_n(x)=T(x)=\prod_{k=1}^{n}(x-x_k)$ be Lagrange interpolation polynomial.

According to Fejer's result , when $|x|\le 1$

$$\sum_{k=1}^{m} l_k^2\,(x)\to 1 \text{ as } m\to\infty \qquad \text{--- (8)}$$

From (7) & (8) it follows that the equation

$\lim\limits_{m\to\infty} H_{2m}(z^2,x)=x^2$ is equivalent to the equation:

$$\lim_{m\to\infty}\sum_{k=1}^{m}\frac{(2-x_k^2)}{x_k\,(1-x_k^2)}\,(x-x_k)\,l_k^2\,(x)=0 \ ,\ |x|\le 1 \qquad \text{----------------------(9)}$$

It can be proved that if $x=\cos\theta$, then

$$\sum_{k=1}^{2m}\frac{l_k^2(x)}{x_k}=\frac{1}{x}\left[1-\frac{\sin 4m\theta\cos\theta}{4m\sin\theta}\right]+\frac{1+x^2}{x^2}\,\frac{T_{2m}(x)T'_{2m}(x)}{4m^2}\text{--------------(10)}$$

From (8)&(10) we can get (9).

To prove (ii), we indicate the proof according to (6) , for ***f (x)=x*** , we have

$$H_{2m}(z,x)=x^2\sum_{k=1}^{2m}\frac{l_k^2(x)}{x_k}-\frac{x^2\,T_{2m}^2(x)}{2m^2}\sum_{k=1}^{2m}\frac{1}{x_k^2\,(x-x_k)}+x^2\,\frac{T_{2m}^2(x)}{4m^2}\sum_{k=1}^{2m}\frac{1}{(x-x_k)}$$

Since $\sum_{j=1}^{2m}\frac{1}{x_j^2}= 4m^2$, we can deduce from this that

$$\mathbf{H}_{2m}(z,x)=x\left[1-\frac{\sin 4m\theta\ \cos\theta}{4m\sin\theta}\right](1+x^2)\frac{\cos 4m\theta}{4m\sin\theta}-2x\,\mathrm{T}_{2m}^2(x)-\frac{\mathrm{T}_{2m}(x)\mathrm{T}_{2m}'(x)}{2m^2}+x^2\left(\frac{\mathrm{T}_{2m}(x)\mathrm{T}_{2m}'(x)}{4m^2}\right)$$
---------------------------------------(11)

By the lemma in [5]&[7] for any $x\in(-1,1)$ there exists a sequence of $\{2m_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\mathrm{T}_{2m_k}^2(x)=1$. Therefore it follows from (11), that

$$\lim_{k\to\infty}\mathrm{H}_{2m_k}(z,x)=-x.$$

Therefore the sequence diverges at every points if $x\neq 0\ \ in\ \ (-1,1)$.

4. Conclusion

To Construct **HFT** interpolation polynomial which converges for $f(x)=x^2$ in **(-1,1)** , while diverges for $f(x)=x\ for\ all\ x\neq 0\ in\ (-1,1)\ \ where\ \ n$ is an even integer number at the node of degree **4m+1**.

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