Research Article

Reserved Domination Number of some Graphs

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Abstract: In this paper the definitions of Reserved domination number and 2-reserved domination number are introduced as for the graph $G=(V,E)$ a subset S of V is called a Reserved Dominating Set (RDS) of G if (i) R be any nonempty proper subset of S ; (ii) Every vertex in $V-S$ is adjacent to a vertex in S . The dominating set S is called a minimal reserved dominating set if no proper subset of S containing R is a dominating set. The set R is called Reserved set. The minimum cardinality of a reserved dominating set of G is called the reserved domination number of G and is denoted by $R_{(k)}-\gamma(G)$ where k is the number of reserved vertices. Using these definitions the 2-reserved domination number for Path graph P_n, Cycle graph C_n, Wheel graph W_n, Star graph S_n, Fan graph $F_{1,n}$, Complete graph K_n and Complete Bipartite graph $K_{m,n}$ are found.

Keywords: Dominating set, Reserved dominating set, 2-Reserved dominating set.

1. Introduction (Times New Roman 10 Bold)

Domination in graphs has wide applications to several fields such as mobile Tower installation, School Bus Routing, Computer Communication Networks, Radio Stations, Locating Radar Stations Problem, Nuclear Power Plants Problem, Modeling Biological Networks, Modeling Social Networks, Facility Location Problems, Coding Theory and Multiple Domination Problems with hierarchical overlay networks. Domination arises in facility location problems, where the number of facilities (e.g., Mobile towers, bus stop, primary health center, hospitals, schools, post office, hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the nearest facility. A similar problem occurs when the maximum distance to a facility is fixed and government or any service provider attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying.

Let $G=(V,E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V-S$ is adjacent to a vertex in S . A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set of G is called the upper domination number of G and is denoted by $\Gamma(G)$.

Consider the situation of installing minimum number of Mobile phone towers so that it will be utilized by all the people living in towns or villages. Here each town or village is considered as separate entity called as vertex. These towns and villages are connected by roads (In this case Ariel distance) called edges. The situation of installing minimum number of Mobile phone Towers is domination problem and this minimum number is domination number.

In real life situation it is not always in practice of installing the Towers only at the public utility pattern. If the installing authorities are very much interested in installing the tower, nearer to their residence or nearer to the residence of VIPs or to their favorites' residence then the concerned person will reserve some of the installation points without any concern about the nearer or proximity of the other towns or villages. Such type installation points are a called reserved installation points. This situation motives the development of the reserved domination points. These reserved points are automatically included in the domination set.

2.Resaserved domination number

Definition: Dominating Set

Let $G=(V,E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V\setminus S$ is adjacent to a vertex in S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set of G is called the upper domination number of G and is denoted by $\Gamma(G)$.

Definition: Reserved Domination set

Let $G=(V,E)$ be a graph. A subset S of V is called a Reserved Dominating Set (RDS) of G if

(i) R be any nonempty proper subset of S.

(ii) Every vertex in $V-S$ is adjacent to a vertex in S.

The dominating set S is called a minimal reserved dominating set if no proper subset of S containing R is a dominating set. The set R is called Reserved set.

The minimum cardinality of a reserved dominating set of G is called the reserved domination number of G and is denoted by $R_{(k)}-\gamma(G)$ where k is the number of reserved vertex.

Remark:

Then $k=2$, we get the 2-reserved domination number of G and is denoted by $R_{(2)}-\gamma(G)$. In this paper let us find the 2-reserved domination number of various graphs.

Definition: Graph Connected by a Bridge

Let G_1, G_2 be any two graphs and G be a graph attained by connecting G_1 and G_2 by a bridge $e=v_1v_2$ where $v_1\in V(G_1)$, $v_2\in V(G_2)$.

3. Some Preliminaries results

Domination Number:

(i) Lollipop Graph: $\gamma(L_{m,n})=\left\lceil\frac{n+2}{3}\right\rceil$.

(ii) Tadpole Graph: $\gamma(T_{m,n})=\left\lfloor\frac{m+2}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor$.

(iii) Pan Graph: $\gamma(P_{n,1})=\left\lceil\frac{n}{3}\right\rceil$.

(iv) Barbell Graph: $\gamma(B_n)=2$.

Definition: Lollipop Graph

The lollipop graph is the graph obtained by joining a complete graph K_m with $m\geq 3$ to a path graph P_n with a bridge. It is denoted by $L_{m,n}$.

Theorem: For the lollipop graph $L_{m,n}$, the reserved domination number is

$$R_{(1)}-\gamma\left(L_{m,n},\mu\right)=\begin{cases}1+\left\lceil\dfrac{n-1}{3}\right\rceil, & \text{if } \mu=u_1\\[2mm] 1+\left\lceil\dfrac{n}{3}\right\rceil, & \text{if } \mu=u_k, k=2,3,4,\ldots,m\\[2mm] 2+\left\lceil\dfrac{k-3}{3}\right\rceil+\left\lceil\dfrac{n-(k+1)}{3}\right\rceil, & \text{if } \mu=v_k, k=1,2,3,\ldots,n\end{cases}$$

Proof:

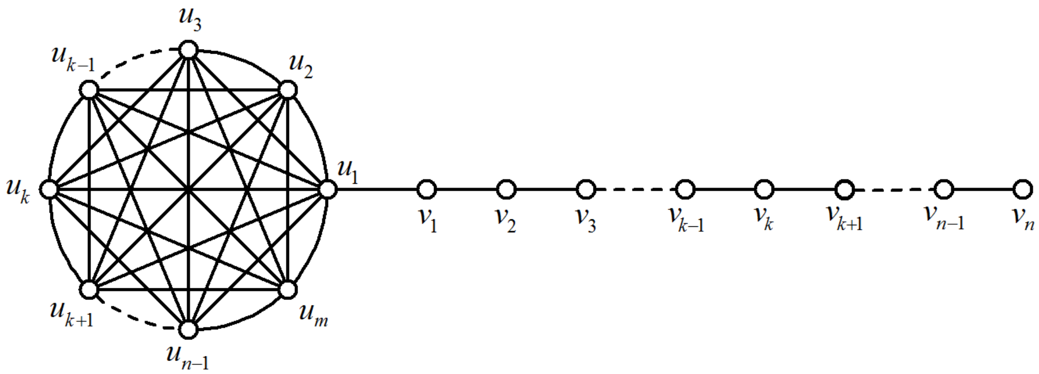

Fig 1: A Lollipop Graph $L_{m,n}$.

For convenience let us consider the lollipop graph $L_{m,n}$ into two entities. One is complete graph K_m with vertices $\{u_1,u_2,u_3,\ldots,u_m\}$ and another one is the path graph P_n with vertices $\{v_1,v_2,v_3,\ldots,v_n\}$.

Case (i): Suppose u_1 is the reserved vertex.

Then u_1 must be in the dominating set and u_1 dominates the vertices $\{u_2,u_3,u_4,\ldots,u_m\}\cup\{v_1\}$. Now it is enough to find the dominating set for the remaining vertices $\{v_2,v_3,\ldots,v_n\}$.

The $L_{m,n}[V_1]$ with $V_1=\{v_2,v_3,\ldots,v_n\}$ is nothing but P_{n-1}.

Hence $R_{(1)}-\gamma\left(L_{m,n},u_1\right)=1+\gamma\left(P_{n-1}\right)$

$$=1+\left\lceil\frac{n-1}{3}\right\rceil.$$

Case (ii): Suppose $u_k, k=2,3,4,\ldots,m$ is the reserved vertex.

u_k Dominates the vertices $u_1,u_2,\ldots,u_{k-1},u_{k+1},\ldots,u_m$ and doesn't dominate the vertices $v_1,v_2,v_3,\ldots,v_n$.

Since the $L_{m,n}[V_2]$ with $V_2=\{v_1,v_2,v_3,\ldots,v_n\}$ is P_n,

$$R_{(1)}-\gamma\left(L_{m,n},\mu\right)=1+\gamma\left(P_n\right)$$

$$=1+\left\lceil\frac{n}{3}\right\rceil,\text{ where } \mu=u_k, k=2,3,\ldots,m.$$

Case (iii): Suppose $v_k, k=1,2,3,\ldots,n$ is the reserved vertex.

To dominate the set $\{u_1,u_2,u_3,\ldots,u_m\}$ it is enough to choose anyone of the vertex from that set. But if we choose u_1 alone then it would dominate the vertex $\{u_2,u_3,\ldots,u_m\}$ as well as $\{v_1\}$. So u_1 must be in the required dominating set. Since v_k is the reserved vertex, it dominates v_{k-1} and v_{k+1}.

Now the vertices which are not dominated while considering the set $\{v_k,u_1\}$ as a subset of the dominating set are $U=\{v_2,v_3,\ldots,v_{k-2},v_{k+2},\ldots,v_n\}$.

$$L_{m,n}[U] = P_{k-3} \cup P_{n-(k+1)}.$$

$$\therefore \gamma\left(L_{m,n}[U]\right) = \gamma\left(P_{k-3}\right) + \gamma\left(P_{n-(k+1)}\right).$$

Hence $R_{(1)} - \gamma\left(L_{m,n}, \mu\right) = \left|\{u_1, v_k\}\right| + \gamma\left(P_{k-3}\right) + \gamma\left(P_{n-(k+1)}\right)$

$$= 2 + \left\lceil \frac{k-3}{3} \right\rceil + \left\lceil \frac{n-(k+1)}{3} \right\rceil, \text{ where } \mu = v_k, k = 1, 2, 3, ..., n.$$

Definition: Tadpole Graph:

The (m,n) Tadpole graph, also called dragon graph denoted by $T_{m,n}$ is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

Theorem: For the Tadpole graph $T_{m,n}$, the reserved domination number is

$$R_{(1)}-\gamma\left(T_{m,n},\mu\right)=\begin{cases}
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=2,3,5,\dots,n-1,n\right)\end{cases}\\
\left\lceil\frac{m-1}{3}\right\rceil+\left\lceil\frac{n+1}{3}\right\rceil, & \text{if } \mu=v_k\left(k=1,4,7,\dots,n-5,n-2\right)
\end{array}\right\} & \text{for } m,n\equiv 0\left(\bmod 3\right)\\
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,4,7,\dots,m-5,m-2\right)\\ v_k\left(k=3,6,9,\dots,n-4,n-1\right)\end{cases}\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=2,3,5,\dots,m-1,m\right)\\ v_k\left(k=1,2,4,\dots,n-2,n\right)\end{cases}
\end{array}\right\} & \text{for } m\equiv 0\left(\bmod 3\right), n\equiv 1\left(\bmod 3\right)\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, \quad \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=1,2,3,\dots,n\right)\end{cases} & \text{for } m\equiv 0\left(\bmod 3\right), n\equiv 2\left(\bmod 3\right)\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, \quad \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=1,2,3,\dots,n\right)\end{cases} & \text{for } m\equiv 1\left(\bmod 3\right), n\equiv 0\left(\bmod 3\right)\\
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=1,3,4,\dots,n-1,n\right)\end{cases}\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, & \text{if } \mu=v_k\left(k=2,5,8,\dots,n-5,n-2\right)
\end{array}\right\} & \text{for } m,n\equiv 1\left(\bmod 3\right)\\
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,2,4,\dots,m-2,m\right)\\ v_k\left(k=2,3,5,\dots,n-2,n\right)\end{cases}\\
\left\lceil\frac{m-1}{3}\right\rceil+\left\lceil\frac{n+1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=3,6,9,\dots,m-4,m-1\right)\\ v_k\left(k=1,4,7,\dots,n-4,n-1\right)\end{cases}
\end{array}\right\} & \text{for } m\equiv 1\left(\bmod 3\right), n\equiv 2\left(\bmod 3\right)\\
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=2,3,5,\dots,n-1,n\right)\end{cases}\\
\left\lceil\frac{m-1}{3}\right\rceil+\left\lceil\frac{n+1}{3}\right\rceil, & \text{if } \mu=v_k\left(k=1,4,7,\dots,n-5,n-2\right)
\end{array}\right\} & \text{for } m\equiv 2\left(\bmod 3\right), n\equiv 0\left(\bmod 3\right)\\
\left.\begin{array}{ll}
\left\lceil \frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=1,3,4,\dots,m-2,m-1\right)\\ v_k\left(k=3,6,9,\dots,n-4,n-1\right)\end{cases}\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, & \text{if } \mu=\begin{cases}u_k\left(k=2,5,8,\dots,m-3,m\right)\\ v_k\left(k=1,2,4,\dots,n-2,n\right)\end{cases}
\end{array}\right\} & \text{for } m\equiv 2\left(\bmod 3\right), n\equiv 1\left(\bmod 3\right)\\
\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, \quad \text{if } \mu=\begin{cases}u_k\left(k=1,2,3,\dots,m\right)\\ v_k\left(k=1,2,3,\dots,n\right)\end{cases} & \text{for } m,n\equiv 2\left(\bmod 3\right)
\end{cases}$$

Proof:

For convenience let us consider the tadpole graph $T_{m,n}$ into two entities. One is cycle graph C_m with vertices $\{u_1,u_2,u_3,\dots,u_m\}$ and another one is the path graph P_n with vertices $\{v_1,v_2,v_3,\dots,v_n\}$.

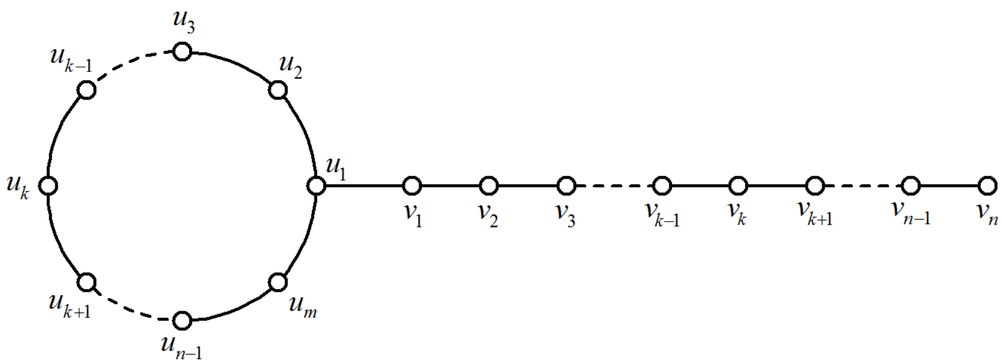

Fig 2: A Tadpole Graph $T_{m,n}$.

Case (i): $m \equiv 0(\bmod 3)$, then $m = 3k$ where $k = 1,2,3,\ldots$.

The possible minimum dominating sets of C_m are

$S_1 = \{u_1,u_4,u_7,\ldots,u_{m-5},u_{m-2}\}$, $S_2 = \{u_2,u_5,u_8,\ldots,u_{m-4},u_{m-1}\}$ and $S_3 = \{u_3,u_6,u_9,\ldots,u_{m-3},u_m\}$.

Sub case (i): $n \equiv 0(\bmod 3)$, then $m = 3k$ where $k = 1,2,3,\ldots$.

a) Let any vertex of the set S_1 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k \in S_1$ then S_1 must be in the required dominating set, also dominated in the vertex v_1. Now it is enough to find the dominating set $\{v_2,v_3,v_4,\ldots,v_n\}$. i.e., P_{n-1}.

Hence $R_{(1)} - \gamma(T_{m,n},\mu) = S_1 \cup \gamma(P_{n-1})$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = S_1.$$

b) Let any vertex of the sets S_2 and S_3 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k \in S_2$ then S_2 must be in the required dominating set. If $u_k \in S_3$ then S_3 must be in the required dominating set. Now it is enough to find the dominating set $\{v_1,v_2,v_3,\ldots,v_n\}$. i.e., P_n.

Hence $R_{(1)} - \gamma(T_{m,n},\mu) = S_2 \cup \gamma(P_n)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = S_2. \left(\because \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{n-1}{3} \right\rceil \right)$$

$R_{(1)} - \gamma(T_{m,n},\mu) = S_3 \cup \gamma(P_n)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = S_3. \left(\because \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{n-1}{3} \right\rceil \right)$$

c) Let any one of the vertex $v_1,v_4,v_7,\ldots,v_{n-5},v_{n-2}$ be a reserved vertex say v_k. Then v_k must be in the required dominating set, also dominated in the vertex u_1. Consider the induced sub graph of the set $\{u_1,v_1,v_2,\ldots,v_n\} = P_{n+1}$. Now it is enough to find the dominating set $\{u_2,u_3,u_4,\ldots,u_m\}$. i.e., P_{m-1}.

Hence $R_{(1)} - \gamma(T_{m,n},\mu) = \gamma(P_{m-1}) + \gamma(P_{n+1})$

$$=\left\lceil\frac{m-1}{3}\right\rceil+\left\lceil\frac{n+1}{3}\right\rceil, \text{ where } \mu=v_1,v_4,v_7,...,v_{n-5},v_{n-2}.$$

d) Let any one of the vertex $v_2,v_5,v_8,...,v_{n-4},v_{n-1}$ be a reserved vertex say v_k. Then v_k must be in the required dominating set. Now it is enough to find the dominating set $\{u_1,u_2,u_3,...,u_m\}$. i.e., C_m.

Hence $R_{(1)}-\gamma(T_{m,n},\mu)=\gamma(C_m)+\gamma(P_n)$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil$$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, \left(\because \left\lceil\frac{n}{3}\right\rceil=\left\lceil\frac{n-1}{3}\right\rceil\right),$$

where $\mu=v_2,v_5,v_8,...,v_{n-4},v_{n-1}$.

e) Let any one of the vertex $v_3,v_6,v_9,...,v_{n-3},v_n$ be a reserved vertex say v_k. Then v_k must be in the required dominating set and also u_1 dominating in the vertex v_1. Now it is enough to find the dominating set $\{u_1,u_2,u_3,...,u_m\}$. i.e., C_m.

Hence $R_{(1)}-\gamma(T_{m,n},\mu)=\gamma(C_m)+\gamma(P_{n-1})$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, \text{ where } \mu=v_3,v_6,v_9,...,v_{n-3},v_n.$$

Combining the results of (a), (b), (c), (d) and (e),

$$\therefore\ R_{(1)}-\gamma(T_{m,n},\mu)=\begin{cases}\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, & \text{where } \mu=\begin{cases}u_k\ (k=1,2,3,...,m)\\ v_k\ (k=2,3,5,...,n-1,n)\end{cases}\\ \left\lceil\frac{m-1}{3}\right\rceil+\left\lceil\frac{n+1}{3}\right\rceil, & \text{where } \mu=v_k\ (k=1,4,7,...,n-5,n-2)\end{cases}.$$

Sub case (ii): $n\equiv 1(\bmod 3)$, then $m=3k+1$ where $k=1,2,3,...$.

a) Let any vertex of the set S_1 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k\in S_1$ then S_1 must be in the required dominating set, also dominated in the vertex v_1. Now it is enough to find the dominating set $\{v_2,v_3,v_4,...,v_n\}$. i.e., P_{n-1}.

Hence $R_{(1)}-\gamma(T_{m,n},\mu)=S_1\cup\gamma(P_{n-1})$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n-1}{3}\right\rceil, \text{ where } \mu=S_1.$$

b) Let any vertex of the sets S_2 and S_3 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k\in S_2$ then S_2 must be in the required dominating set. If $u_k\in S_3$ then S_3 must be in the required dominating set. Now it is enough to find the dominating set $\{v_1,v_2,v_3,...,v_n\}$. i.e., P_n.

Hence $R_{(1)}-\gamma(L_{m,n},\mu)=S_2\cup\gamma(P_n)$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, \text{ where } \mu=S_2.$$

$R_{(1)}-\gamma(T_{m,n},\mu)=S_3\cup\gamma(P_n)$

$$=\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{n}{3}\right\rceil, \text{ where } \mu=S_3.$$

c) Let any one of the vertex $v_1, v_4, v_7, ..., v_{n-3}, v_n$ be a reserved vertex say v_k. Then v_k must be in the required dominating set, also dominated in the vertex u_1. Consider the induced sub graph of the set $\{u_1, v_1, v_2, ..., v_n\} = P_{n+1}$. Now it is enough to find the dominating set $\{u_2, u_3, u_4, ..., u_m\}$. i.e., P_{m-1}.

Hence $R_{(1)} - \gamma(T_{m,n}, \mu) = \gamma(P_{m-1}) + \gamma(P_{n+1})$

$$= \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \left(\because \left\lceil \frac{m-1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil \right)$$

where $\mu = v_1, v_4, v_7, ..., v_{n-3}, v_n$.

d) Let any one of the vertex $v_2, v_5, v_8, ..., v_{n-2}, v_n$ be a reserved vertex say v_k. Then v_k must be in the required dominating set. Now it is enough to find the dominating set $\{u_1, u_2, u_3, ..., u_m\}$. i.e., C_m.

Hence $R_{(1)} - \gamma(T_{m,n}, \mu) = \gamma(C_m) + \gamma(P_n)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{ where } \mu = v_2, v_5, v_8, ..., v_{n-2}, v_n.$$

e) Let any one of the vertex $v_3, v_6, v_9, ..., v_{n-4}, v_{n-1}$ be a reserved vertex say v_k. Then v_k must be in the required dominating set and also u_1 dominating in the vertex v_1. Now it is enough to find the dominating set $\{u_1, u_2, u_3, ..., u_m\}$. i.e., C_m.

Hence $R_{(1)} - \gamma(T_{m,n}, \mu) = \gamma(C_m) + \gamma(P_{n-1})$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = v_3, v_6, v_9, ..., v_{n-4}, v_{n-1}.$$

Combining the results of (a), (b), (c), (d) and (e),

$$\therefore R_{(1)} - \gamma(T_{m,n}, \mu) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = \begin{cases} u_k\,(k = 1, 4, 7, ..., m-5, m-2) \\ v_k\,(k = 3, 6, 9, ..., n-4, n-1) \end{cases} \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \quad \text{where } \mu = \begin{cases} u_k\,(k = 2, 3, 5, ..., m-1, m) \\ v_k\,(k = 1, 2, 4, ..., n-2, n) \end{cases} \end{cases}.$$

Sub case (iii): $n \equiv 2(\text{mod}\, 3)$, then $m = 3k + 2$ where $k = 1, 2, 3, ...$.

a) Let any vertex of the set S_1 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k \in S_1$ then S_1 must be in the required dominating set, also dominated in the vertex v_1. Now it is enough to find the dominating set $\{v_2, v_3, v_4, ..., v_n\}$. i.e., P_{n-1}.

Hence $R_{(1)} - \gamma(T_{m,n}, \mu) = S_1 \cup \gamma(P_{n-1})$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{where } \mu = S_1. \left(\because \left\lceil \frac{n-1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil \right)$$

b) Let any vertex of the sets S_2 and S_3 be a reserved vertex say u_k. Then u_k must be in the required dominating set, if $u_k \in S_2$ then S_2 must be in the required dominating set. If $u_k \in S_3$ then S_3 must be in the required dominating set. Now it is enough to find the dominating set $\{v_1, v_2, v_3, ..., v_n\}$. i.e., P_n.

Hence $R_{(1)} - \gamma\left(L_{m,n}, \mu\right) = S_2 \cup \gamma\left(P_n\right)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{ where } \mu = S_2.$$

$$R_{(1)} - \gamma\left(T_{m,n}, \mu\right) = S_3 \cup \gamma\left(P_n\right)$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{ where } \mu = S_3.$$

c) Let any one of the vertex $v_1, v_4, v_7, ..., v_{n-4}, v_{n-1}$ be a reserved vertex say v_k. Then v_k must be in the required dominating set, also dominated in the vertex u_1. Consider the induced sub graph of the set $\{u_1, v_1, v_2, ..., v_n\} = P_{n+1}$. Now it is enough to find the dominating set $\{u_2, u_3, u_4, ..., u_m\}$. i.e., P_{m-1}.

Hence $R_{(1)} - \gamma\left(T_{m,n}, \mu\right) = \gamma\left(P_{m-1}\right) + \gamma\left(P_{n+1}\right)$

$$= \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \left(\because \left\lceil \frac{m-1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil, \left\lceil \frac{n+1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil \right)$$

where $\mu = v_1, v_4, v_7, ..., v_{n-4}, v_{n-1}$.

d) Let any one of the vertex $v_2, v_5, v_8, ..., v_{n-3}, v_n$ be a reserved vertex say v_k. Then v_k must be in the required dominating set. Now it is enough to find the dominating set $\{u_1, u_2, u_3, ..., u_m\}$. i.e., C_m.

Hence $R_{(1)} - \gamma\left(T_{m,n}, \mu\right) = \gamma\left(C_m\right) + \gamma\left(P_n\right)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{ where } \mu = v_2, v_5, v_8, ..., v_{n-3}, v_n.$$

e) Let any one of the vertex $v_3, v_6, v_9, ..., v_{n-2}, v_n$ be a reserved vertex say v_k. Then v_k must be in the required dominating set and also u_1 dominating in the vertex v_1. Now it is enough to find the dominating set $\{u_1, u_2, u_3, ..., u_m\}$. i.e., C_m.

Hence $R_{(1)} - \gamma\left(T_{m,n}, \mu\right) = \gamma\left(C_m\right) + \gamma\left(P_{n-1}\right)$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil,$$

$$= \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \left(\because \left\lceil \frac{n-1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil \right)$$

where $\mu = v_2, v_5, v_8, ..., v_{n-3}, v_n$.

Combining the results of (a), (b), (c), (d) and (e),

$$\therefore R_{(1)} - \gamma\left(T_{m,n}, \mu\right) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \text{ where } \mu = \begin{cases} u_k\left(k = 1, 2, 3, ..., m\right) \\ v_k\left(k = 1, 2, 3, ..., n\right) \end{cases}.$$

Case (ii): $m \equiv 1(\bmod 3)$, then $m = 3k + 1$ where $k = 1, 2, 3, \ldots$.

i) $n \equiv 0(\bmod 3)$, then $m = 3k$, where $k = 1, 2, 3, \ldots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 2(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil,\ \text{where } \mu = \begin{cases} u_k\,(k=1,2,3,\dots,m) \\ v_k\,(k=1,2,3,\dots,n) \end{cases}$$

ii) $n \equiv 1(\text{mod}\,3)$, then $m = 3k+1$, where $k = 1,2,3,\dots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 1(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = \begin{cases} u_k\,(k=1,2,3,\dots,m) \\ v_k\,(k=1,3,4,\dots,n-1,n) \end{cases} \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \quad \text{where } \mu = v_k\,(k=2,5,6,\dots,n-5,n-2) \end{cases}.$$

iii) $n \equiv 2(\text{mod}\,3)$, then $m = 3k+2$, where $k = 1,2,3,\dots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 0(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \quad \text{where } \mu = \begin{cases} u_k\,(k=1,2,4,\dots,m-2,m) \\ v_k\,(k=2,3,5,\dots,n-2,n) \end{cases} \\ \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil, \text{ where } \mu = \begin{cases} u_k\,(k=2,6,9,\dots,m-4,m-1) \\ v_k\,(k=1,4,7,\dots,n-4,n-1) \end{cases} \end{cases}$$

Case (iii): $m \equiv 2(\text{mod}\,3)$, then $m = 3k+2$ where $k = 1,2,3,\dots$.

i) $n \equiv 0(\text{mod}\,3)$, then $m = 3k$, where $k = 1,2,3,\dots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 0(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \quad \text{where } \mu = \begin{cases} u_k\,(k=1,2,3,\dots,m) \\ v_k\,(k=2,3,5,\dots,n-1,n) \end{cases} \\ \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil, \text{ where } \mu = v_k\,(k=1,4,7,\dots,n-5,n-2) \end{cases}.$$

ii) $n \equiv 1(\text{mod}\,3)$, then $m = 3k+1$, where $k = 1,2,3,\dots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 1(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, \text{ where } \mu = \begin{cases} u_k\,(k=1,3,4,\dots,m-2,m-1) \\ v_k\,(k=3,6,9,\dots,n-4,n-1) \end{cases} \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil, \quad \text{where } \mu = \begin{cases} u_k\,(k=2,5,8,\dots,m-3,m) \\ v_k\,(k=1,2,4,\dots,n-2,n) \end{cases} \end{cases}.$$

iii) $n \equiv 2(\text{mod}\,3)$, then $m = 3k+2$, where $k = 1,2,3,\dots$.

We can also get the same result for $m \equiv 0(\text{mod}\,3);\ n \equiv 2(\text{mod}\,3)$.

$$\therefore\ R_{(1)} - \gamma(T_{m,n},\mu) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil,\ \text{where } \mu = \begin{cases} u_k\,(k=1,2,3,\dots,m) \\ v_k\,(k=1,2,3,\dots,n) \end{cases}.$$

Definition: Pan Graph

The pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. It is denoted by $P_{n,1}$.

Theorem: For the Pan graph $P_{n,1}$, $n \geq 3$ the reserved domination number is

$$R_{(1)} - \gamma\left(P_{n,1}, \mu\right) = \begin{cases} 1 + \left\lceil \dfrac{k-1}{3} \right\rceil + \left\lceil \dfrac{n-(k+2)}{3} \right\rceil, & \text{if } \mu = u_k \left(k = 1, 2, 3, \ldots, n\right) \\ 1 + \left\lceil \dfrac{n-1}{3} \right\rceil, & \text{if } \mu = v_1 \end{cases}$$

Proof:

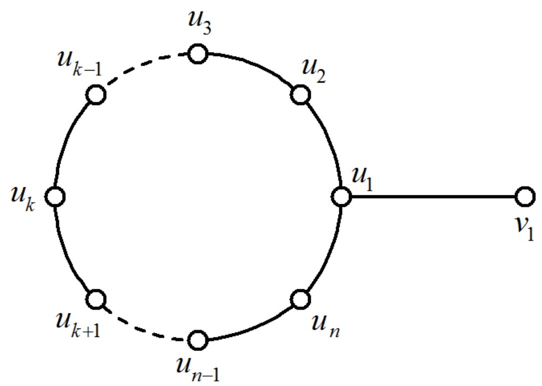

Fig 3: A Pan Graph $P_{n,1}$.

For convenience let us consider the pan graph $P_{n,1}$ into two entities. One is cycle graph C_n with vertices $\{u_1, u_2, u_3, \ldots, u_n\}$ and another one is the singleton graph K_1 with vertex $\{v_1\}$.

Case (i): Suppose $u_k, k = 1, 2, 3, \ldots, n$ is the reserved vertex.

In order to dominate the vertex v_1, the vertex u_1 must chose in the dominating set.

So the minimal reserved dominating set with the vertex $u_k \left(k = 1, 2, 3, \ldots, n\right)$ of the pan graph $P_{n,1}$ is the minimal 2-reserved dominating set with the vertices $\{u_1, v_k\}$ of the cycle graph C_n. Hence

$$\left.\begin{array}{c} R_{(1)} - \gamma\left(P_{n,1}, \mu\right) \\ \left(\text{where } \mu = u_k, k = 1, 2, 3, \ldots, n\right) \end{array}\right\} = R_{(2)} - \gamma\left(C_n, \{u_1, u_k\}\right)$$

$$= 2 + \left\lceil \frac{k-1-3}{3} \right\rceil + \left\lceil \frac{(n+1)-(k+3)}{3} \right\rceil$$

$$= 2 + \left\lceil \frac{k-1}{3} \right\rceil - 1 + \left\lceil \frac{n-k-2}{3} \right\rceil$$

$$= 1 + \left\lceil \frac{k-1}{3} \right\rceil + \left\lceil \frac{n-(k+2)}{3} \right\rceil.$$

Case (ii): Suppose v_1 is the reserved vertex.

Then v_1 must be in the dominating set and v_1 dominates the vertex u_1. Now it is enough to find the dominating set for the remaining vertices $\{u_2, u_3, \ldots, u_n\}$.

The $P_{n,1}[V]$ with $V = \{u_2, u_3, \ldots, u_n\}$ is nothing but P_{n-1}.

Hence $R_{(1)} - \gamma\left(P_{n,1}, v_1\right) = 1 + \gamma\left(P_{n-1}\right)$

$$=1+\left\lceil \frac{n-1}{3} \right\rceil.$$

Definition: Barbell Graph

The barbell graph is the simple graph obtained by joining two copies of complete graph K_n by a bridge (where $n \geq 3$). It is denoted by B_n.

Remark: For the barbell graph B_n, $n \geq 3$ the reserved domination number is $R_{(1)} - \gamma(B_n, \mu) = 2$, if
$$\mu = \begin{cases} u_k\,(k=1,2,3,...,n) \\ v_k\,(k=1,2,3,...,n) \end{cases}.$$

5. 2-reserved domination number:

Definition: **Path graph:**

The path graph P_n is a tree with two nodes of vertex degree 1, and the other $n-2$ nodes of vertex degree 2.

Result: For the path graph P_2, the 2-reserved domination number is $R_{(2)} - \gamma(P_2) = 2$.

Result: For the path graph P_n, $n \geq 3$ the 2-reserved domination number is
$$R_{(2)} - \gamma(P_n, \{v_k, v_l\}) = 2 + \left\lceil \frac{k-2}{3} \right\rceil + \left\lceil \frac{l-k-3}{3} \right\rceil + \left\lceil \frac{n-(l+1)}{3} \right\rceil, \text{ if } k < l \; (k,l=1,2,3,...,n).$$

Definition: **Cycle graph:**

A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3, if the graph is simple) connected in a closed chain. The cycle graph with *n* vertices is called C_n. The number of vertices in C_n equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.

Result: For the cycle graph C_n, $n \geq 3$ the 2-reserved domination number is
$$R_{(2)} - \gamma(C_n, \{v_k, v_l\}) = 2 + \left\lceil \frac{l-k-3}{3} \right\rceil + \left\lceil \frac{(n+k)-(l+3)}{3} \right\rceil, \text{ if } k < l \; (k,l=1,2,3,...,n).$$

Definition: **Wheel Graph**

A wheel graph is a graph W_n formed by connecting a single universal vertex to all vertices of a cycle. To denote a wheel graph with $n+1$ vertices $(n \geq 3)$, which is formed by connecting a single vertex to all vertices of a cycle of length n.

Result: For the wheel graph W_n, $n \geq 5$ the 2-reserved domination number is
$$R_{(2)} - \gamma(W_n, \mu) = \begin{cases} 2 & \text{if } \mu = \{u_1, v_l\} \\ 3 & \text{if } \mu = \{v_k, v_l\}. \end{cases}$$

Definition: **Star Graph**

A star graph S_n is the complete bipartite graph $K_{1,n}$, a tree with one internal node and n leaves (but no internal nodes and $k+1$ leaves when $k \leq 1$).

Result: For the star graph $S_n = K_{1,n}$, $n \geq 3$ the 2-reserved domination number is
$$R_{(2)} - \gamma(S_n, \mu) = \begin{cases} 2 & \text{if } \mu = \{u_1, v_l\} \\ 3 & \text{if } \mu = \{v_k, v_l\}. \end{cases}$$

Definition: **Fan graph**

A fan graph $F_{1,n}$ is defined as the graph join K_1+P_n, where K_1 is the singleton graph on 1 node and P_n is the path graph on n nodes.

Result: For the fan graph $F_{1,n}$, $n\geq 7$ the 2-reserved domination number is,

$$R_{(2)}-\gamma\left(F_{1,n},\mu\right)=\begin{cases}2, \text{ if } \mu=\{u_1,v_l\}\\ 3, \text{ if } \mu=\{v_k,v_l\}\end{cases}.$$

Definition: **Complete bipartite graph**

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both end points in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. That is, it is a bipartite graph (V_1,V_2,E) such that for every two vertices $v_1\in V_1$ and $v_2\in V_2$, v_1v_2 is an edge in E. A complete bipartite graph with partitions of size $|V_1|=m$ and $|V_2|=n$, is denoted $K_{m,n}$.

Result: For the complete bipartite graph $K_{m,n}$, $m,n\geq 3$ the 2-reserved domination number is

$$R_{(2)}-\gamma\left(K_{m,n},\mu\right)=\begin{cases}2, \text{ if } \mu=\{u_k,v_l\}\\ 3, \text{ if } \mu=\{u_k,u_l\} \text{ and } \{v_k,v_l\}\end{cases}$$

Remark: For the complete graph K_n, $n\geq 3$ the 2-reserved domination number is $R_{(2)}-\gamma\left(K_n,\{v_k,v_l\}\right)=2$.

Theorem: For the lollipop graph $L_{m,n}$, the 2-reserved domination number is

$$R_{(2)}-\gamma\left(L_{m,n},\mu\right)=\begin{cases}\begin{cases}1+\left\lceil\frac{l-3}{3}\right\rceil+\left\lceil\frac{n-(l+1)}{3}\right\rceil, & \text{if } \mu=\{u_1,v_l\},(l=1,2,3,\dots,n)\\ 1+\left\lceil\frac{l-2}{3}\right\rceil+\left\lceil\frac{n-(l+1)}{3}\right\rceil, & \text{if } \mu=\{u_k,v_l\},(k=2,3,4,\dots,m;\ l=1,2,3,\dots,n)\end{cases}\\ \begin{cases}2+\left\lceil\frac{n-1}{3}\right\rceil, & \text{if } \mu=\{u_1,u_l\},(l=2,3,4,\dots,m)\\ 2+\left\lceil\frac{n}{3}\right\rceil, & \text{if } \mu=\{u_k,u_l\},k<l,(k,l=2,3,4,\dots,m)\end{cases}\\ 3+\left\lceil\frac{k-3}{3}\right\rceil+\left\lceil\frac{l-k-3}{3}\right\rceil+\left\lceil\frac{n-(l+1)}{3}\right\rceil, \quad \text{if } \mu=\{v_k,v_l\},k<l,(k,l=1,2,3,\dots,n)\end{cases}$$

Proof:

Case (i): $\mu=\{u_1,v_l\}$.

Let u_1 and $v_l\left(l=1,2,3,\dots,n\right)$ be the 2-reserved vertices. As u_1 and v_l must be chosen in the dominating set and u_1 dominates the vertices $\{u_2,u_3,u_4,\dots,u_m\}\cup\{v_1\}$. And v_l dominates the vertices v_{l-1} and v_{l+1}.

So now it is enough to find the domination number of the paths $P_{(1)}$ and $P_{(2)}$,

where $P_{(1)}=L_{m,n}[V_1]$ with $V_1=\{v_2,v_3,\dots,v_{l-2}\}$

$P_{(2)}=L_{m,n}[V_2]$ with $V_2=\{v_{l+2},v_{l+3},\dots,v_n\}$.

The length of the path $P_{(1)} = l-3$.

$$\gamma\left(P_{(1)}\right) = \left\lceil \frac{l-3}{3} \right\rceil.$$

The length of the path $P_{(2)} = n-(l+1)$.

$$\gamma\left(P_{(2)}\right) = \left\lceil \frac{n-(l+1)}{3} \right\rceil.$$

So now,

$$R_{(2)} - \gamma\left(L_{m,n}, \{u_1, v_l\}\right) = \left|\{u_1\}\right| + \gamma\left(P_{(1)}\right) + \left|\{v_l\}\right| + \gamma\left(P_{(2)}\right)$$

$$= 1 + \left\lceil \frac{l-3}{3} \right\rceil + 1 + \left\lceil \frac{n-(l+1)}{3} \right\rceil$$

$$= 2 + \left\lceil \frac{l-3}{3} \right\rceil + \left\lceil \frac{n-(l+1)}{3} \right\rceil.$$

Case (ii): $\mu = \{u_k, v_l\}$.

Let $u_k\ (k = 2,3,4,...,m)$ and $v_l\ (l = 1,2,3,...,n)$ be the 2-reserved vertices. As u_k and v_l must be chosen in the dominating set and u_k dominates the vertices $\{u_1, u_2, u_3, ..., u_{k-1}, u_{k+1}, ..., u_m\}$. And v_l dominates the vertices v_{l-1} and v_{l+1}.

So now it is enough to find the domination number of the paths $P_{(3)}$ and $P_{(4)}$,

where $P_{(3)} = L_{m,n}[V_3]$ with $V_3 = \{v_1, v_2, ..., v_{l-2}\}$

$P_{(4)} = L_{m,n}[V_2]$ with $V_2 = \{v_{l+2}, v_{l+3}, ..., v_n\}$.

The length of the path $P_{(3)} = l-2$.

$$\gamma\left(P_{(3)}\right) = \left\lceil \frac{l-2}{3} \right\rceil.$$

The length of the path $P_{(4)} = n-(l+1)$.

$$\gamma\left(P_{(4)}\right) = \left\lceil \frac{n-(l+1)}{3} \right\rceil.$$

So now,

$$R_{(2)} - \gamma\left(L_{m,n}, \{u_k, v_l\}\right) = \left|\{u_k\}\right| + \gamma\left(P_{(3)}\right) + \left|\{v_l\}\right| + \gamma\left(P_{(4)}\right)$$

$$= 1 + \left\lceil \frac{l-2}{3} \right\rceil + 1 + \left\lceil \frac{n-(l+1)}{3} \right\rceil$$

$$= 2 + \left\lceil \frac{l-2}{3} \right\rceil + \left\lceil \frac{n-(l+1)}{3} \right\rceil.$$

Case (iii): $\mu = \{u_1, u_l\}$.

Let u_1 and $u_l\ (l = 2,3,4,...,m)$ be the 2-reserved vertices. As u_1 and u_l must be chosen in the dominating set and u_1 dominates the vertices $\{u_2, u_3, u_4, ..., u_m\} \cup \{v_1\}$. Now it is enough to find the dominating set for the remaining vertices $\{v_2, v_3, ..., v_n\}$.

The $L_{m,n}[V_4]$ with $V_4 = \{v_2, v_3, ..., v_n\}$ is nothing but P_{n-1}.

Hence $R_{(2)} - \gamma(L_{m,n}, \{u_1, u_l\}) = 2 + \gamma(P_{n-1})$

$$= 2 + \left\lceil \frac{n-1}{3} \right\rceil.$$

Case (iv): $\mu = \{u_k, u_l\}$.

Let $k < l$ and $u_k\ (k = 2,3,4,...,m)$ and $u_l\ (l = 2,3,4,...,m)$ be the 2-reserved vertices. As u_k and u_l must be chosen in the dominating set and u_k dominates the vertices $u_1, u_2, ..., u_{k-1}, u_{k+1}, ..., u_{l-1}, u_l, u_{l+1}, ..., u_m$. Now it is enough to find the dominating set for the remaining vertices $\{v_1, v_2, v_3, ..., v_n\}$.

The $L_{m,n}[V_5]$ with $V_5 = \{v_1, v_2, v_3, ..., v_n\}$ is nothing but P_n.

Hence $R_{(2)} - \gamma(L_{m,n}, \{u_k, u_l\}) = 2 + \gamma(P_n)$

$$= 2 + \left\lceil \frac{n}{3} \right\rceil.$$

Case (v): $\mu = \{v_k, v_l\}$.

Let $k < l$ and $v_k\ (k = 1,2,3,...,n)$ and $v_l\ (l = 1,2,3,...,n)$ be the 2-reserved vertices.

To dominate the set $\{u_1, u_2, u_3, ..., u_m\}$ it is enough to choose anyone of the vertex from that set. But if we choose u_1 alone then it would dominate the vertex $\{u_2, u_3, ..., u_m\}$ as well as $\{v_1\}$. So u_1 must be in the required dominating set. Since v_k, v_l be the 2-reserved vertices, it dominates $v_{k-1}, v_{k+1}, v_{l-1}$ and v_{l+1}.

Now the vertices which are not dominated while considering the set $\{u_1, v_k, v_l\}$ as a subset of the dominating set are $U = \{v_2, v_3, ..., v_{k-2}, v_{k+2}, v_{k+3}, ..., v_{l-2}, v_{l+2}, v_{l+2}, ..., v_n\}$.

$$L_{m,n}[U] = P_{k-3} \cup P_{l-k-3} \cup P_{n-(l+1)}.$$

$$\therefore \gamma(L_{m,n}[U]) = \gamma(P_{k-3}) + \gamma(P_{l-k-3}) + \gamma(P_{n-(l+1)}).$$

Hence $R_{(2)} - \gamma(L_{m,n}, \{v_k, v_l\}) = |\{u_1, v_k, v_l\}| + \gamma(P_{k-3}) + \gamma(P_{l-k-3}) + \gamma(P_{n-(l+1)})$

$$= 3 + \left\lceil \frac{k-3}{3} \right\rceil + \left\lceil \frac{l-k-3}{3} \right\rceil + \left\lceil \frac{n-(l+1)}{3} \right\rceil.$$

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