Some Properties On μ-Metrics Induced By An Intuitionistic Fuzzy Metric Spaces

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Abstract: The idea of Intuitionistic Fuzzy Metric Space introduced by Park (2004). In this paper, a new concept of upper μ-metrics and lower μ-metrics by which the distance between two points are calculated upto the degree of correctness parameter μ of crisp metrics induced by a Intuitionistic Fuzzy Metric Space is introduced also discuss some properties of μ-metrics on Intuitionistic fuzzy metric spaces.

Key words: μ-metric, Intuitionistic fuzzy metric space, Topology.2010 Mathematics Subject Classification: 05C72, 54E50, 03F55

1.Introduction

George and Veeramani[4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek[6] with a view to obtain a Hausdroff topology on fuzzy metric spaces which have very important applications in quantum particle particularly in connection with both string and E-infinity theory. In 2004, Park[7] defined the concept of Intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms.

Several researchers have shown interest in the Intuitionistic fuzzy set theory and successfully applied in many fields, it can be found in [5,9,10,12,13,14,15]. Fuzzy application in almost every direction of mathematics such as airthmetic, topology, graph theory, probability theory, logic etc.

In this paper , the concept of μ-metrics induced by an Intuitionistic fuzzy metric spaces are introduced and also discuss some properties of μ-metrics on Intuitionistic fuzzy metric spaces.

2.Preliminaries

Definition 2.1:[16]

Let X be a nonempty set. A ***fuzzy set*** A in X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) | x \in X\}$.

Definition 2.2:[4]

The 3-tuple $(A, M, *)$ is said to be a ***fuzzy metric space*** if A be a non empty set and $*$ be a continuous t-norm. A fuzzy set A^2 x $(0, \infty)$ is called a fuzzy metric on A if $a, b, c \in A$ and $s, t > 0$, the following condition holds

1. $M(a, b, t) = 0$
2. $M(a, b, t) = 1$ if and only if $a = b$
3. $M(a, b, t) = M(b, a, t)$
4. $M(a, b, t + s) \geq M(a, b, t) * M(a, b, s)$
5. $M(a, b, \bullet) : (0, +\infty) \to [0, 1]$ is left continuous

The function $M(a, b, t)$ denote the degree of nearness between a and b with respect to t respectively.

Definition 2.3:[1][2]

Let a set E be fixed. An ***IFS*** A in E is an object of the following $A = \{(x, \mu_A(x), \upsilon_A(x)), x \in E\}$ Where the functions $\mu_A(x) : E \to [0, 1]$ and $\upsilon_A(x) : E \to [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$: $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$,When $\upsilon_A(x) = 1 - \mu_A(x)$ for all $x \in E$ is an ordinary fuzzy set. In addition, for each IFS A in E, if $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$. Then $\mu_A(x)$ is called the degree of indeterminacy of X to A or called the degree of hesitancy of X to A. It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in E$.

Definition 2.4:[7]

A 5-tuple $(A, M, N, *, \circ)$ is said to be an ***Intuitionistic fuzzy metric space*** if A is an arbitrary set, $*$ is a continuous t-norm, $\circ$ is a continuous t- conorm and, M, N are fuzzy sets on $A^2 \times [0, \infty)$ satisfying the conditions:

1. $M(a, b, t) + N(a, b, t) \leq 1$, for all $a, b \in A$ and ; $t > 0$

2. $M(a,b,0) = 0$, for all $a,b \in A$
3. $M(a,b,t) = 1$, for all $a,b \in A$ and $t > 0$ if and only if $a = b$
4. $M(a,b,t) = M(b,a,t)$, for all $a,b \in A$ and $t > 0$
5. $M(a,b,t) * M(b,c,s) \leq M(a,c,t+s)$, for all $a,b,c \in A$ and $s,t > 0$
6. $M(a,b,\bullet): [0,\infty) \to [0,\infty]$ is left continuous, for all $a,b \in A$
7. $\lim_{t\to\infty} M(a,b,t) = 1$, for all $a,b \in A$ and $t > 0$
8. $N(a,b,0) = 1$, for all $a,b \in A$
9. $N(a,b,t) = 0$, for all $a,b \in A$ and $t > 0$ if and only if $a = b$
10. $N(a,b,t) = N(b,a,t)$, for all $a,b \in A$ and $t > 0$
11. $N(a,b,t) \circ N(b,c,s) \geq N(a,c,t+s)$, for all $a,b,c \in A$ and $s,t > 0$
12. $N(a,b,\bullet): [0,\infty) \to [0,1]$ is right continuous, for all $a,b \in A$
13. $\lim_{t\to\infty} N(a,b,t) = 0$, for all $a,b \in A$.

The functions $M(a,b,t)$ and $N(a,b,t)$ denote the degree of nearness and the degree of non-nearness between a and b w.r.t. t respectively.

3.PROPERTIES OF μ-METRICS INDUCED BY AN INTUITIONISTIC FUZZY METRIC SPACE

Definition 3.1:

Let $(A,M,N,*,\circ)$ be a Intuitionistic fuzzy metric space and $\mu \in (0,1)$, Let $\Omega_{M,N;\mu}(a,b)$ and $\omega_{M,N;\mu}(a,b)$ be defined by $\Omega_{M,N;\mu}(a,b) = \inf\{t \in R : M(a,b,t) > \mu, N(a,b,t) < \mu\}$,

$\omega_{M,N;\mu}(a,b) = \sup\{t \in R : M(a,b,t) < \mu, N(a,b,t) > \mu\}$

Theorem 3.2:

Let $(A,M,N,*,\circ)$ be a Intuitionistic fuzzy metric space. For each $\mu \in (0,1)$, $\Omega_{M,N;\mu}$ and $\omega_{M,N;\mu}$ are metric on A.

Proof:

For each pair a,b, $M(a,b,t)$ & $N(a,b,t)$ is an increasing function and decreasing function of t respectively, we observe that $\{t \in R : M(a,b,t) > \mu, N(a,b,t) < \mu\}$ is an interval with left end $\Omega_{M,N;\mu}(a,b)$ and right end $+\infty$.

Clearly, $\Omega_{M,N;\mu}(a,b) \geq 0$ and $\Omega_{M,N;\mu}(a,b) = \Omega_{M,N;\mu}(b,a)$ for all $a,b \in A$. If $a = b$ then $M(a,b,t) = 1, N(a,b,t) = 0$ for all $t > 0$, which implies $\{t: M(a,b,t) > \mu, N(a,b,t) < \mu\} = (0,\infty)$

and hence $\Omega_{M,N;\mu}(a,b) = 0$.

Conversely, suppose that $\Omega_{M,N;\mu}(a,b) = 0$ and $a \neq b$. since $M(a,b,t)$ & $N(a,b,t)$ is right & left continuous at 0 respectively and $M(a,b,0) = 0, N(a,b,0) = 1$, there exists $t_0 > 0$ such that

$\{M(a,b,t) < \mu, N(a,b,t) > \mu\}$,

this implies that $t_0 \notin \{t: M(a,b,t) > \mu, N(a,b,t) < \mu\}$ and hence $\Omega_{M,N;\mu}(a,b) \geq t_0 > 0$, which is a contradiction.Thus we have proved that $\Omega_{M,N;\mu}(a,b) = 0$ if and only if $a = b$.

Let $a,b,c \in A$. If any two of a,b and c are equal, then it follows that $\Omega_{M,N;\mu}(a,c) \leq \Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c)$.

So we assume that a,b and c are pairwise distinct. We have,

$$M\left(a,b,\Omega_{M,N;\mu}(a,b) + \varepsilon/2\right) > \mu$$
$$N\left(a,b,\Omega_{M,N;\mu}(a,b) + \varepsilon/2\right) < \mu$$
$$M\left(b,c,\Omega_{M,N;\mu}(b,c) + \varepsilon/2\right) > \mu$$
$$N\left(b,c,\Omega_{M,N;\mu}(b,c) + \varepsilon/2\right) < \mu$$

And hence $M\left(a,c,\Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c) + \varepsilon\right) > \mu$,

$$N\left(a,c,\Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c) + \varepsilon\right) < \mu$$

This implies that

$$\left(\Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c) + \varepsilon\right) \in \{t: M(a,c,t) > \mu, N(a,c,t) < \mu\}$$

Which shows that $\Omega_{M,N;\mu}(a,c) \leq \Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c) + \varepsilon$ t.

This is true for all $\varepsilon > 0$, we have $\Omega_{M,N;\mu}(a,c) \leq \Omega_{M,N;\mu}(a,b) + \Omega_{M,N;\mu}(b,c)$ and $\Omega_{M,N;\mu}$ is a metric.

Similarly, we can show that $\omega_{M,N;\mu}$ is a metric.

Theorem 3.3:

Let M,N be a Intuitionistic fuzzy metric on A. Then for any $a,b \in A$, $\Omega_{M,N;\mu}(a,b) = \omega_{M,N;\mu}(a,b)$ f and only if the set $\{t : M(a,b,t) = \mu, N(a,b,t) = \mu\}$ contains atmost one element.

Proof:

Let $G = \{t : M(a,b,t) = \mu, N(a,b,t) = \mu\}$,

$$G_l = \{t : M(a,b,t) < \mu, N(a,b,t) > \mu\}$$

$$G_u = \{ t : M(a,b,t) > \mu, N(a,b,t) < \mu \}$$

Suppose that G contains two elements t_1 and t_2 with $t_1 < t_2$, then $t_1 \notin G_l$, $t_2 \notin G_u$ and hence $\omega_{M,N;\mu}(a,b) \le t_1 < t_2 \le \Omega_{M,N;\mu}(a,b)$, which implies that $\omega_{M,N;\mu}(a,b) \neq \Omega_{M,N;\mu}(a,b)$.

Conversely, if $\omega_{M,N;\mu}(a,b) < \Omega_{M,N;\mu}(a,b)$, then we choose a real number g such that $\omega_{M,N;\mu}(a,b) < g < \Omega_{M,N;\mu}(a,b)$.

Therefore it follows that $g \notin G_l \subseteq (-\infty, \omega_{M,N;\mu}(a,b)]$ and $g \notin G_u = [\Omega_{M,N;\mu}(a,b), +\infty)$ and hence $g \in G$. Thus G contains uncountable elements.

Lemma 3.4:

Let $(A, M, N, *, \circ)$ be a Intuitionistic fuzzy metric space. If $0 < \mu_1 < \mu_2 < 1$, then $\Omega_{M,N;\mu_1} \le \Omega_{M,N;\mu_2}$ and $\omega_{M,N;\mu_1} \le \omega_{M,N;\mu_2}$.

Proof:

Since $\mu_1 < \mu_2$, we have $\{t: M(a,b,t) < \mu_1, N(a,b,t) > \mu_1\} \subseteq \{t: M(a,b,t) < \mu_2, N(a,b,t) > \mu_2\}$ and $\{t: M(a,b,t) > \mu_1, N(a,b,t) < \mu_1\} \supseteq \{t: M(a,b,t) > \mu_2, N(a,b,t) < \mu_2\}$. So, it follows that $\Omega_{M,N;\mu_1}(a,b) \le \Omega_{M,N;\mu_2}(a,b)$ and $\omega_{M,N;\mu_1}(a,b) \le \omega_{M,N;\mu_2}(a,b)$.

Example 3.5:

For $a, b \in \mathcal{R}$, if $M(a,b,t) = \begin{cases} \frac{t}{t+|a-b|}, & t > 0 \\ 0, & t \le 0 \end{cases}$ and

$$N(a,b,t) = \begin{cases} \frac{|a-b|}{t+|a-b|}, & t > 0 \\ 1, & t = 0 \end{cases}$$

Then $M \& N$ are Intuitionistic fuzzy metric on $\mathcal{R}$. Let us take $a = 2$ and $b = 1$. Then $M(a,b,t) = \frac{t}{t+1}$, $N(a,b,t) = \frac{1}{t+1}$, $\Omega_{M,N;\mu}(a,b) = \frac{\mu}{1-\mu}$. $\Omega_{M,N;\mu}(a,b) \to \infty$ as $\mu \to 1$.

Remark 3.6:

To characterize the Intuitionistic fuzzy metrics, for which the limits $\lim_{\mu\to1} \Omega_{M,N;\mu}$ *and* $\lim_{\mu\to1} \omega_{M,N;\mu}$ exist. We introduce a particular class of Intuitionistic fuzzy metric spaces, which satisfy the finite distance condition (FD) for every pair (a,b), there exists $t_{a,b}$ such that $M(a,b,t_{a,b}) = 1$, $N(a,b,t_{a,b}) = 0$.

Definition 3.7:

An Intuitionistic fuzzy metric space $(A, M, N, *, \circ)$ is said to be an ***FD- Intuitionistic fuzzy metric space*** if it satisfies the condition, for every pair (a,b), there exist $t_{a,b}$ such that $M(a,b,t_{a,b}) = 1, N(a,b,t_{a,b}) = 0$.

Definition 3.8:

Let $(A, M, N, *, \circ)$ be a FD-Intuitionistic fuzzy metric space. We define the actual metric induced by the Intuitionistic fuzzy metric M, N by $d_{M,N}(a,b) = \lim_{\mu\to1} \Omega_{M,N;\mu}(a,b)$ provided the limit exists for all $a, b \in A$.

Lemma 3.9:

Let $(A, M, N, *, \circ)$ be a Intuitionistic fuzzy metric space. If $0 < \mu_1 < \mu_2 < 1$, then $\Omega_{M,N;\mu_1}(a,b) \le \omega_{M,N;\mu_2}(a,b), \forall a, b \in A$.

Proof:

If $\omega_{M,N;\mu_2}(a,b) < \Omega_{M,N;\mu_1}(a,b)$, then we choose $t_0 \in \mathcal{R}$ such that $\omega_{M,N;\mu_2}(a,b) < t_0 < \Omega_{M,N;\mu_1}(a,b)$. Hence, we have $t_0 \notin (-\infty, \omega_{M,N;\mu_2}(a,b)) = \{t: M(a,b,t) < \mu_2, N(a,b,t) > \mu_2\}$ and

$$t_0 \notin (\Omega_{M,N;\mu_1}(a,b), \infty) = \{t: M(a,b,t) > \mu_1, N(a,b,t) < \mu_1\}$$

Therefore $\mu_1 \ge M(a,b,t_0) \ge \mu_2$, $\mu_1 \le N(a,b,t_0) \le \mu_2$,

Which is a contradiction.

Hence $\Omega_{M,N;\mu_1}(a,b) \le \omega_{M,N;\mu_2}(a,b)$.

Theorem 3.10:

Let $(A, M, N, *, \circ)$ be a Intuitionistic fuzzy metric space. Then the following are equivalent.

(i) $(A, M, N, *, \circ)$ is an (FD) Intuitionistic fuzzy metric space.

(ii) $\lim_{\mu\to1} \omega_{M,N;\mu}(a,b)$ exists for all pairs (a,b).

(iii) $\lim_{\mu\to1} \Omega_{M,N;\mu}(a,b)$ exists for all pairs (a,b).

Proof:

(i)⇒(ii) The condition (FD) states that for every pair (a,b) of points in A, there exists $t_{a,b}$ such that $M(a,b,t_{a,b}) = 1 > \mu, N(a,b,t_{a,b}) = 0 < \mu \;\forall \mu \in (0,1)$, and hene $\Omega_{M,N;\mu}(a,b) = \inf\{ t : M(a,b,t) > \mu, N(a,b,t) < \mu\} \le t_{a,b}$. Since $\omega_{M,N;\mu}(a,b) \le \Omega_{M,N;\mu}(a,b) \le t_{a,b}$, for all $\mu \in (0,1)$, and $\omega_{M,N;\mu}(a,b)$ increases with μ, we see that $\lim_{\mu\to1} \omega_{M,N;\mu}(a,b)$ exists.

(ii)⇒(iii) Let $\mu \in (0,1)$ be arbitrary. We choose μ' between μ and 1. Then by lemma 3.9, we have $\Omega_{M,N;\mu}(a,b) \leq \omega_{M,N;\mu'}(a,b) \leq \lim_{\mu\to 1} \omega_{M,N;\mu}(a,b)$, as $\omega_{M,N;\mu}(a,b)$ increases with μ. Since $\Omega_{M,N;\mu}(a,b)$ increases as μ increases and bounded above, we conclude that $\lim_{\mu\to 1} \Omega_{M,N;\mu}(a,b)$ exists.

(iii)⇒(i) Assume that $\lim_{\mu\to 1} \Omega_{M,N;\mu}(a,b)$ exists and let it be t_0. Since $\Omega_{M,N;\mu}(a,b)$ increases as μ increases, we have $t_0 + 1 > \Omega_{M,N;\mu}(a,b)$ for all $0 < \mu < 1$. Hence $M(a,b,t_0+1) > \mu, N(a,b,t_0+1) < \mu$, for all $0 < \mu < 1$. Thus $M(a,b,t_0+1) = 1, N(a,b,t_0+1) = 0$.

Corollary 3.11:

Let $(A, M, N, *, \circ)$ be a FD-Intuitionistic fuzzy metric space. Then for any $a, b \in A$, $\lim_{\mu\to 1} \omega_{M,N;\mu}(a,b) = \lim_{\mu\to 1} \Omega_{M,N;\mu}(a,b)$.

Proof:

Let $a, b \in A$ be fixed arbitrarily. By the condition (FD), both of the limits exist using $\omega_{M,N;\mu}(a,b) \leq \Omega_{M,N;\mu}(a,b), \forall \mu \in (0,1)$. We get that $\lim_{\mu\to 1} \omega_{M,N;\mu}(a,b) \leq \lim_{\mu\to 1} \Omega_{M,N;\mu}(a,b)$. On the other hand from lemma 3.9, we have that $\Omega_{M,N;\mu_1}(a,b) \leq \omega_{M,N;\mu_2}(a,b)$, whenever $\mu_1 < \mu_2$. If we allow μ_1 to 1, then μ_2 also tends to 1 and hence we get the reverse inequality $\lim_{\mu_1\to 1} \Omega_{M,N;\mu_1}(a,b) \leq \lim_{\mu_2\to 1} \omega_{M,N;\mu_2}(a,b)$.

4.Conclusion

In this paper, we have discussed μ-metrics induced by an Intuitionistic fuzzy metric and proved that the existence of the μ-metrics induced by an Intuitionistic fuzzy metric M, N is characterized by the FD condition on M, N. We have also provided two different approximations of the metric induced from the Intuitionistic fuzzy metric, through upper μ-metrics and lower μ-metrics.

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