Research Article

A Two-State Retrial Queueing Model with Feedback having Two Non-Identical Parallel Servers

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Abstract: This paper presents a two-state retrial queueing model with feedback having two non-identical paallel servers. Primary and secondary customers follow Poisson process. The service times of both the servers are exponentially distributed with different service rates. Explicit transient probabilities are obtained for exact number of arrivals and departures from the system when both, one or none are busy. Some performance measures are also obtained. Numerical and graphical solutions are obtained.

1. Introduction

Apart from standard queueing models there exists a new class of queueing models known as retrial queueing models. In many real life systems like in telecommunication systems and in computer networks, the customers who do not get immediate service on arrival to a system retry for service after a random amount of time. As there is no waiting space for the arriving customers so they join the virtual queue called orbit or pool and retry from there as retrial customers or secondary calls. The primary customers and secondary customers both follow Poisson process. [3], [11] and [1] did initial work on retrial queues.

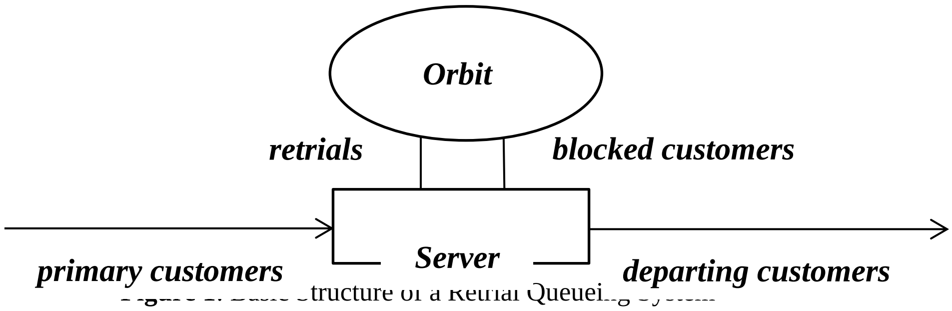

Figure 1: Basic Structure of a Retrial Queueing System

Sometimes due to dissatisfaction, the customers the seek service again in order to get a satisfied service which results in feedback concept. For example: when a message faces a failed transmission in multiple access telecommunication systems, it can be sent again.

[2] published `A discrete-time retrial queueing system with starting failures, Bernoulli feedback and general retrial times'. [6] analyzed `A single server feedback retrial queue with collisions'. [9] studied `Transient and numerical solution of a feedback queueing system with correlated departures'.

In some systems various servers possess different service rates depending on the requirements and other reasons, these servers are called heterogeneous or non-identical servers. Real time examples of heterogeneous servers can be seen in banks, telecommunication centers. Here the same type of job is rendered by different servers with different service rates. [8] analyzed `Busy Period Analysis of a Markovian Feedback Queueing Model with servers having unequal service rate' where busy period of the system was obtained using generating function technique. [4] studied markovian system with two heterogeneous servers and constant retrial rate under a threshold policy.

The concept of two-state was introduced by [7] in `Some new results for the M/M/1 queue' in which the solution for exactly `*i*' number of arrivals, `*j*' number of departures were obtained over a time interval *t*. [10] published `Performance Analysis of a Two-State Queueing Model with Retrials' where the transient state probabilities were obtained.

The section wise description of the paper is as follows:

The model is described mathematically in section 2 where the difference-differential equations of the system are given. The transient state probabilities are obtained in section 3. In section 4 some important performance measures are derived. The numerical and graphical results are obtained in section 5. Busy period distribution of the system and its numerical and graphical representation is given in section 6 and 7 respectively. Finally, the paper is concluded in section 8.

2. Model Description

The detailed description of the present model is given as follows:

- The Arrival Process: The arrival of primary customers follow Poisson process with parameter λ.
- An arriving customer joins the first server with probability $a_1$and second with a_2.

- The Retrial Process: On arrival of a customer if any of the servers is free, it is served immediately. Otherwise, the customer joins the orbit and calls repeatedly until any of the servers is free. The retrial customers also follow Poisson process with parameter θ.

- The Service Process: Service times follow exponential distribution with parameters μ_1 for first server and μ_2 for second server.

- The Feedback Rule: If a customer after getting service feels unsatisfied it may join the orbit with probability γ else leaves the system with probability 1- γ.

The arrival of primary and secondary calls, departures and service times are statistically independent.

Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}\,(s) = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k - l}\, e^{a_k t}}{(m_k - l)!(l-1)!} \times \left(\frac{d}{dp}\right)^{l-1} \left(\frac{Q(p)}{P(p)}\right) (p - a_k)^{m_k} \ \forall \text{ p}= a_k, \ \ a_i \neq a_k \text{ for i} \neq \text{k}$$

where,

$P(p) = (p - a_1)^{m_1} (p - a_2)^{m_2} \ldots\ldots\ldots.(p - a_n)^{m_n}$

$Q(p)$ is a polynomial of degree < $m_1 + m_2 + m_3 + \cdots\cdots + m_n - 1$

The Laplace inverse of

$\bar{N}_{n_1,n_2,n_3}^{a,b,c}(s) = \frac{1}{(s+a)^{n_1}\,(s+b)^{n_2}\,(s+c)^{n_3}}$ is

$$N_{n_1,n_2,n_3}^{a,b,c}(\mathrm{t}) = \sum_{l=1}^{n_3} \sum_{m=1}^{l} \frac{e^{-at}\ t^{n_3 - l}\ (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}} (n_1 + \mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}} (n_2 + \mathrm{g_2})\right)}{(n_3 - l)!\,(m-1)!\ (b-a)^{n_2+m-1}\ (c-a)^{n_1+l-m}}$$
$$+ + \sum_{l=1}^{n_2} \sum_{m=1}^{l} \frac{e^{-bt}\ t^{n_2 - l}\ (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}} (n_1 + \mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}} (n_3 + \mathrm{g_2})\right)}{(n_2 - l)!\,(m-1)!\ (a-b)^{n_3+m-1}\ (c-b)^{n_1+l-m}}$$
$$+ \ + \sum_{l=1}^{n_1} \sum_{m=1}^{l} \frac{e^{-ct}\ t^{n_1 - l}\ (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}} (n_2 + \mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}} (n_3 + \mathrm{g_2})\right)}{(n_1 - l)!\,(m-1)!\ (a-c)^{n_3+m-1}\ (b-c)^{n_2+l-m}}$$

If $L^{-1}\{\mathrm{f(s)}\} = F(\mathrm{t})$ and $L^{-1}\{\mathrm{g(s)}\} = G(\mathrm{t})$, *then*

$L^{-1}\{\mathrm{f(s)\ g(s)}\} = \int_0^t F(\mathrm{u})G(\mathrm{t-u})\,\mathrm{du} = F * G,$

$F * G$ is called the convolution of F and G.

2.1. The Two-Dimensional State Model

2.1.1. Definitions

$P_{i,j,0}(t)$= Probability that there are exactly *i* number of arrivals, *j* number of departures by time *t* and servers are free.

$P_{i,j,1,k}(t)$ = Probability that there are exactly *i* number of arrivals, *j* number of departures by time *t* from the system and k^{th} *(k = 1 or 2)* server is busy.

$P_{i,j,2}(t)$= Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t* and both the servers are busy.

$P_{i,j}(t)$= Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t*.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t), \forall\, i,j; i \geq j$$
$$P_{i,j,1}(t) = P_{i,j,1,1}(t) + P_{i,j,1,2}(t)$$
$$P_{i,j,0}(t) = 0, i < j; P_{i,j,1,k}(t) = 0 (k = 1\ or\ 2), i < j+1; P_{i,j,2}(t) = 0, i < j+2$$

Initially

$$P_{0,0,0}(0) = 1;$$
$$P_{i,j,0}(0) = 0, \forall\, i,j\ \big(i \neq 0\ \&j \neq 0 (simultaneously)\big);\ P_{i,j,1,k}(0) = 0 (k = 1\ or\ 2), \forall\, i,j;\ P_{i,j,2}(0) = 0, \forall i,j$$

2.2. The Difference-Differential equations governing the system:

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu_1(1-\gamma)P_{i,j-1,1,1}(t) + \mu_1\gamma P_{i,j,1,1}(t) + \mu_2(1-\gamma)P_{i,j-1,1,2}(t) + \mu_2\gamma P_{i,j,1,2}(t); \qquad i \geq j \geq 0 \qquad (1)$$

$$\frac{d}{dt}P_{i,j,1,1}(t) = -(\lambda + \mu_1 + (i-j-1)\theta)P_{i,j,1,1}(t) + \lambda a_1 P_{i-1,j,0}(t) + (i-j)\theta a_1 P_{i,j,0}(t) + \mu_2(1-\gamma)P_{i,j-1,2}(t) + \mu_2\gamma P_{i,j,2}(t); \qquad i > j \geq 0 \qquad (2)$$

$$\frac{d}{dt}P_{i,j,1,2}(t) = -(\lambda + \mu_2 + (i-j-1)\theta)P_{i,j,1,2}(t) + \lambda a_2 P_{i-1,j,0}(t) + (i-j)\theta a_2 P_{i,j,0}(t) + \mu_1(1-\gamma)P_{i,j-1,2}(t) + \mu_1\gamma P_{i,j,2}(t); \qquad i > j \geq 0 \qquad (3)$$

$$\frac{d}{dt}P_{i,j,2}(t) = -(\lambda + \mu_1 + \mu_2)P_{i,j,2}(t) + (i-j-1)\theta\{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\} + \lambda\{P_{i-1,j,1,1}(t) + P_{i-1,j,1,2}(t)\} + \lambda(1-\delta_{i-2,j})P_{i-1,j,2}(t); \quad i \geq 2, i > j \geq 0 \quad (4)$$

where

$$\delta_{i-2,j} = \begin{cases} 1; & i-2 = j \\ 0; & otherwise \end{cases}$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \qquad Re(s) > 0$$

and using initial condition in equations (1) to (4), we have

$$\left.\begin{aligned} (s+\lambda)\bar{P}_{0,0,0}(s) &= \bar{P}_{0,0,0}(0) \\ (s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) &= \mu_1(1-\gamma)\bar{P}_{i,j-1,1,1}(s) + \mu_1\gamma\bar{P}_{i,j,1,1}(s) + \\ &\quad \mu_2(1-\gamma)\bar{P}_{i,j-1,1,2}(s) + \mu_2\gamma\bar{P}_{i,j,1,2}(s) \end{aligned}\right\} \qquad i \geq j > 0 \qquad (5)$$

$$(s + \lambda + \mu_1 + (i-j-1)\theta)\bar{P}_{i,j,1,1}(s) = \lambda a_1\bar{P}_{i-1,j,0}(s) + (i-j)\theta a_1\bar{P}_{i,j,0}(s) + \mu_2(1-\gamma)\bar{P}_{i,j-1,2}(s) + \mu_2\gamma\bar{P}_{i,j,2}(s); \qquad i > j \geq 0 \qquad (6)$$

$$(s + \lambda + \mu_2 + (i-j-1)\theta)\bar{P}_{i,j,1,2}(s) = \lambda a_2\bar{P}_{i-1,j,0}(s) + (i-j)\theta a_2\bar{P}_{i,j,0}(s) + \mu_1(1-\gamma)\bar{P}_{i,j-1,2}(s) + \mu_1\gamma\bar{P}_{i,j,2}(s); \qquad i > j \geq 0 \qquad (7)$$

$$(s + \lambda + \mu_1 + \mu_2)\bar{P}_{i,j,2}(s) = (i-j-1)\theta\{\bar{P}_{i,j,1,1}(s) + \bar{P}_{i,j,1,2}(s)\} + \lambda\{\bar{P}_{i-1,j,1,1}(s) + \bar{P}_{i-1,j,1,2}(s)\} + \lambda(1-\delta_{i-2,j})\bar{P}_{i-1,j,2}(s); \qquad i \geq 2, i > j \geq 0 \qquad (8)$$

3. Solution of the Problem

Solving equations (5) to (8) recursively, we get:

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \qquad (9)$$

$$\bar{P}_{1,1,0}(s) = \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_1}\left(\frac{\lambda a_1}{(s+\lambda)^2}\right) + \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_1}\left(\frac{\theta a_1}{s+\lambda}\right)\bar{P}_{1,0,0}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_2}\left(\frac{\lambda a_2}{(s+\lambda)^2}\right) + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_2}\left(\frac{\theta a_2}{s+\lambda}\right)\bar{P}_{1,0,0}(s) \qquad (10)$$

$$\bar{P}_{1,0,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1}\left(\frac{1}{s+\lambda}\right) + \frac{\theta a_1}{s+\lambda+\mu_1}\bar{P}_{1,0,0}(s) \qquad (11)$$

$$\bar{P}_{1,0,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2}\left(\frac{1}{s+\lambda}\right) + \frac{\theta a_2}{s+\lambda+\mu_2}\bar{P}_{1,0,0}(s) \qquad (12)$$

$$\bar{P}_{i,0,0}(s) = \frac{\mu_1\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1,1}(s) + \frac{\mu_2\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1,2}(s); i \geq 1 \qquad (13)$$

$$\bar{P}_{i,i,0}(s) = \frac{1}{s+\lambda}\left[\frac{\lambda a_1\mu_1(1-\gamma)}{s+\lambda+\mu_1} + \frac{\lambda a_2\mu_2(1-\gamma)}{s+\lambda+\mu_2}\right]\bar{P}_{i-1,i-1,0}(s) + \frac{1}{s+\lambda}\left[\frac{\theta a_1\mu_1(1-\gamma)}{s+\lambda+\mu_1} + \frac{\theta a_2\mu_2(1-\gamma)}{s+\lambda+\mu_2}\right]\bar{P}_{i,i-1,0}(s) + \frac{\mu_1(1-\gamma)\mu_2(1-\gamma)}{s+\lambda}\left[\frac{1}{s+\lambda+\mu_1} + \frac{1}{s+\lambda+\mu_2}\right]\bar{P}_{i,i-2,2}(s);\ i \geq 2 \quad (14)$$

$$\bar{P}_{i,i-1,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_1}{s+\lambda+\mu_1}\bar{P}_{i,i-1,0}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1}\bar{P}_{i,i-2,2}(s);\ i \geq 2 \quad (15)$$

$$\bar{P}_{i,i-1,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_2}{s+\lambda+\mu_2}\bar{P}_{i,i-1,0}(s) + \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_2}\bar{P}_{i,i-2,2}(s);\ i \geq 2 \quad (16)$$

$$\bar{P}_{i,1,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i-1,1,0}(s) + \frac{(i-1)\theta a_1}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,1,0}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,0,2}(s) + \frac{\mu_2\gamma}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,1,2}(s);\ i \geq 3 \quad (17)$$

$$\bar{P}_{i,1,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i-1,1,0}(s) + \frac{(i-1)\theta a_2}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,1,0}(s) + \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,0,2}(s) + \frac{\mu_1\gamma}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,1,2}(s);\ i \geq 3 \quad (18)$$

$$\bar{P}_{i,2,0}(s) = \frac{\mu_1(1-\gamma)}{s+\lambda+(i-2)\theta}\bar{P}_{i,1,1,1}(s) + \frac{\mu_1\gamma}{s+\lambda+(i-2)\theta}\bar{P}_{i,2,1,1}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+(i-2)\theta}\bar{P}_{i,1,1,2}(s) + \frac{\mu_2\gamma}{s+\lambda+(i-2)\theta}\bar{P}_{i,2,1,2}(s);\ i \geq 3 \quad (19)$$

$$\bar{P}_{i,j,2}(s) = \sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j,1,1}(s) + \bar{P}_{j+k,j,1,2}(s)\};\ i \geq j+2, j \geq 1 \quad (20)$$

where

$$\eta'_k = \begin{cases} 1; & k = 1 \\ 1 + \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k = 2\ to\ i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k = i-j \end{cases}$$

$$\bar{P}_{i,j,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1+(i-j-1)\theta}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta a_1}{s+\lambda+\mu_1+(i-j-1)\theta}\bar{P}_{i,j,0}(s) + \frac{\mu_2\gamma}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j,1,1}(s) + \bar{P}_{j+k,j,1,2}(s)\}\right] + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\omega'_k(s)\{\bar{P}_{j+k,j-1,1,1}(s) + \bar{P}_{j+k,j-1,1,2}(s)\}\right];\quad i \geq j+2, j \geq 2 \quad (21)$$

where

$$\eta'_k = \begin{cases} 1; & k = 1 \\ 1 + \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k = 2\ to\ i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k = i-j \end{cases}$$

$$\omega'_k = \begin{cases} 1; & k = 0 \\ 1 + \frac{k\theta}{s+\lambda+\mu_1+\mu_2}; & k = 1\ to\ i-j-1 \\ \frac{k\theta}{s+\lambda+\mu_1+\mu_2}; & k = i-j \end{cases}$$

$$\bar{P}_{i,j,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2+(i-j-1)\theta}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta a_2}{s+\lambda+\mu_2+(i-j-1)\theta}\bar{P}_{i,j,0}(s) + \frac{\mu_1\gamma}{s+\lambda+\mu_2+(i-j-1)\theta}\left[\sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j,1,1}(s)+\bar{P}_{j+k,j,1,2}(s)\}\right] + \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_2+(i-j-1)\theta}\left[\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\omega'_k(s)\{\bar{P}_{j+k,j-1,1,1}(s)+\bar{P}_{j+k,j-1,1,2}(s)\}\right]; \quad i \geq j+2, j \geq 2 \quad (22)$$

where

$$\eta'_k = \begin{cases} 1; & k=1 \\ 1+\frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k=2 \text{ to } i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}; & k=i-j \end{cases}$$

$$\omega'_k = \begin{cases} 1; & k=0 \\ 1+\frac{k\theta}{s+\lambda+\mu_1+\mu_2}; & k=1 \text{ to } i-j-1 \\ \frac{k\theta}{s+\lambda+\mu_1+\mu_2}; & k=i-j \end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu_1(1-\gamma)}{s+\lambda+(i-j)\theta}\bar{P}_{i,j-1,1,1}(s) + \frac{\mu_1\gamma}{s+\lambda+(i-j)\theta}\bar{P}_{i,j,1,1}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+(i-j)\theta}\bar{P}_{i,j-1,1,2}(s) + \frac{\mu_2\gamma}{s+\lambda+(i-j)\theta}\bar{P}_{i,j,1,2}(s); i > j > 2 \quad (23)$$

Taking Laplace Inverse of equations (9) to (23) we get:

$$P_{0,0,0}(t) = e^{-\lambda t} \quad (24)$$

$$P_{1,1,0}(t) = \left[\lambda a_1\mu_1(1-\gamma)e^{-(\lambda+\mu_1)t} + \lambda a_2\mu_2(1-\gamma)e^{-(\lambda+\mu_2)t}\right]te^{-\lambda t} + \left[\theta a_1\mu_1(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \theta a_2\mu_2(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{1,0,0}(t) \quad (25)$$

$$P_{1,0,1,1}(t) = \lambda a_1 e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \theta a_1 e^{-(\lambda+\mu_1)t} * P_{1,0,0}(t) \quad (26)$$

$$P_{1,0,1,2}(t) = \lambda a_2 e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\} + \theta a_2 e^{-(\lambda+\mu_2)t} * P_{1,0,0}(t) \quad (27)$$

$$P_{i,0,0}(t) = \mu_1\gamma e^{-(\lambda+i\theta)t}P_{i,0,1,1}(t) + \mu_2\gamma e^{-(\lambda+i\theta)t}P_{i,0,1,2}(t); i \geq 1 \quad (28)$$

$$P_{i,i,0}(t) = \left[\lambda a_1\mu_1(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \lambda a_2\mu_2(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i-1,i-1,0}(t) + \left[\theta a_1\mu_1(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \theta a_2\mu_2(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i,i-1,0}(t) + \mu_1(1-\gamma)\mu_2(1-\gamma)\left[e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i,i-2,2}(t); \quad i \geq 2 \quad (29)$$

$$P_{i,i-1,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1)t} * P_{i-1,i-1,0}(t) + \theta a_1 e^{-(\lambda+\mu_1)t} * P_{i,i-1,0}(t) + \mu_2(1-\gamma)e^{-(\lambda+\mu_1)t} * P_{i,i-2,2}(t); i \geq 2 \quad (30)$$

$$P_{i,i-1,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2)t} * P_{i-1,i-1,0}(t) + \theta a_2 e^{-(\lambda+\mu_2)t} * P_{i,i-1,0}(t) + \mu_1(1-\gamma)e^{-(\lambda+\mu_2)t} * P_{i,i-2,2}(t); i \geq 2 \quad (31)$$

$$P_{i,1,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i-1,1,0}(t) + (i-1)\theta a_1 e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,1,0}(t) + \mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,0,2}(t) + \mu_2\gamma e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,1,2}(t); i \geq 3 \quad (32)$$

$$P_{i,1,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i-1,1,0}(t) + (i-1)\theta a_2 e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,1,0}(t) + \mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,0,2}(t) + \mu_1\gamma e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,1,2}(t); i \geq 3 \quad (33)$$

$$P_{i,2,0}(t) = \mu_1(1-\gamma)e^{-(\lambda+(i-2)\theta)t} * P_{i,1,1,1}(t) + \mu_2(1-\gamma)e^{-(\lambda+(i-2)\theta)t} * P_{i,1,1,2}(t) + \mu_1\gamma e^{-(\lambda+(i-2)\theta)t} * P_{i,2,1,1}(t) + \mu_2\gamma e^{-(\lambda+(i-2)\theta)t} * P_{i,2,1,2}(t); i \geq 3 \quad (34)$$

$$P_{i,j,2}(t) = \lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+\mu_1+\mu_2)t} * \{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\} + \sum_{k=2}^{i-j-1}\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+\mu_1+\mu_2)t} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\} + \sum_{k=2}^{i-j-1}\lambda^{i-j-k}(k-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+\mu_1+\mu_2)t} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\} + (i-j-1)\theta e^{-(\lambda+\mu_1+\mu_2)t} * \{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\}; i \geq j+2, j \geq 1 \quad (35)$$

$$P_{i,j,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)t} * P_{i,j,0}(t) + \mu_2\gamma\lambda^{i-j-1}e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-2}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r-1}}\right\} * \{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k}(k-1)\theta\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\} + (i-j-1)\theta\mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\left\{\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right\} * \{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\} + \mu_2(1-\gamma)\lambda^{i-j}e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r}}\right\} * \{P_{j,j-1,1,1}(t) + P_{j,j-1,1,2}(t)\} + \mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\} + \mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}k\theta\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r+1}}\right\} * \{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\} + \mu_2(1-\gamma)(i-j)\theta e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\left\{\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right\} * \{P_{i,j-1,1,1}(t) + P_{i,j-1,1,2}(t)\}; i \geq j+2, j \geq 2 \quad (36)$$

$$P_{i,j,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)t} * P_{i,j,0}(t)$$
$$+ \mu_1\gamma\lambda^{i-j-1}e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-2}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r-1}}\right\}$$
$$* \{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\}$$
$$+ \mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}}\right.$$
$$\left. - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\}$$
$$+ \mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}(k-1)\theta\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}}\right.$$
$$\left. - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r+1}}\right\} * \{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\}$$
$$+ \mu_1\gamma(i-j-1)\theta e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\left\{\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right\} * \{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\}$$
$$+ \mu_1(1-\gamma)\lambda^{i-j}e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r}}\right\}$$
$$* \{P_{j,j-1,1,1}(t) + P_{j,j-1,1,2}(t)\}$$
$$+ \mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}}\right.$$
$$\left. - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\}$$
$$+ \mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}(k\theta)\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}}\right.$$
$$\left. - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r+1}}\right\} * \{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\}$$
$$+ \mu_1(1-\gamma)(i-j)\theta e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\left\{\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right\}$$
$$* \{P_{i,j-1,1,1}(t) + P_{i,j-1,1,2}(t)\};\ i \geq j+2, j \geq 2 \qquad (37)$$

$$P_{i,j,0}(t) = \mu_1(1-\gamma)e^{-(\lambda+(i-j)\theta)t} * P_{i,j-1,1,1}(t) + \mu_1\gamma e^{-(\lambda+(i-j)\theta)t} * P_{i,j,1,1}(t) + \mu_2(1-\gamma)e^{-(\lambda+(i-j)\theta)t} * P_{i,j-1,1,2}(t) + \mu_2\gamma e^{-(\lambda+(i-j)\theta)t} * P_{i,j,1,2}(t);\ \ i > j > 2 \qquad (38)$$

4. Some Important Performance Measures

1. The Laplace transform of $\bar{P}_{i.}(s)$ is given as:

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i}\bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}};\quad i > 0$$

and its Laplace Inverse is:

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

which proves the basic assumption that primary arrivals follow Poisson process.

2. The probability that exactly *j* customers depart from the system by time *t* is given as:

$$\bar{P}_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

3. Summing equations (9)-(23) over *i* and *j* we get:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1,1}(s) + \bar{P}_{i,j,1,2}(s) + \bar{P}_{i,j,2}(s)\} = \frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)\} = 1$$

which is the verification of our results.

4. Define $Q_{n,m}(t) =$ Probability that there are exactly *n* customers in the orbit when *m(m=0,1,2)* servers are busy at time *t*.

When server is free, it is represented by probability $Q_{n,0}(t)$

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

The number of customers in the orbit, in this case are calculated with the following formula:

n = (number of arrivals − number of departures)

When one server is busy (m=1), it is represented by the probability $Q_{n,m,k}$(t).

$$Q_{n,m,k}(t) = \sum_{j=0}^{\infty} P_{j+n+m,j,m,k}(t); \quad k = 1,2$$

The number of customers in the orbit in this case is calculated by the following formula:

n = (number of arrivals − number of departures − m)

When both the servers are busy (m=2), it is represented by the probability $Q_{n,m}$(t).

$$Q_{n,m}(t) = \sum_{j=0}^{\infty} P_{j+n+m,j,m}(t)$$

The number of customers in the orbit in this case is calculated by the following formula:

n = (number of arrivals − number of departures − m)

Using above definitions in (1) to (4) and let μ_1=$\mu_2$1, γ=0 the equations we get are:

$(\lambda + n\theta)Q_{n,0} = Q_{n,1,1} + Q_{n,1,2}; \quad n \geq 0$ (39)

$(\lambda + n\theta + 1)Q_{n,1,1} = \lambda a_1 Q_{n,0} + (n+1)\theta a_1 Q_{n+1,0} + Q_{n,2}; \; n \geq 0$ (40)

$(\lambda + n\theta + 1)Q_{n,1,2} = \lambda a_2 Q_{n,0} + (n+1)\theta a_2 Q_{n+1,0} + Q_{n,2}; \; n \geq 0$ (41)

$(\lambda + 2)Q_{n,2} = \lambda\{Q_{n,1,1} + Q_{n,1,2}\} + (n+1)\theta\{Q_{n+1,1,1} + \{Q_{n+1,1,2}\} + \lambda(1 - \delta_{n,0})Q_{n-1,2}; \quad n \geq 0$ (42)

Using $Q_{n,1,1} + Q_{n,1,2} = Q_{n,1}$ and let $a_1 = a_2 = \frac{1}{2}$ and adding equations (40) and (41) we get:

$(\lambda + n\theta)Q_{n,0} = Q_{n,1}; \quad n \geq 0$ (43)

$(\lambda + n\theta + 1)Q_{n,1} = \lambda Q_{n,0} + (n+1)\theta Q_{n+1,0} + 2Q_{n,2}; \; n \geq 0$ (44)

$(\lambda + 2)Q_{n,2} = \lambda Q_{n,1} + (n+1)\theta Q_{n+1,1} + \lambda(1 - \delta_{n,0})Q_{n-1,2}; \quad n \geq 0$ (45)

which coincides with the results (1)-(3) of [5].

5. Numerical Solution and Graphical Representation

The Numerical solutions are generated using MATLAB programming for the case ρ=0.6, η=0.5, γ=0.4, r_1=0.6, r_2=0.4, a_1=0.5 and a_2=0.5. Observing the below tables for various time instants it could be seen that the sum of probabilities approaches to 1.

Table 1: At *t*=1

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{1,0,1,1}$	$P_{1,0,1,2}$	$P_{2,1,1,2}$	$P_{2,0,2}$	$P_{3,0,2}$	Sum
0.5488	0.0238	0.0422	0.1257	0.1376	0.0117	0.063	0.0127	0.9655

Table 2: At *t*=5

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{1,0,1,1}$	$P_{2,0,1,1}$	$P_{2,1,1,1}$

0.0498	0.0193	0.0622	0.0241	0.0388	0.0163	0.014	0.029	0.0108	0.0353

$P_{3,1,1,1}$	$P_{3,2,1,1}$	$P_{4,2,1,1}$	$P_{5,2,1,1}$	$P_{1,0,1,2}$	$P_{2,0,1,2}$	$P_{2,1,1,2}$	$P_{3,1,1,2}$	$P_{3,2,1,2}$	$P_{4,1,1,2}$
0.0166	0.0198	0.0105	0.009	0.0389	0.0157	0.0502	0.0247	0.0279	0.0107

$P_{4,2,1,2}$	$P_{4,3,1,2}$	$P_{5,1,1,2}$	$P_{5,2,1,2}$	$P_{5,3,1,2}$	$P_{2,0,2}$	$P_{3,0,2}$	$P_{3,1,2}$	$P_{4,0,2}$	$P_{4,1,2}$
0.0153	0.0088	0.0086	0.0136	0.0106	0.0454	0.04	0.0466	0.0248	0.0415

$P_{4,2,2}$	$P_{5,0,2}$	$P_{5,1,2}$	$P_{5,2,2}$	$P_{5,3,2}$	Sum
0.0216	0.021	0.0415	0.0319	0.0098	0.9046

Table 3: At *t*=10

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,2,0}$	$P_{5,3,0}$	$P_{5,4,0}$
0.0025	0.0089	0.0159	0.0103	0.018	0.0122	0.0142	0.0112	0.0354	0.0716

$P_{5,5,0}$	$P_{3,2,1,1}$	$P_{4,2,1,1}$	$P_{5,2,1,1}$	$P_{4,3,1,1}$	$P_{5,3,1,1}$	$P_{5,4,1,1}$	$P_{2,1,1,2}$	$P_{3,2,1,2}$	$P_{4,2,1,2}$	$P_{4,3,12}$
0.0719	0.0113	0.0087	0.0122	0.023	0.0448	0.0447	0.0095	0.0164	0.013	0.0176

$P_{5,1,1,2}$	$P_{5,2,1,2}$	$P_{5,3,1,2}$	$P_{5,4,1,2}$	$P_{3,1,2}$	$P_{4,1,2}$	$P_{4,2,2}$	$P_{5,0,2}$	$P_{5,1,2}$
0.0106	0.0359	0.0695	0.0691	0.0126	0.0173	0.0199	0.0115	0.0491

$P_{5,2,2}$	$P_{5,3,2}$	Sum
0.0874	0.0678	0.9243

Table 4: At *t*=20

$P_{0,0,0}$	$P_{1,0,0}$	$P_{4,4,0}$	$P_{5,0,0}$	$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{5,3,1,1}$	$P_{5,4,1,1}$	$P_{5,2,1,2}$
0	0	0.018	0	0.0116	0.1071	0.6125	0.0143	0.065	0.0038

$P_{5,3,1,2}$	$P_{5,4,1,2}$	$P_{5,2,2}$	$P_{5,3,2}$	Sum
0.0232	0.1131	0.0099	0.0252	0.9875

Table 5: At *t*=30

$P_{0,0,0}$	$P_{2,1,0}$	$P_{4,3,0}$	$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{5,3,1,1}$	$P_{5,4,1,1}$	$P_{5,3,1,2}$	$P_{5,4,1,2}$
0	0	0	0.0116	0.1071	0.6125	0.0143	0.065	0.0232	0.1131

$P_{2,0,2}$	$P_{5,2,2}$	$P_{5,3,2}$	Sum
0	0.0099	0.0252	0.9819

Table 6: At *t*=40

$P_{1,1,0}$	$P_{3,2,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{5,3,1,1}$	$P_{5,4,1,2}$	$P_{3,1,2}$	$P_{4,2,2}$	Sum
0	0	0.0055	0.9849	0	0.0062	0	0	0.9966

The probabilities against time are graphically represented in following figures:

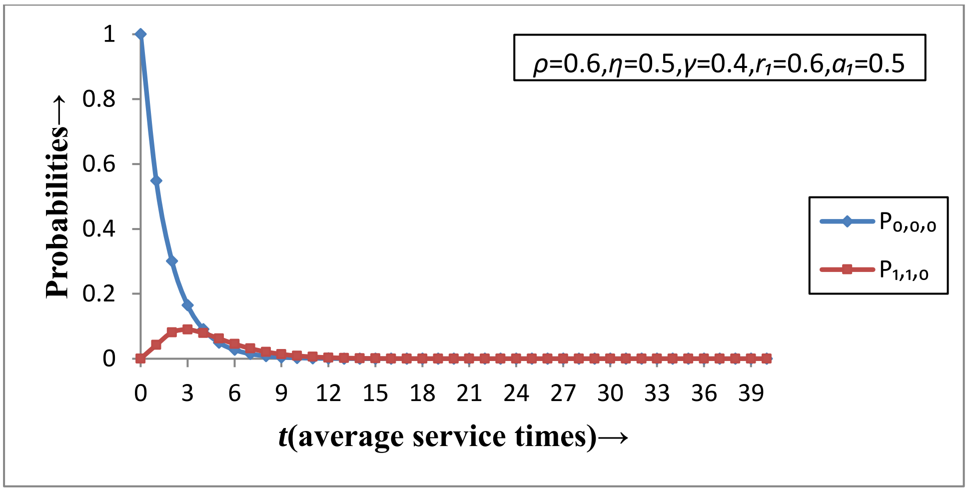

Figure 1

In Figure 1 the probabilities $P_{0,0,0}$ and $P_{1,1,0}$ are plotted against time t (average service times) for the case ρ=0.6, η=0.5, γ=0.4, a_1=0.6(a_2=1-a_1), r_1=0.5(r_2=1-r_1). It is interpreted from the graph that the probability $P_{0,0,0}$ rapidly decreases from initial value 1 for t=0 whereas probability $P_{1,1,0}$ increases in starting from initial value 0 for t=0 and then decreases gradually.

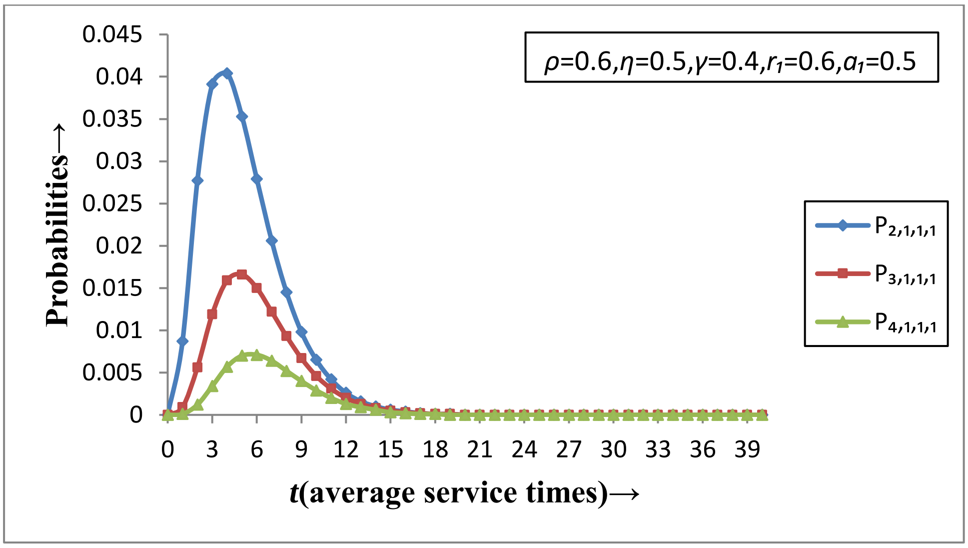

Figure 2

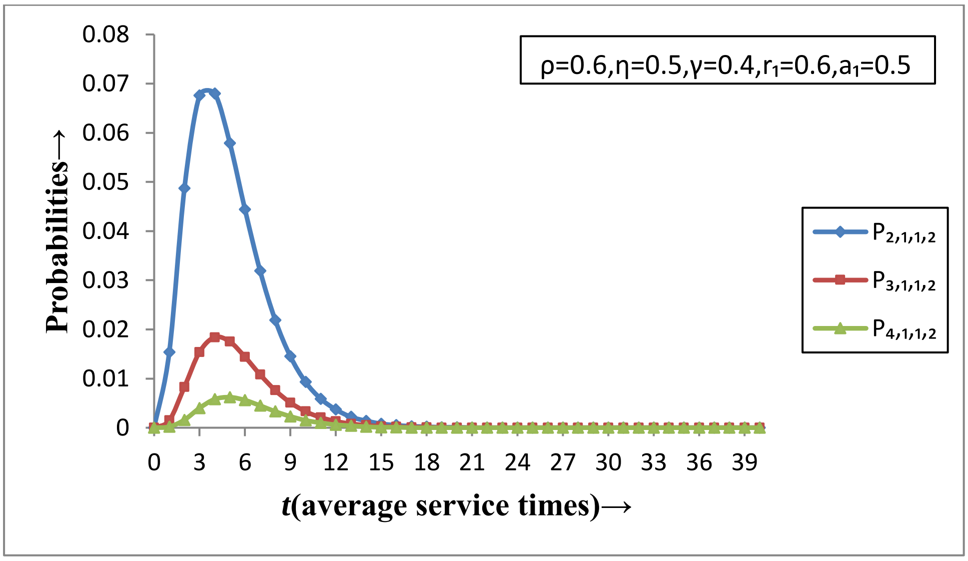

Figure 3

Figures 2 and 3 depict the probabilities $P_{2,1,1}$, $P_{3,1,1}$ and $P_{4,1,1}$ for both the servers 1 and 2 against time *t*. From both the figures it is clearly interpreted that probabilities start increasing from 0 at *t*=0 in the beginning and then start decreasing. Also, the curve peaks are higher for lower number of arrivals. If we compare both the graphs the probabilities are higher for first server than second because of the difference in r_1 and r_2.

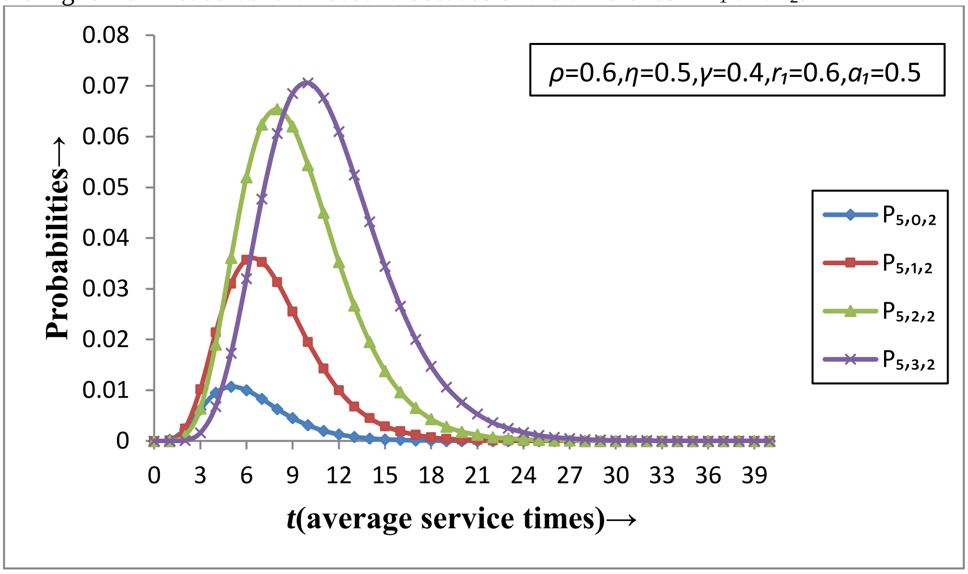

Figure 4

Figure 4 shows comparison between probabilities $P_{5,0,2}$, $P_{5,1,2}$, $P_{5,2,2}$ and $P_{5,3,2}$ against *t*. Beginning with value 0 at *t*=0 the probabilities increases rapidly to their highest values and then decreases gradually. Also, the probabilities are higher for larger number of departures when both the servers are busy.

6. Busy Period Distribution

Using some numerical results obtained through MATLAB programming, the busy period of the server as well as system is discussed in this section.

The probability that the server is busy is given as:

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} \left(P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)\right) \quad (46)$$

The probability that the system is busy is given as:

$$P(System\ is\ busy) = \sum_{i>j\geq 0} \left(P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)\right) \quad (47)$$

7. Numerical and Graphical Representation of Busy Period

Following Bunday's work and using MATLAB programming the numerical results are found. Here the probabilities for server busy as well as system busy are obtained which are presented in the table below for various values of ρ keeping η, γ, r_1, r_2, a_1 and a_2 same in each case.

t	Probability(System Busy)			Probability(Server Busy)		
	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.2302	0.4074	0.5438	0.2133	0.3809	0.5126
2	0.3722	0.6061	0.7532	0.3299	0.5497	0.6954
3	0.4675	0.7174	0.8507	0.4036	0.6407	0.7779
4	0.5351	0.7859	0.9023	0.4546	0.6959	0.8178
5	0.5851	0.8306	0.9305	0.4921	0.7305	0.8322
6	0.6230	0.8602	0.9433	0.5207	0.7503	0.8273
7	0.6524	0.8783	0.9430	0.5426	0.7573	0.8065
8	0.6751	0.8858	0.9297	0.5592	0.7523	0.7731
9	0.6924	0.8826	0.9039	0.5709	0.7362	0.7300
10	0.7048	0.8686	0.8664	0.5781	0.7099	0.6802

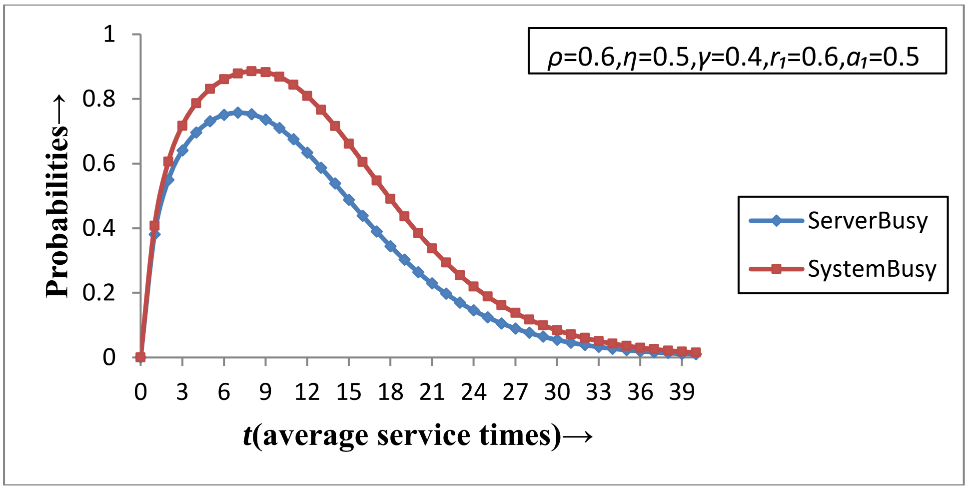

Figure 5

In Figure 5 the probabilities of server busy and system busy are plotted for the case ρ=0.6, η=0.5, γ=0.4, r_1=0.5(r_2=1-r_1) and a_1=0.6(a_2=1-a_1). In the beginning both curves increase rapidly and then decreases. The probability of System busy remains higher than probability of Server busy which is required.

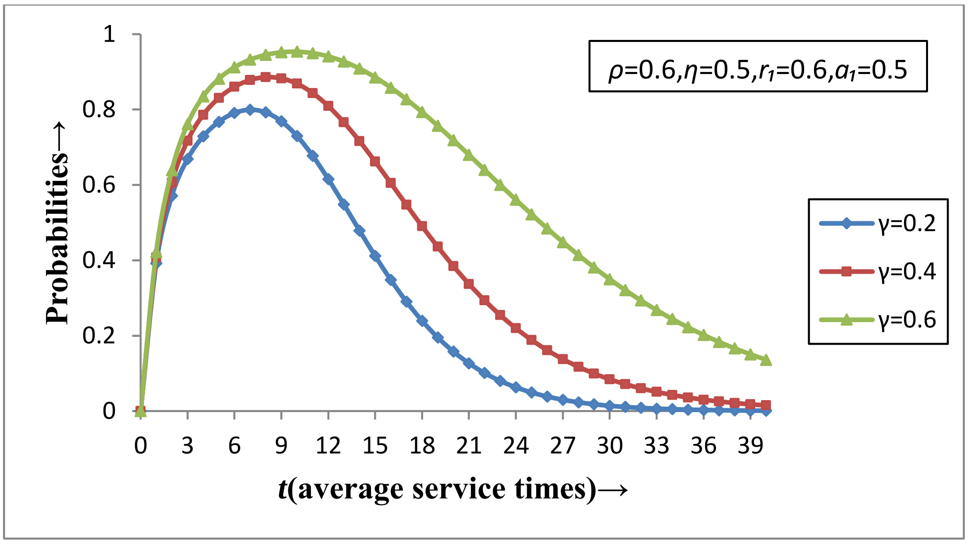

Figure 6

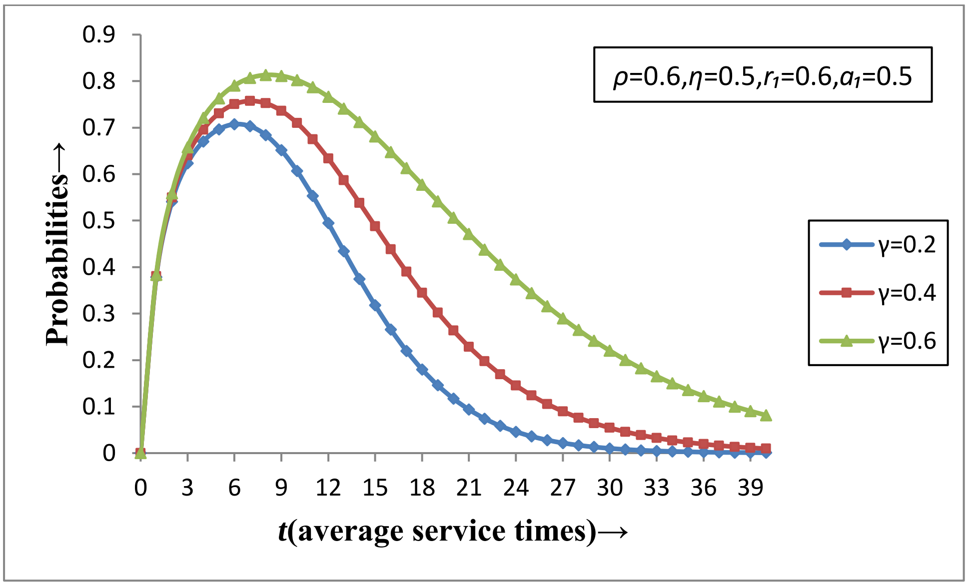

Figure 7

The effect of change of γ on System busy and Server busy is studied in figure 6 and 7 keeping other parameters as ρ=0.6, η=0.5, r_1=0.6(r_2=1-r_1) and a_1=0.5(a_2=1- a_1) against time *t*. The probabilities start increasing from 0 at *t*=0 and then start decreasing gradually and these are higher for larger values of γ.

8. Conclusion

We considered a system with feedback having two non-identical parallel servers in this paper which can be implemented in modeling of computer and communication systems. The time dependent probabilities are obtained when system is busy or free. The numerical and graphical results are presented which shows the influence of change in arrival rate, retrial rate and feedback factor. The busy period distribution and its numerical and graphical representation are also given.

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